Exact Eigenstates of a Relativistic Spin less Charged Particle in a Homogeneous Magnetic Field

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(*Submitted* on 26/3/2011)

Abstract

Abstract: In this article we work out the exact eigenvalues and eigenfunctions of a spin less charged particle placed in a uniform magnetic field that has azimuthal symmetry. The possible applications are mentioned.

PACS: 03.65.Ge Keywords: Bound states; Landau Levels; Uniform Magnetic field

1. Introduction

In non-relativistic quantum mechanics, for a spin less charged particle placed in a uniform magnetic field, one obtains the well-known Landau-levels [1]. There are several applications of that result [2]. In many instances, charged particles can have relativistic high velocities [3], necessitating relativistic correct expressions for Landau-levels. In this article we deal with a relativistic spin less charged particle placed in a uniform magnetic field. We work out the exact energy eigenvalues and eigenfunctions. The formulae reproduce the Landau levels in the non-relativistic limit, as they should. Our result can find applications in astronomical bodies like white dwarfs and neutron stars.

Klein-Gordon particle in a uniform magnetic field

The modern theory of interaction of fields and particles demands the exact solutions that describe the quantum states of charged spin less particles in the external electromagnetic fields. Such solutions are very much useful to analyze and characterize these particles in the external fields.

The relativistic relation that connects energy and momentum of a free particle is given by

$$E^2 = P^2c^2 + m^2c^4$$

where E includes rest mass energy mc^2.

Replacing E and P by their corresponding operators

$$E \rightarrow i\hbar \frac{\partial}{\partial t}$$

and

$$P \rightarrow -i\hbar\, \nabla$$

and then operating on a wave function $\psi(\boldsymbol{r}, t)$, we get

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \vec{\nabla}^2 \psi + m^2 c^4 \psi \tag{1}$$

The above equation is the well known Klein-Gordon equation for a free particle. The Klein-Gordon equation for a charged spin less particle placed in an electromagnetic field with potentials $\left(\phi, \frac{\vec{A}}{c}\right)$ is obtained by using the minimum coupling rule [4] as:

$$\left[i\hbar\frac{\partial}{\partial t}-e\phi\right]^2\psi(\vec{r},t)=\left(-i\hbar c\vec{\nabla}-e\vec{A}\right)^2\psi(\vec{r},t)+m^2c^4\psi(\vec{r},t)$$

in the transmitted light composed of multiple degrees of elliptical polarization.

For a uniform magnetic field H, which is chosen to be along z direction, we can choose the potentials as:

$$\phi=0,\ H_x=H_y=0,\quad A_z=0,\ A_x=-\frac{yH}{2},\ A_y=\frac{xH}{2} \tag{3}$$

For a static magnetic field,

$$\text{div}\,\vec{A}=0$$

Thus for a time-independent magnetic field the energy eigenvalues and eigenfunctions may be obtained by solving the following equation: For two space dimensions,

$$E^2\psi(\vec{r})=\left[-\hbar^2c^2\nabla^2-eHcL_z+\frac{e^2H^2}{4}\left(x^2+y^2\right)+m^2c^4\right]\psi(\vec{r}) \tag{4}$$

In terms of cylindrical coordinates $(\vec{r},\phi,z)$ this equation simplifies to

$$E^2\psi(\vec{r},\phi,z)=-\hbar^2c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}\right]\psi(\vec{r},\phi,z)+ieHc\,\hbar\frac{\partial}{\partial\phi}\psi(\vec{r},\phi,z)$$
$$+\frac{e^2H^2}{4}r^2\psi(\vec{r},\phi,z)+m^2c^4\psi(\vec{r},\phi,z)$$

This equation could be solved by the method of separation of variables using ansatz

$$\psi(\vec{r},\phi,z)=R(r)\Phi(\phi)Z(z) \tag{6}$$

Taking $\Phi\sim e^{il\phi}$ and $Z(z)\sim e^{ikz}$,

clearly the motion along the z-direction, the direction of the magnetic field is that of a free particle. For simplicity, we can put k = *0* without any loss of generality when we are considering the bound states.

