BASICS OF RENORMALISATION GROUP-DIVIDE AND CONQUER

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A first introduction to basic features of renormalisation group applied to critical phenomena is presented. As an illustration, 1D Ising model and Gaussian model are considered .Understanding of universality and scheme to compute critical exponents are also given.

I INTRODUCTION

Developments in Science owes itself to a miracle: miracle being phenomena in different length scales decouple.Without this feature, science would not have even taken off.

Let us understand this statement by taking fluids as illustration. Fluids have properties like viscosity, surface tension, and all of these features of fluids have been well studied. Now consider the description of fluids at different length scales. At $1m$ distance, description is as a continuous medium in terms of density and velocity and obeying Euler equations. At $10^{-5}m$ description will be as granular material.At $10^{-10}m$ it is described by atoms/molecules following quantum mechanics.At 10^{-15} description is in terms of nucleus applying strong interaction physics.At still smaller distance of $10^{-34}m$, laws of still unknown quantum gravity effects will show up.

Fortunately to study fluids at $1m$ scale we do not have to know,the still to be discovered quantum gravity laws at $10^{-34}m$.If that were so, even Newton might have to wait for quantum gravity to be solved ,to understand basic properties of matter.This is the miracle alluded to above: phenomena at different scales decouple, and each can be studied independently.Physics at each short distance scale only contributes to the values of the parameters in the succeeding larger scale.If those parameters are taken from experiments, then they can be studied independently. In the example above, strong interaction effects provide nuclear parameters, atomic physics provide atomic and molecular parameters.The molecular parameters provide macroscopic parameters of properties of matter like viscosity.

Difficulty arises when different length scales do not decouple.This happens close to critical point in continuous (or second order)phase transition.[1]In this transition order parameter increases from zero at T_c *continuously* to its maximum value at $T = 0$ as temperature is reduced.Recall that near T_C there are fluctuations in the order parameters in length scale given by correlation length ξ.Correlation length is the *maximum* distance to which spins are correlated.This means that fluctuations in the order parameter are from distance 0 to ξ.At T_C, since the correlation length diverges, the fluctuations are from 0 to all the way up to infinity.The degrees of freedom at different fluctuation length scale are entangled.Thus degree of freedoms associated with all length scales have to be taken into account.

This feature is also seen well in fluid system at criticality. The order parameter in this case is the density difference between liquid and vapour.Near T_c the fluctuations of all length scale shows up by the presence of liquid drops and vapor bubbles of varying sizes, all mingled within each other.Fluid has not made up its mind whether to condense or not.This density fluctuations is experimentally seen in scattering of light.When the fluid is scattered by light,it looses its transparent nature and there will be a white milkish patch ,due to scattering of wavelengths of order few thousand Angstrom. Since this is much larger than lattice spacing, lattice cannot be the cause.It happens only at critical temperature.Hence the large fluctuations are the culprits causing it. This is known as critical opalescence. Mathematically this is due to divergence in the density-density correlation function, equivalently compressibility, at critical point.

Similar feature also occurs in quantum field theory.Virtual particles of arbitrary energy scales are emitted and absorbed owing to uncertanity principle.There is fluctuation in energy in all scales contributing to the loop diagram.

The problems of these kind needs renormalisation group (RG) method to handle.This procedure, to handle these kinds of problem was developed by K.G.Wilson, who was awarded the Nobel prize for this contribution[2]. The method is to systematically eliminate the degree of freedoms at short distances and obtain effective theory for long distance.This will, as we will see, provide relation between parameters of theory at one length scale to another scale. More broadly speaking , Wilson scheme provides long distance effective description, wherein, short distance effects have been taken care of by suitable redefinition of the parameters valid for long distance.

How is this problem avoided in most of the system?For most of the systems, the correlation length is only a small number and whole system is superposition of small systems with very small correlation length. Since the de-

grees of freedom within the correlation length is small , approximation methods works.

