
On Magnetic analogue of Clausius-Mossotti equation

Ankita Niranjana and Bikash K. Padhi

Department of Physics, Indian Institute of Technology,
Delhi, New Delhi-110016, India

((Submitted 14-07-2013))

Abstract

In this paper the magnetic equivalent of the Clausius - Mossotti equation is re-derived for a modern reader. The equation relates the relative permeability of a diamagnetic substance to the magnetic atomic polarizability of the atoms. The historical background and importance of this equation and its relation to the Clausius-Mossotti equation is also discussed briefly.

1. Introduction

Due to the symmetric structure of the Maxwell equations there are several equations in electrostatics which have got their magnetic equivalents. For instance the electric and magnetic fields produced by an isolated electric charge and a hypothetical magnetic monopole, respectively, follow the inverse square law. Again the expression for the electric field and magnetic field produced by an electric dipole and a magnetic dipole, respectively, both vary according to the inverse cube law [1,2]. Motivated by such striking symmetries between the two, derivation of an equation which would be a magnetic analogue of the Clausius-Mossotti equation is carried here. This equation enables us to find the extent of magnetization of a diamagnetic substance in terms of a certain microscopic characteristic of it. The equation holds good only for diamagnetic substances.

Standard textbooks [1-3] on electromagnetism or solid state physics often discuss the derivation and applications of the standard Clausius-Mossotti equation. However, none seem to extend the idea to the magnetic case. However, it is the 'magnetic analogue' which was developed first by Poisson and then people like Faraday, Mossotti and Clausius extended this idea to the

electric case. So in the last section we discuss this *misunderstood* history of this pair of equations in more detail.

The derivation presented here is much simpler for a modern reader to follow as compared to that done in the original works of Poisson and others in the beginning of 19th century. It is done simply by creating a model which is the magnetic equivalent of the electrostatic model, usually used to derive the Clausius-Mossotti equation [1-3]. For convenience the derivation is divided into two sections. In the first section the average value of the microscopic magnetic field produced by a tiny ideal magnetic dipole over a sphere of radius R is calculated. Then the expression for this average field is used in the second section to get to the desired equation. The reader may note that the derivation of average magnetic field over a sphere is somewhat unconventional.

2. Average magnetic field due to a dipole over a sphere

The average electric field inside a sphere ($\langle \mathbf{E} \rangle_{\text{sphere}}$) of radius R due to a tiny dipole of dipole moment \mathbf{p} present *anywhere* inside it is given by [4, 5]

$$\langle \mathbf{E} \rangle_{\text{sphere}} = \frac{-\mathbf{p}}{4\pi\epsilon_0 R^3}. \quad (1)$$

This may be deduced as follows. Consider a fictitious sphere of radius R centered at the origin. A tiny charge q is located at position \mathbf{r}_0 inside the sphere. The average field produced by it over the volume of the sphere, by using Coulomb's law is

$$\langle \mathbf{E} \rangle_{\text{sphere}} = \frac{q}{4\pi\epsilon_0 V} \iiint_{\text{sphere}} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} dV. \quad (2)$$

V is the volume of the sphere.

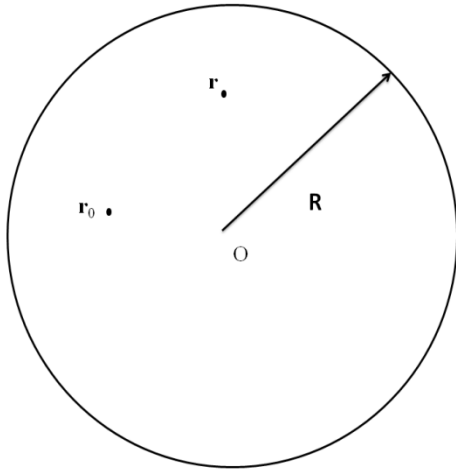


FIG.1: Average field over a sphere due to a point charge q at \mathbf{r}_0

Now the electric field at the point \mathbf{r}_0 (cf. Fig.1) due to a uniformly charged sphere, centered at the origin, of charge density ρ is

$$\mathbf{E}_{\text{sphere}} = \frac{\rho}{4\pi\epsilon_0} \iiint_{\text{sphere}} \frac{\mathbf{r}_0-\mathbf{r}}{|\mathbf{r}-\mathbf{r}_0|^3} dV. \quad (3)$$