The equations for $\Phi(\phi)$ and $R(r)$ will be:

$$\frac{d^2\Phi}{d\phi^2}=-l^2\Phi \tag{7}$$

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right]R(r)+\left[\frac{E^2-m^2c^4}{\hbar^2c^2}-\frac{l^2}{r^2}-a^2r^2+2al\right]R(r)=0 \tag{8}$$

where

$$a=\frac{eH}{2\hbar c} \tag{9}$$

l has to be an integer due to periodic boundary condition $\Phi(\phi)=\Phi(\phi+2\pi)$. The radial equation can be reduced to the following convenient dimension less form by substitution $\rho=ar^2$

$$\left[\rho\frac{d^2}{d\rho^2}+\frac{d}{d\rho}+\left(\lambda+\frac{l}{2}\right)-\frac{\rho}{4}-\frac{l^2}{4\rho}\right]R=0 \tag{10}$$

Where $\lambda=\dfrac{E^2-m^2c^4}{4a\hbar^2c^2}$ (11)

Using the asymptotic conditions of the radial function

For large ρ equation (10) reduces to

$$\frac{d^2R}{d\rho^2}-\frac{1}{4}R\cong 0$$

$$R_{\pm} \sim e^{\pm\frac{\rho}{2}}$$

Only the negative power is acceptable from physical considerations.
For small ρ equation (10) reduces to

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{l^2}{4\rho^2}R \cong 0 \qquad (12)$$

Taking $R \sim r^q$, we find for q the following equation:

$$q^2 - \frac{l^2}{4} \cong 0$$

Hence

$$q = \pm\frac{l}{2} \qquad (13)$$

Once again, from physical considerations only $q = \frac{l}{2}$ is acceptable.
Hence putting

$$R(\rho) = \rho^{l/2} e^{-\rho/2} L(\rho)$$

We obtain the following equation for $L(\rho)$:

$$\rho\frac{d^2L}{d\rho^2} + [(l+1)-\rho]\frac{dL}{d\rho} + \left[\lambda + l - \frac{1}{2}\right]L = 0 \qquad (14)$$

Power series solution of the following type for the above equation can be obtained:

$$L(\rho) = \sum_{\nu=0}^{\infty} a_\nu \rho^\nu \qquad (15)$$

The coefficients a_ν satisfy the following recurrence relation

$$a_{\nu+1} = \frac{\left(\nu + \frac{1}{2} - l - \lambda\right)}{(\nu+1)(\nu+l+1)} a_\nu \qquad (16)$$

Unless the power series terminates for some finite value of ν we get a diverging solution for $R\ (\rho)$, similar to the solution obtained for the Hydrogen atom problem.
Let the highest value of ν that terminates the series be s, then

$$\lambda = \left(s - l + \frac{1}{2}\right) \qquad (17)$$

Putting

$$s - l = n \qquad (18)$$

We get

$$\lambda = (n+1/2) \qquad (19)$$

Where n is a positive integer.
Substituting for λ from equation (11), we find the allowed energy eigenvalues as:

$$E_n = \left[m^2c^4 + 2e\hbar cH\left(n + \frac{1}{2}\right)\right]^{\frac{1}{2}} \qquad (20)$$

These are the quantized energies of the charged particle.
We may expand the expression for energy in powers of $\frac{\hbar\Omega}{mc^2}$, where $\Omega = \frac{eH}{mc}$ is the cyclotron frequency in the magnetic field.
results have a good agreement with the theoretical predictions based on Eq. (1).
Then

$$E_n = mc^2 + \hbar\Omega\left(n + \frac{1}{2}\right) - \frac{1}{2}\frac{\hbar^2\Omega^2}{mc^2}\left(n + \frac{1}{2}\right)^2 + ... \qquad (21)$$

While the first term is the rest energy, the second term is precisely the non- relativistic Landau term. The third term may be taken as the relativistic correction to the Landau energy levels.
The normalized eigenfunctions may be expressed as:

$$\psi(\vec{r},\phi,z) = C_{nl}\left(ar^2\right)^{\frac{1}{2}} e^{-\frac{ar^2}{2}} L^l_{n+2l}\left(ar^2\right)e^{il\phi} \qquad (22)$$

Results and discussion

We have obtained exact expressions for the

energy eigenvalues and eigenfunctions for a relativistic spin less particle in a magnetic field.
It is possible to apply our results to high-speed charged particles orbiting such astronomical bodies like white dwarfs and neutron stars. Our results may find use in understanding degeneracy [5] of particles of matter under relativistic conditions.

References:

[1] Pokrovsky V. L. (2009), *Modern Physics*, Physics-Uspekhi, **52**, 1169-1176.
[2] Tesanovic Z. and Sacramento P. D. (1998), *Landau levels and Quasiparticle spectrum of Landau levels of extreme type-II super-conductors*, Phys. Rev. Letters, **80,** 1521-1524.
[3] Hyman A. T. (1997), *Relativistic charged particle motion in a constant field according to Lorentz force law*, **65**, 195-198.
[4] Hoagland H. (1978), *Minimal electromagnetic coupling in elementary quantum mechanics,* J. Phys. A: Math. Gen. **11.**
[5] Kippenhahn R. and Weigert A. (1994), *Stellar structure and evolution*, Springer-verlag, First edition.