In the following ,first we illustrate the RG procedure for 1d Ising model, followed by Gaussian model(defined below).Then the general procedure of RG will be abstracted, explaining Scaling and Universality. Finally we conclude how RG has given a new vantage point to understand quantum field theory.Some earlier reviews on RG are [3,4,5]. There are several excellent books on Renormalisation group and critical phenomena.A few recent ones include,[6,7]

II REAL SPACE RG OF 1D ISING MODEL

The 1D Ising model is defined by the Hamiltonian

$$H = -J\sum_i s_i s_{i+1} + \frac{H}{2}(s_i + s_{i+1}) \tag{1}$$

whose partition function is

$$Z = \sum_{s_1}\sum_{s_2}..\prod_i \exp ks_i s_{i+1} - \frac{h}{2}(s_i + s_{i+1}) \tag{2}$$

where $k = \beta J, h = H\beta$ The above equation(2) can be written as

$$Z = \sum_{s_i}\prod_i K(s_i, s_{i+1}) \tag{3}$$

where$K(s_i, s_{i+1}) = \exp(ks_i s_{i+1} - \frac{h}{2}(s_i + s_{i+1}))$ Note the parameters of the model are temperature T, Magnetic field H. Instead of summing over spins at all sites at one go, the spirit of RG is to first sum spin degrees at all even sites only.[8]

$$\sum_{s_2=\pm} K(s_1, s_2)K(s_2, s_3) \tag{4}$$

The above will be a function only of s_1, s_3. It is

$$\begin{aligned} &= \exp\{k(s_1 + s_3) - (h/2)(s_1 + s_3 + 2)\} \\ &+ \exp\{-k(s_1 + s_3) - (h/2)(s_1 + s_3 - 2) \equiv \tilde{K}(s_1) \end{aligned} \tag{5}$$

The other even sites elimination by summing over it will also have similar structure. Next we demand that equation(3) to have the same structure as the original Ising model but for a different set of parameters k', h'.

$$K(s_1.s_3) = \exp k'(s_1 s_3) - h'/2(s_1 + s_3) \tag{6}$$

Similar structure follows for all other even site spins. Next we get an explicit relation between old parameters (before even site spins were eliminated) and new ones (after they eliminated). $\tilde{K}(s_1, s_3)$ is a $2X2$ symmetric matrix with s_1, s_3 taking $\pm$ value.

$$\tilde{K}(+1,+1) = \exp(2k - 2h) + \exp -2k \tag{7}$$
$$\tilde{K}(+1,-1) = K(-1,+1) = \exp -h + \exp h \tag{8}$$
$$\tilde{K}(-1,-1) = \exp(-2k) + \exp(2k + 2h) \tag{9}$$

Similarly matrix elements of the symmetric $2X2$ matrix equation(6) are

$$K(+1,+1) = \exp(k' - h') \tag{10}$$
$$K(+1,-1) = \exp -k' \tag{11}$$
$$K(-1,-1) = \exp(k' + h') \tag{12}$$

By equating equations(7) and (10) we get the relation between primed parameters and unprimed ones:

$$\begin{aligned} \exp(-2h') &= \exp(-2h)\frac{cosh(2k - h)}{cosh(2k + h)} \\ \exp(4k') &= \frac{cosh(2k - h)cosh(2k + h)}{cosh^2 h} \end{aligned} \tag{13}$$

Thus there is a reduction of degrees of freedom(elimination of $N/2$ even spins) and concomitant change in parameters.

The second step is to rescale the distance in units of lattice spacings to bring it back to the original system.With even site spins eliminated the lattice spacing between the remaining spins is *twice* the original spacing.To compare with the original system , we must rescale the distance by *half* so that the lattice spacing is the same.Though in general scaling may require scaling spin degrees also , in this example there is no necessity for it. This process can be repeated with parameters changing under each iteration. Next we find the fixed point of the transformation.