A comparison of (2) and (3) reveals that the average field due to a charge q at \mathbf{r}_0 equals the field at \mathbf{r}_0 due to a uniformly charged sphere of charge density $\rho = -q/V$. The latter, however, using Gauss' divergence theorem turns out to be

$$\mathbf{E} = \frac{\rho}{3\epsilon_0} \mathbf{r}_0. \quad (4)$$

Thus,

$$\langle \mathbf{E} \rangle_{\text{sphere}} = -\frac{q}{4\pi\epsilon_0 R^3} \mathbf{r}_0. \quad (5)$$

Now if we place another charge $-q$ at $\mathbf{r}_0 - \mathbf{d}$ inside the sphere, then the net average field inside the sphere due to the pair of charges (of dipole moment $\mathbf{p} = q\mathbf{d}$) can be obtained by using (1).

One may recall that the electric field at position \mathbf{r} due to an ideal electric dipole of moment \mathbf{p} located at the origin is [4] (in SI unit)

$$\mathbf{E}_{\text{dip}} = \frac{1}{4\pi\epsilon_0 r^3} (3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}) - \frac{\mathbf{p}}{3\epsilon_0} \delta(\mathbf{r}). \quad (6)$$

Similarly the expression for magnetic field in case of a magnetic dipole \mathbf{m} located at the origin is given by [5] (in SI unit)

$$\mathbf{B}_{\text{dip}} = \frac{\mu_0}{4\pi r^3} (3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}) + \frac{2\mu_0\mathbf{m}}{3} \delta(\mathbf{r}). \quad (7)$$

The delta function $\delta(\mathbf{r})$ terms that appear in equations (6) and (7) are of great importance (6, 7). If one carries out the volume average of \mathbf{E}_{dip} over a sphere of radius R , the contribution due to the first term in (6) vanishes and one obtains the value of $\langle \mathbf{E}_{\text{dip}} \rangle$ the same as (1), which is due to the delta function term only. To put it mathematically,

$$\iiint_{\text{sphere}} \frac{(3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p})}{r^3} dV = 0, \quad (8)$$

and,

$$\begin{aligned} \langle \mathbf{E}_{\text{dip}} \rangle &= \frac{1}{V} \iiint_{\text{sphere}} \frac{-\mathbf{p}}{3\epsilon_0} \delta(\mathbf{r}) dV \\ &= \frac{-\mathbf{p}}{\left(\frac{4\pi R^3}{3}\right)3\epsilon_0} = \frac{-\mathbf{p}}{4\pi\epsilon_0 R^3}. \end{aligned} \quad (9)$$

One can note that the coefficient of the delta term in (6) is nothing but the average electric field over a sphere multiplied by the volume of the sphere, or it is the volume integral of the electric field of the dipole over a sphere of any size.

For the magnetic case, a similar derivation of (8) can be carried out by using tiny circulating current loops in place of charge distribution.

