Analyticity and Cauchy-Riemann Equations

S. Sivakumar[1]

[1]Theoretical Studies section
Indira Gandhi Centre for Atomic Research
Kalpakkam 603 102 INDIA
siva@igcar.gov.in

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Abstract

Cauchy-Riemann equations relate the real and imaginary parts of a complex-analytic function. These equations originate from the requirement for the uniqueness of the derivative, independent of the line or direction along which the limit is taken. This note attempts to make a presentation of the derivation in which the "direction of approach" is made explicit. Some interesting implications of the notion of analyticity are given.

What is the analytic function whose real part is x? It is $x+iy+c$, where c is a constant. Is there an analytic function whose real part is x^2? There is none. If we choose two functions $u(x,y)$ and $v(x,y)$ randomly, it is likely that $f(x,y) = u + iv$ is not an analytic function. Complex analyticity is about defining the derivative of a complex-valued function of the complex variable $z = x + iy$. Formally, the complex-valued function f is said to be analytic at a point $z \in C$ if

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}, \qquad (1)$$

exists and is unique irrespective of the direction along which δz approaches zero[1]. This requirement for uniqueness gives rise to the well known Cauchy-Riemann (CR) equations relating the real and imaginary parts of $f(z)$. To ensure uniqueness, two independent directions in the complex $x - y$ plane are chosen and the equality of the derivatives along the respective directions is demanded.

We recall that to define the derivative of a function of a single real variable, the left-sided and right-sided derivatives are to be equal[2]. In the complex plane, the number of directions are unlimited. The conventional choices for the two directions are along the x and y axes respectively. In fact, any two axes which are inclined at a nonzero angle are fine to establish the need for the CR equations. It would be better if this "direction of approach" is made to appear explicitly in the derivation of the CR conditions and in this note we attempt that.

With $f(z) = u(x,y) + iv(x,y)$, u and v being the real and imaginary parts respectively, and $\delta z = \delta x + i\delta y$,

$$f(z+\delta z) = f(x+\delta x + i(y+\delta y)) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y). \tag{2}$$

Expanding $u(x+\delta x, y+\delta y)$ and $v(x+\delta x, y+\delta y)$ about (x,y) and rearranging[3], we get

$$f(x+\delta x, y+\delta y) - f(x,y) = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \epsilon_1 \delta x + \epsilon_2 \delta y + i\left[\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \epsilon_3 \delta x + \epsilon_4 \delta y\right]. \tag{3}$$

The factors ϵ_1, ϵ_2, ϵ_3 and ϵ_4 vanish as δz approaches zero. These factors are included to represent the contribution from the higher order terms in δx and δy to the expansion of the function.

To bring out the dependence of the derivative on the direction of approach, it is advantageous to adapt the polar representation[4] of complex numbers to write $\delta z = r\exp(i\theta)$, so that $\delta x = r\cos\theta$ and $\delta y = r\sin\theta$. Here θ represents the angle δz makes with x-axis and this angle can be interpreted as the direction of approaching z as $\delta z \to 0$. In Fig. 1, three of the infinitely many possible directions are indicated. Approaching along the x-axis and y-axis correspond to choosing $\theta = 0$ and $\pi/2$ respectively. Using the polar representation of δz in the definition of the derivative of $f(z)$ yields

$$f'(z) = \lim_{r\exp(i\theta)\to 0} \frac{r\left[\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} + i\cos\theta\frac{\partial v}{\partial x} + i\sin\theta\frac{\partial v}{\partial y}\right]}{r\exp(i\theta)},. \tag{4}$$

Since $\exp(-i\theta)$ does not vanish for any real θ, the limit corresponds to $r \to 0$. Therefore,

$$f'(z) = \frac{\cos\theta\frac{\partial u}{\partial x} + i\sin\theta\frac{\partial v}{\partial y} + \sin\theta\frac{\partial u}{\partial y} + i\cos\theta\frac{\partial v}{\partial x}}{\exp(i\theta)}. \tag{5}$$

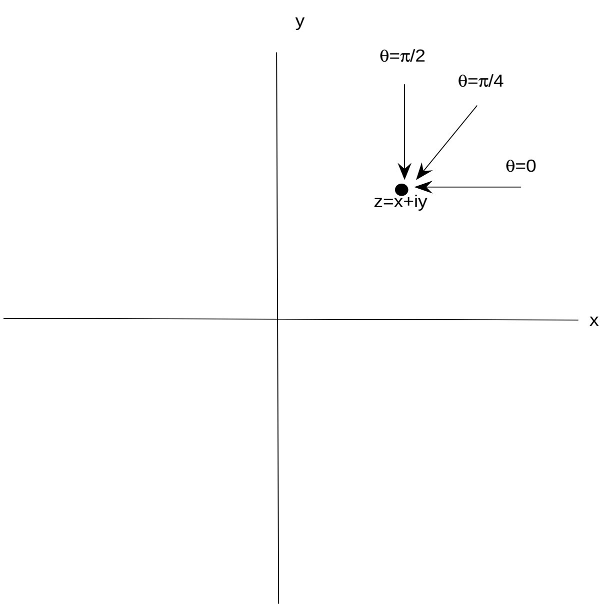

Figure 1: Some possible directions for approaching z as $\delta z \to 0$.

This ratio is independent of r. However, it depends on θ, the direction of approaching z. For the uniqueness of the derivative, the limit should be independent of θ. That is possible only if the factor $\exp(i\theta)$ in the denominator is cancelled by an identical term in the numerator. It is readily recognized that if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{6}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{7}$$

the required cancellation occurs in Eq. 5 and the limit is independent of θ. Consequently, the derivative $f'(z)$ is uniquely defined. The general form, given in Eq. 5, itself is simple enough to identify the conditions for the uniqueness of the derivative. The relations given in Eq. 6 and Eq. 7 are the CR equations. We note that the various ϵ factors in Eq. 3 do not appear in the expression for $f'(z)$ since they become zero if δz approaches zero.

Now, to see why x^2 cannot be the real part of an analytic function, let us assume that $f = x^2 + iv(x, y)$ is an analytic function. We need to find the imaginary part v. The CR conditions imply

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x, \quad (8)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0. \quad (9)$$

The second equation implies that v is a function of y alone, so that its partial derivative with respect to x vanishes. So, the partial derivative of $v(y)$ with respect ot y is also a function of y alone. But the first equation implies that the partial derivative of $v(y)$ depends on x, which is inconsistent. So, the CR conditions cannot be satisfied if the real part of an analytic function is x^2. We, therefore, conclude that an analytic function cannot have x^2 as its real part. In fact, it is straightforward to extend the argument to establish that there is no analytic function, other than $x + iy$, whose real part is solely a function of x.

An exercise to illustrate the severe restriction imposed by the analyticity is to consider those functions whose real part is a sum of the form $f(x) + g(y)$. Though this form appears to be rather nonrestrictive, the CR conditions give the real part to be $a(x^2 - y^2) + bx + cy$ and the corresponding imaginary part is $2axy - cx + by$, where a, b and c are real constants. It is equally possible to choose a specific form for the imaginary part of an analytic function, and that would determine its real part.

References

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[4] F. J. Flanigan, *ibid.*, p 123.