It is convenient to define $x = \exp(-4k)$ and $y = \exp(2h)$ then equation(13) is

$$\begin{aligned} x' &= f(x,y) = x\frac{(1+y)^2}{(1+yx)(y+x)} \\ y' &= g(x,y) = y\frac{x+y}{1+xy} \end{aligned} \tag{14}$$

These equations provide the recurrence relations between the parameters. Next we have to identify the fixed point of the transformation.These are the solutions of the equation :$x' = x, y' = y$ i.e,

$$\begin{aligned} x &= x\frac{(1+y)^2}{(1+yx)(y+x)} \\ y &= y\frac{x+y}{1+xy} \end{aligned} \tag{15}$$

The solutions are

1.
$$x = 1, y \tag{16}$$
which is $T \to \infty$, describing paramagnetic phase and has zero correlation length and hence not a critical point.

2.
$$x = 0, y = 1 \tag{17}$$
This is $T = 0, H = 0$, which is a critical point and has infinite correlation length.This critical fixed point is what we will be interested in.

We will study the behavior of the fixed point of critical point.Taylor expand the recurrence relation equation (14)around critical fixed point $x^\star = 0, y^\star = 1$.

$$x' = x^\star + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \tag{18}$$
$$y' = y^\star + \frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial y}\delta y \tag{19}$$

Where $\delta x = x - x^\star$ and $\delta y = y^\star - y$, similarly for $\delta x'$ and $\delta y'$. This gives using equation(15)

$$\delta x' = 4\delta x \tag{20}$$
$$\delta y' = 2\delta y \tag{21}$$

This gives the deviation from fixed point after each iteration.Since ,by summing over all even sites spins, we have effectively increased the distance by 2 the above is actually

$$\delta x' = 2^2\delta x \tag{22}$$
$$\delta y' = 2\delta y \tag{23}$$

In general if b units were scaled for a variable A,then we have $\delta A' = b^{l_A}\delta A$

For Ising model this gives $l_t = 2$ and $l_h = 1$.As we shall see this can be used to calculate the critical exponents.

As the RG transformations were done on spins on a lattice sites in 1d coordinate space, this is real space RG. Next we consider RG in momentum space as opposed to real space.

III MOMENTUM SPACE RG OF GAUSSIAN MODEL

Gaussian model is defined by the Hamiltonian

$$\beta H = \int d^d x \frac{1}{2}[\bigtriangledown\phi.\bigtriangledown\phi + r_0\phi.\phi] \tag{24}$$

Note the fields appear quadratically in the Hamiltonian, hence the name Gaussian.[5] In momentum space ,this Hamiltonian is

$$\beta H = \frac{1}{2}\int \frac{d^d k}{[2\pi]^d}(k^2 + r_0)\phi(k)\phi(k)^* \tag{25}$$

$$Z = \int \prod_k d\phi(k)\exp(-\beta H) \qquad where \;\; 0 < k < \Lambda \tag{26}$$

Divide k into two regions $0 < k < \Lambda/s$ and $\Lambda/s < \Lambda$ Note this is a sharp division of wave vectors into two divisions. Divide the field $\phi(k) = \phi'(k) + \sigma(k)$ where
$\phi(k) = \phi'(k) \quad 0 < k < \Lambda/s$
$\phi'(k) = 0 \qquad \Lambda/s < k < \Lambda$
$\sigma(k) = 0 \quad 0 < k < \Lambda/s$
$\phi(k) = \sigma(k) \quad \Lambda/s < k < \Lambda$
Also $\int dk\phi'(k)\sigma(k) = 0$ since there is no overlap in non vanishing region .Define $\bar{dk} = \frac{d^d k}{(2\pi)^d}$

$$\beta H = \frac{1}{2}\int \bar{dk}|\phi' + \sigma|^2(k^2 + r_0) \tag{27}$$
$$= \frac{1}{2}\int \bar{dk}(\phi'\phi' + \sigma\sigma)(k^2 + r_0) \tag{28}$$
$$Z = \int d\phi' e^{-\int \bar{dk}\frac{1}{2}(k^2+r_0)|\phi'|^2}\int d\sigma e^{-\int \bar{dk}\frac{1}{2}(k^2+r_0)|\sigma(k)|^2} \tag{29}$$

Observe there is no interference between low and high wave vector modes and they factorize as we have only Gaussian model.If interaction like ϕ^4 is added, this feature will fail.