Thus, the average value of the magnetic field produced by a magnetic dipole, inside a sphere of radius R can be calculated as,

$$\langle \mathbf{B} \rangle_{\text{sphere}} = \frac{1}{V} \iiint_{\text{sphere}} \mathbf{B}_{\text{dip}}(\mathbf{d}) dV'. \quad (10)$$

Invoking equation (7) and by using (8) with \mathbf{p} replaced by \mathbf{m} , we obtain

$$\langle \mathbf{B} \rangle_{\text{sphere}} = \frac{\mu_0}{2\pi R^3} \mathbf{m}. \quad (11)$$

3. Derivation of the analogue equation

When an external magnetic field \mathbf{B} is applied to a diamagnetic substance, the electronic orbits of the atoms get modified and they acquire a magnetic moment $\Delta\mathbf{m}$. Classical [8] or semi classical [9] derivations of induced magnetization in a diamagnetic substance upon application of an external magnetic field \mathbf{B} give us a relation which tells us that induced dipole moment $\Delta\mathbf{m}$ and applied magnetic field \mathbf{B} are

linearly proportional, provided the field is not too strong.

$$\Delta\mathbf{m} = \alpha_m \mathbf{B}. \quad (12)$$

If we consider the diamagnetic substance to be spatially *homogeneous* then a uniform external field applied on it can be assumed to magnetize every atomic dipole by the same amount. Thus a proportionality constant α_m can be assigned to the *entire* diamagnetic substance, which may be called ‘magnetic polarizability’, in analogy with electric polarizability. Another advantage of spatial homogeneity is - a uniform applied field \mathbf{B} will generate a uniform dipole moment per unit volume, \mathbf{M} where \mathbf{M} and \mathbf{B} are related to \mathbf{H} and μ_r by the macroscopic definitions (for linear materials)

$$\mathbf{M} = \chi_m \mathbf{H}. \quad (13)$$

Or,

$$\mathbf{M} = \frac{\chi_m}{\mu_0(1+\chi_m)} \mathbf{B}. \quad (14)$$

χ_m is the magnetic susceptibility of the diamagnetic substance. Microscopically, the magnetization vector can be expressed as the cumulative sum of all the tiny atomic magnetic dipoles

$$\mathbf{M} = \sum_{i=1}^N \Delta\mathbf{m}_i. \quad (15)$$

Here N is the number density of atoms in the substance (hence an integer). Eq. (15) can be rewritten as

$$\mathbf{M} = \sum_{i=1}^N \Delta\mathbf{m}_i = N\Delta\mathbf{m}_i \quad (16)$$

From (12), (14) and (16) one may be tempted to

conclude that $\frac{\chi_m}{\mu_0(1+\chi_m)} = N\alpha_m$.

However, the field vector \mathbf{B} appearing in (12) is not the same field vector that appears in equation (14). The \mathbf{B} appearing in (14) is the *total macroscopic field* in the medium and the \mathbf{B} in (12) is due to everything *except* the particular

atomic dipole under consideration. So (12) can be rewritten as

$$\Delta \mathbf{m} = \alpha_m (\mathbf{B} - \mathbf{B}_{\text{self}}). \quad (17)$$

Here $(\mathbf{B} - \mathbf{B}_{\text{self}})$ is the local field present in the vicinity of the atomic dipole that is located at the center of a fictitious sphere (cf. Fig. 1) and \mathbf{B}_{self} is the average field over the sphere due to the dipole itself. It is reasonable to question why the average field over the sphere is taken instead of the field at the center of the sphere where the dipole is located. However, average magnetic field due to all dipoles outside the sphere is same as the field they produce at the center [10].

The radius of the sphere is related to the number density of atoms as

$$N \left(\frac{4\pi R^3}{3} \right) = 1. \quad (18)$$

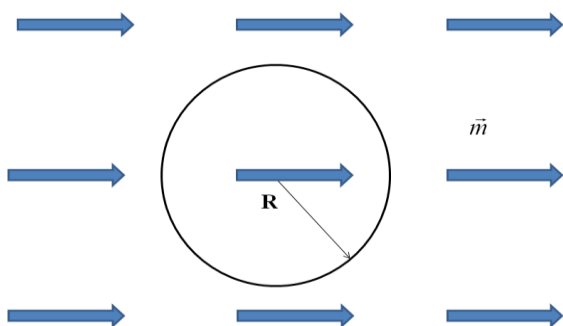


FIG. 2: The arrangement of dipoles. The sphere represents the fictitious boundary of each of the induced atomic dipoles \mathbf{m} in the diamagnetic substance.