Denote by $Z_>(Z_<)$ contribution of $\sigma(k)(\phi')$ field. We are after the long wave vector modes.The contribution of $Z_>$ is only for the free energy and not for the recursion relation between parameter r_0 .Hence we can ignore them.

$$Z = Z_> \int \prod_{k=0}^{\Lambda/s} \exp - \int \bar{dk}\frac{1}{2}(k^2 + r_0)|\phi'|^2 \tag{30}$$

The next part of the recipe of RG is to rescale k such that it goes over to the same range i.e, between 0&Λ. k scales as $k \to sk$ so that rescaled k ranges from 0to Λ. The third part of RG is, ϕ' has to be scaled correspondingly so that the k^2 coefficient remains $\frac{1}{2}$.

$$\phi'(k) \to \phi(k) = (1/z)'\phi(k) \tag{31}$$

Scaling $k \to k' = sk$, we regain the same range for k' ie; $0 \to \Lambda$.This scales H to

$$\bar{d}'ks^{-d}(k'^2 s^{-2} + r_0)z^2\phi'('k) \tag{32}$$

Keeping the coefficient of kinetic energy term (without dimensional parameter) invariant gives

$$k'^2\phi' s^{-d-2}z^2 = k'^2\phi' \tag{33}$$
$$\text{gives} \quad z = s^{\frac{d}{2}+1} \tag{34}$$

$$r_0 \to r_0' = s^{-d}z^2 = s^{-d+d+2}r_0 = s^2 r_0 \equiv r(s) \tag{35}$$

This equation(35) relates the (only) parameter in the theory at different scales.For infinitismal change of momentum scale

$$\begin{aligned} r(s+ds) &= (s+ds)^2 r_0 \\ r(s) + \frac{dr}{ds}ds + .. &= s^2 r_0 + 2sr_0 ds + .. \\ \frac{dr(s)}{ds} &= 2sr_0 \\ s\frac{dr}{ds} &= 2s^2 r_0 \\ s\frac{dr}{ds} &= 2r(s) \\ \frac{dr}{d\ln s} &= 2r(s) \end{aligned}$$

This equation represents $'$flow $'$of the r_0 under RG transformation.

The fixed point of the transformation is solution of$\frac{dr}{d\ln s} = 0$.ie $r \equiv r^* = 0$ This fixed point is Gaussian fixed point.Once the RG transformation reaches this point , r will remain stay put-ie;fixed.

Two remarks are in order

1)In Gaussian model no new terms are generated under RG transformation. In this case it is similar to 1d Ising model. In general, new terms will be generated, with coefficients dependent on wave vectors too.

2)When we refer as coupling constant, the word constant makes us take them to be a universal number like Plancks constant or π.But thats not correct, as the extension 'constant' is a misnomer.It is more a coupling function , whose value depends on the length scale at which it is measured.The recursion relation provides the change in the value of the so called constant as we change the scale at which we observe.Electrons charge, as listed in tables, is related to the coupling constant at zero k .

IV GENERAL FEATURES

GENERAL FEATURES

The issues RG tries to explain include:a) Universality of critical exponents b)calculation scheme for calculating exponents.

Recipe for RG

Ingredients Required:

a)Hamiltonian with an order parameter (or field)

b) a method for introducing cut-off (there is no unique choice: for real space , it is lattice, for momentum space, it can be sharp cut-off, like theta function, or smooth cut-off

c) a scheme for taking degrees of freedom associated with short distances/large wave-vectors.If the scheme has a controlled approximation it is better.

Procedure:

Take the given Hamiltonian with the chosen cut-off method. Integrate the short distance degrees of freedom by applying the scheme chosen: this can be ,in real space by majority rule/avarage spin for the block spin, in k-space perturbatively integrate over the high momentum modes of the field. Let the momenta integrated by between $\Lambda > k > \Lambda/s$. The resulting Hamiltonian contains fields for modes of momenta of only $k < \Lambda/s$.But we cannot compare this with the starting system , as they are defined in reduced range of k-space.To bring them back to their original range, *scale* momentum /coordinate. This ,in general, needs *rescaling of field*, ie change in the magnitude of the field. The net result of these three operations is that the original Hamiltonian,with a given set of couplings, transforms to a Hamiltonian with different coupling constant and (in general)new set of couplings .This provides the relationship between the two set of parameters.Take the relationship between the original set of couplings and new set of couplings.

Serve it hot for consumption!

Classification of Scaling variables

Given an order parameter field and a symmetry , we can consider the most general Hamiltonian involving them consistent with the symmetry.Each term will have a coupling parameter.Note here though the coefficients are referred to as 'coupling constants', it is better to regard them as parameters in the theory.Define a Hamiltonian space , which is the space of coupling constants.If one wishes to call Hamiltonians with different couplings as different theories,then this space is a 'theory space'.For eg; scalar order parameter case,

$$H = |\bigtriangledown \phi|^2 + r\phi^2 + u\phi^4 + g_1\phi^6 + g_2|\bigtriangledown \phi|^2\phi^2 + ... \quad (36)$$

Thus in general it has infinite number of couplings.To make it better to handle, we will consider a subspace , which is m dimensional, with couplings $K_\alpha = \{K_1, K_2, ..K_m\}$.This set of values of K can be represented as a point in m−dimensional space. Under RG transformation, the set of couplings will change to another value, which can be pictured as a point in theory space moving to a different space.

$$K_\alpha \to K'_\alpha = R_\alpha(K_1, ..K_m) \quad (37)$$

Thus, by repeated RG transformation $K \to K' \to K''$.. The transformation is said to reach a *fixed point* if $K \to K^* \to K^*$. ie point in theory space stops moving further under RG flow.K^*_α are the couplings at the fixed point. The correlation length scales under RG transformation, since one of the action of RG is to scale the

distances/momenta. Correlation length in extrinsic unit like cm is invariant.But in units of intrinsic length scale changes.$\xi(K) \to \xi'(K')$.At fixed point,$\xi(K^*) \to \xi(K^*) = l^{(-d)}\xi(K^*)$. ξ under scaling must remain invariant.No finite number will be invariant.Only $0, \infty$ will remain so.Hence fixed point physics must correspond to these two values of ξ.The value 0 correspond to stable bulk phase as in that phase, degrees of freedom have only short range correlation.$\xi = \infty$ case represents the (unstable) critical phase.

Given the fixed point,the couplings are expanded about it.

$$K'_\alpha = R_\alpha(K) \quad (38)$$
$$K^*_\alpha + \delta K'_a = R_\alpha(K^* + \delta K) \quad (39)$$
$$= R_\alpha(K^*) + \frac{dR_\alpha}{dK_\beta}\delta K_\beta \quad (40)$$
$$= K^* + \frac{dR_\alpha}{dK_\beta}\delta K_\beta \quad (41)$$
$$\text{define} \quad \frac{dR_\alpha}{\delta K_\beta} \equiv M_{\alpha\beta}(l) \quad (42)$$
$$\delta K^* = M(l)\delta K \quad (43)$$