The average magnetic field produced by the dipole (moment) would be

$$\langle \mathbf{B} \rangle_{\text{sphere}} = \frac{\mu_0}{2\pi R^3} \Delta \mathbf{m}, \quad (19)$$

Now, consider any atom, or magnetic dipole with index 'i' in the substance. \mathbf{B}_{self} of this atom is then given by

$$\mathbf{B}_{\text{self},i} = \frac{\mu_0}{2\pi R^3} \Delta \mathbf{m}_i. \quad (20)$$

Plugging this value in (17), one gets

$$\Delta \mathbf{m}_i = \alpha_m \left(\mathbf{B} - \frac{\mu_0}{2\pi R^3} \Delta \mathbf{m}_i \right). \quad (21)$$

Rearranging the terms we get,

$$\left(1 + \frac{\alpha_m \mu_0}{2\pi R^3} \right) \Delta \mathbf{m}_i = \alpha_m \mathbf{B}. \quad (22)$$

Now summation of both sides of (22) for N terms is taken and then the results of (18) is used to get

$$\left(1 + \frac{\alpha_m \mu_0}{2\pi R^3} \right) \mathbf{M} = \alpha_m N \mathbf{B}. \quad (23)$$

Further invoking (14) to eliminate \mathbf{B} and \mathbf{M} from the previous equation we get

$$\alpha_m = \frac{3}{N\mu_0} \left[\frac{\chi_m}{3 + \chi_m} \right]. \quad (24)$$

Using $\chi_m = \mu_r - 1$

$$\alpha_m = \frac{3}{N\mu_0} \left[\frac{\mu_r - 1}{\mu_r + 2} \right]. \quad (25)$$

Here μ_r is the relative permeability of the diamagnetic substance. The relation (25) is the *magnetic analogue* of Clausius-Mossotti equation.

For a diamagnetic substance, μ_r is less than unity. Hence, (25) tells us that α_m will always be negative. This reflects the fact that in a diamagnetic substance, the applied field \mathbf{B} and the change in magnetic moment $\Delta \mathbf{m}$ are always antiparallel. To put it in another way, in the presence of an external magnetic field, each atom picks up a little extra dipole moment and these increments are all antiparallel to the field direction.

4. Brief historical survey of the equation

In this section we briefly discuss the historical development of the Clausius-Mossotti relation and its magnetic analogue which unfortunately lacks a different name for itself. The scientific figures responsible the development of his equation are S.D. Poisson (France), M. Faraday (England), J.C. Maxwell (Scotland), O.F. Mossotti (Italy), R. Clausius (Germany) and H.A. Lorentz (The Netherlands). We discuss their contributions chronologically.

The history of this equation begins in 1824, when Poisson presented his book [11] at a meeting of the French Academy in which he had carried out a detail mathematical analysis of the *problem of magnetic induction*. In his classic text on Electricity and Magnetism [12], Maxwell mentions that “the mathematical theory of magnetic induction was first given by Poisson...” In order to explain the phenomenon of magnetic induction Poisson hypothesized that an imaginary ‘magnetic matter’ or ‘magnetic fluid’ is *confined to certain* molecules of the magnetic substance. That molecule is magnetized in which the *two opposite kinds* of magnetic matter, which are present precisely in *equal* quantity, are separated towards *opposite poles* of the molecule. He called such molecules ‘magnetic elements’ of the substance and examined the particular case in which these elements are spherical and are uniformly distributed throughout the substance. He calculated the ratio ‘K’ called Poisson’s Magnetic Coefficient - the ratio of the volume of magnetic elements to the whole volume of the

substance. This turned out to be $\left[\frac{\mu_r - 1}{\mu_r + 2} \right]$, the factor that appears in the expression for magnetic polarizability in Eq. (25). Maxwell ruled out the validity of such a hypothesis by using the experimental works of Thalen. However, he concludes [13], “... the value of

Poisson’s mathematical investigation remains unimpaired, as they don’t rest on his hypothesis.” Later on, to explain this phenomenon of magnetic induction Ampere hypothesized that the magnetism of a molecule is due to an electric current that already exists in it which constantly circulates in some closed path within the molecules of the magnet, and must not flow from one molecule to another. These are the two alternative pictures of a magnetic dipole.