Note $M_{\alpha\beta}$ is $m \times m$ matrix and is not assured to be symmetric.Let $V^{(\sigma)}, l^\sigma(l)$ be the right eigenvector and corresponding eigen value.$\sigma = 1, ..m$.
Expanding $\dot{K}$ in terms of the eigenvectors

$$\delta K_\alpha = a^{(\sigma)}_{\alpha\beta} V^{(\sigma)}_\beta$$
$$\delta K'_\alpha = a'^{(\sigma)}_{\alpha\beta} V^{(\sigma)}_\beta \quad (44)$$
$$a'^{(\sigma)} = l^{(\sigma)}(l) a^{(\sigma)} \quad (45)$$

Now what can we say about l^σ,eigenvalues of M? The RG transformation obeys three axioms of group: a) existence of identity-$l = 1$ gives $M = I$
b)Existence of product rule .If represents Matrices associated with integrating momentas $\Lambda \to \Lambda/l_1 \to \Lambda/l_2$which can be achieved directly $\Lambda \to \Lambda/(l_1 l_2)$. Hence $M(l_1)M(l_2) = M(l_1 l_2)$
c)Ofcourse associativity is satisfied as it is a matrix. Importantly inverse does *not* exist.Once we trace out some degrees of freedom , we cannot uniquely fix the original configuration.This is like given a matrix we can tell its trace, but if I give you trace of a matrix, can you tell me a unique matrix which has this trace?

Hence the transformation does not form a group in the precise sense.It is sometimes referred to as semi-group.

These three properties of the group restricts the eigenvalue $l^\sigma(l) = l^{d_\sigma}$ The eigenvalue is given by d_σ. The eigenvector V^σ is called **Relevent** if the associated eigenvalue $d_\sigma > 0$, **Irrelevant** if $d_s < 0$, **Margininal** if $d_\sigma = 0$ Why are they called so?By repeated RG transformation , for relevent eigenvector its contribution to couplings K_α will go on increasing as $d_\sigma > 0$.Hence they are relavent. Similarly contribution of irrelevent eigenvectors will be decreasing.Hence it is irrelevent. Marginal variable does not contribute at linear order and hence to look at it contribution one should go beyond linear order.
Let $K_1, K_2..K_n (n < m)$ be the couplings which are relevant and the remaining irrelevant. For simplicity we consider there is no marginal variable. This classification of relevant,irrelevant and marginal will turn out to be important to understand universality and calculation of exponents.These definitions are with respect to a fixed point.The same variable with respect to a different fixed point can change their relevancy.

Universality

1. To understand, how systems with different T_C and different microscopics share the same exponents near criticality.

2. Definition:*critical surface* : In the m dimensional theory space, critical surface is a subspace, defined by setting all relevent couplings to zero.This is like in 3d space we can define a 2d surface,xy plane, by setting $z = 0$.The main feature of the Critical surface is the fixed point is contained in it. Since relevant parameters are zero and the effect of irrelevant ones will die down ,any point on the critical surface will flow under RG to the fixed point.This is the importance of the critical surface. The dimension of the critical surface is $m - n$.The number of conditions required to define this critical surface is known as co-dimension, which is n here.In the case of magnetic system, we will see that the relevant variables are temperature and the magnetic field, and hence the codimension of the critical surface is two.

3. On the theory space we are going to define two kinds of transformation: one a physical transformation ,which can be done in principle by a "knob"in experiment.This transformation will bring the system to its critical point. The second is a mathematical RG transformation, which cannot be achieved by experimentalists by tuning "knobs" of apparatus, but it exits only as theorists construction.

4. Consider a line piercing the critical surface at a point P.This change is effected by physical transformation, and the system is tuned to be at the point P .For eg in magnetic system this physical transformation is achieved by tuning the system to be at its $T = T_C \quad H = 0$ .Similarly a different system will be piercing the critical

surface at a different point Q. The critical surface contains fixed point.Hence both the points P, Q which are on the critical surface, and are at their respective critical point, under RG flow to the fixed point.(fig-1)
What happens if the system is slightly off the critical surface? This is done by the physical transformation by tuning the system close to the critical point. Under RG transformation this point p' will flow close to the fixed point.Since the system is not exactly on critical surface, the relevant variable is not exactly zero, but close to it.Near the fixed point , due to their relevance , RG effects amplify their effect, and the trajectory moves away from the fixed point.Similar behavior is expected for the system Q' which represent the system Q slightly away from the critical point. For eg; these two points can represent elements Ni and iron.(fig-2) In fact for all points slightly off the critical surface will undergo similar behavior.

This explains that all points on the critical surface will have universal behavior as their long distance effective description will be governed by the same fixed point.The critical exponents are provided by the eigenvalues of *relevant* variable of this fixed point. This has been seen in earlier examples.
To summarise: all systems on the critical surface,ie Hamiltonians having these $m - n$ irrelevant couplings will all share the same exponents and have the same long distance behavior.The microscopic parameters which distinguish the different systems sharing the same universal class, are irrelevant in the RG sense, explaining universality. It explains the universality by staying close to T_C , it is the long distance "cooperative behavior" and not short distance dirty details that are dominating.

SCALING

1. At criticality there are fluctuations of order parameter at all length scales.Hence there is no specific length scale which shows up. To whatever length scale we zoom in the system will appear the same. The scale invariance shows up in this way.

2. Mathematically it shows up in correlation function $G(\mathbf{x}, \mathbf{y})$. Correlation function must be decreasing function of the distance between the points.It can be exponentially falling or power law dependance. But exponential functions are not scale invariant.

$$\exp(\lambda x) \neq \lambda \exp(x) \tag{46}$$

But power law functions have this property.

$$\frac{1}{(\lambda x)^p} = \lambda^{-p} \frac{1}{x^p} \tag{47}$$

This means there is self-similarity as the distances are scaled. Correlation function at *criticality* is a power law function.Mathematically ,this is seen as

$$G(\mathbf{x}, \mathbf{y}) \sim \frac{\mathbf{1}}{|\mathbf{x} - \mathbf{y}|^{\mathbf{d-2}+\eta}} \tag{48}$$

Here η is a critical exponent which characterizes the power law behavior. It is zero for Gaussian model as different modes of the field decouple. When it is non-zero, a)it shows how fluctuation at lattice level(or at momentum cut-off)gets coupled with long distance fluctuation and b)it reflects in failure of naive dimensional analysis of fields .Fields then have dimensions different from the engineering dimension;they get "anomalous dimension". Physically the power law behavior is seen as the presence of clusters of up and down spins of all sizes or in fluid system as drops of liquid and bubbles of gas of all sizes. This behavior is also similar to the structure of some naturally occurring objects like: clouds, river basin,..There is self similarity of structure at all length scales.These are known as fractal structure.
Thus there is a deep connection between scale invariance-power law behavior-criticality.

3. Singular part of free energy close to criticality has scaling form.This follows from RG point of view.Consider ,1d Ising model as illustration. Free energy per unit site (which, incidentally is an intensive quantity)is $f \propto \frac{1}{N} \ln Z_N$
Under RG first note that $Z_N(K)$ is invariant, where K denote couplings. Let us split the N degrees of freedoms into N' and N''. Out of N sites N' has been first traced.

$$f(K) = \frac{1}{N} \ln[Tr_N e^{-H(K)}] \tag{49}$$

$$= \frac{1}{N} \cdot \frac{N'}{N'} \ln Z_{N'}(K') = \frac{N'}{N} f(K') \tag{50}$$

$$\text{Since} \quad \frac{N'}{N} = b^{-d}. \tag{51}$$

$$\tag{52}$$

Thus f has the required scaling behavior.After b^d degrees of freedoms have been summed out ,$K \to K_b$, where K_b is given in terms of K by RG equation.

$$f(K) = b^{-d} f(K_b) \tag{53}$$

For specific case of couplings being temperature$t$and magnetic field h, (which are relevant parameters in Ising model)

$$f(t,h) = (b^{-d})^l f(t_l, h_l) \tag{54}$$

calculation of exponents

1. Recall that from the linear analysis of the RG transformation for parameter, say K, about the fixed point , we got $K_b = b^{y_K}$. Hence $t_b = b^{y_t}, h_b = b^{y_h}$.The quantities y_t, y_h are calculable from RG transformation and are > 0 as they are relevant . We will see that the exponents are given in terms of them ,using scaling form of free energy.

   From the scaling form of the free energy

$$\begin{aligned} f(t,h) &= b^{-d} f(b^{y_t}t, b^{y_h}h) \\ \text{Choose } b \text{ such that} \quad b &= t^{-1/y_t} \\ f(t,h) &= t^{d/y_t}\Phi(t^{-y_h/y_t}h) \end{aligned} \tag{55}$$

   As illustration few of critical exponents are calculated below:

2. Exponent β.

$$\begin{aligned} m(t,h) \equiv \frac{\partial f}{\partial h} &= t^{d/y_t}\Phi' \quad t^{-y_h/y_t} \\ m(t,0) &\sim t^{d/y_t - y_h/y_t} \\ \beta = \frac{d-y_h}{y_t} & \end{aligned} \tag{56}$$

3. Exponent γ

$$\begin{aligned} \chi &\equiv \frac{\partial m}{\partial h} \\ &= t^{\frac{d-y_h}{y_t}}\Phi'' \quad t^{-y_h/y_t} \\ \chi(t,0) &\sim t^{\frac{d-2y_h}{y_t}} \\ \gamma &= \frac{d-2y_h}{y_t} \end{aligned} \tag{57}$$

4. Exponent δ
   Exponent $\delta$is defined by $M \sim h^{1/\delta}$ at $t = 0$. consider

$$\begin{aligned} M &= \frac{\partial f}{\partial h} \\ &= t^{d/y_t}\Phi' \quad t^{-y_h/y_t}. \end{aligned} \tag{58}$$

   We cannot put $t = 0, h \neq 0$ in the above as M itself vanishes,which is not correct.As $x \to 0 \quad \Phi'(x) \to x^q$ with q to be determined such that M is finite. As $t \to 0$

$$\begin{aligned} M &= t^{(d-y_h)/y_t}\left(\frac{h^q}{t^{qy_h/y_t}}\right) \\ &= t^{(d-y_h)/y_t - qy_h/y_t} h^q \end{aligned} \tag{59}$$

$$\begin{aligned} \text{demanding power of } t = 0 \quad q &= \frac{d-y_h}{y_h} \\ \text{hence} \quad \delta^{-1} &= \frac{d-y_h}{y_h} \end{aligned} \tag{60}$$

Conclusion

RG is a framework to understand the long distance behavior of a system.The theory valid at long distance is obtained by systematically eliminating the short distance effects and incorporating them as change in the parameters defining the theory. Thus the seemingly insurmountable difficult problem of all length scales coupled,is won by diving into smallest length scale and conquering them one by one: *divide and conquer* is the policy! .
RG has also shed light on quantum field theory.In the olden days ,in quantum field theory,'good'(ie renormalisable) theories were expected to be valid for all length scales upto zero diatance.A cut-off was introduced more as a convenient intermediary to absorb certain infinities. But RG has provided a vantage point to understand some of the behavior.Now cut-off is considered as a necessity, above which the theory is not defined. Some times it is possible to lump the effects of short distances as change in finite number of parameters.They are, in older language 'good' theories.In such theories their region of validity will be decided by experiments.Sometimes it is not possible and it may require infinite number of parameters.These were considered earlier as "bad" (non-renormalizable)theories. But even these theories can have a predictive power to an arbitrary level of accuracy.This insensitiveness of the long distance phenomena to short distance behavior (which is is universality in critical phenomena) is presently seen as renormalisablity feature in quantum field theory.
RG is a broad framework and it can be considered as giving a theory of writing theories.Its application to more areas ,one can say,is in beginning stages.

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