Thus, using the hypothesis of Poisson and Ampere, and the magnetic analogue of Clausius-Mossotti equation, the problem of magnetic induction was completely resolved. Note all this happened about half a century *before* the development of the Clausius-Mossotti equation for dielectrics.

After a few years of Poisson’s formulation, Faraday [14] for the first time [15] applied Poisson’s idea to dielectrics. It was Mossotti who studied the problem in greater detail and presented it in his memoirs [16]. He introduced the ‘cavity method’ [17] which he later developed in his second book [18]. Meanwhile, Clausius was also studying the same problem [19]. For the first time, he explicitly wrote the formula of what is now famous as the Clausius-Mossotti equation, as called by Lorentz. It may be noted that all of them attacked the same problem using different approaches.

Coming back to our derivation, the approach that is followed in this paper (use of local field or Lorentz field) significantly departs from that used by Poisson yet resembles the one used by H.A. Lorentz [20] and L.V. Lorenz [21] in their derivation of the Lorentz-Lorenz equation (used in optics).

So it’s evident that historically, Poisson’s equation plays a more fundamental role as compared to that by the Clausius- Mossotti equation.

5. Concluding remarks

In the derivation the use of the relation between magnetic field \mathbf{B} and dipole moment $\Delta\mathbf{m}$ is independent of whether the scenario is quantum mechanical or classical. The only assumptions are that the substance must be a linear, homogeneous, isotropic diamagnetic substance for our equation to be valid. One can check the equation very easily by putting relative magnetic permeability = 1, which is true for non-magnetic substances. For this case, the polarizability comes out to be zero, which is quite obvious as a non-magnetic substance would not respond to an external magnetic field.

One must recognize that it is the self field term \mathbf{B}_{self} which actually leads us to such an equation for diamagnetic substances. We may consider the case of paramagnetic materials to see whether the self field term leads us to any such equation or not. In this case, the atoms possess a net dipole moment even in the absence of an external magnetic field. When a magnetic field is applied, magnetization arises due to the reorientation of the atomic dipoles. The potential energy of a dipole of dipole moment \mathbf{m} in presence of a magnetic field \mathbf{B} is $\phi = -\mathbf{m}\cdot\mathbf{B}$. As we have said above the \mathbf{B} here should be written as $\mathbf{B}-\mathbf{B}_{\text{self}}$. And using (11) and (18), \mathbf{B}_{self} can be written as $\frac{-2\mu_0 N}{3}\mathbf{m}$. Hence

$$\begin{aligned}\phi &= -\mathbf{m}\cdot\mathbf{B} + \frac{2\mu_0 N}{3}\mathbf{m}\cdot\mathbf{m} \\ &= -mB \cos \theta + \frac{2\mu_0 N}{3}m^2.\end{aligned}\quad (26)$$

The energy distribution of atoms can be adequately described by the classical Maxwell-Boltzmann statistics. Thus, the number of atoms dN whose energy lies between ϕ and $\phi+d\phi$ is given by

$$\begin{aligned}dN &= A \exp\left(\frac{-\phi}{K_B T}\right) d\phi \\ &= A \exp\left(\frac{2N\mu_0 m^2}{3}\right) \exp\left(\frac{\mathbf{m}\cdot\mathbf{B}}{K_B T}\right) mB \sin \theta d\theta.\end{aligned}\quad (27)$$

Here T is the temperature of the system. A is the normalization constant that can be found out by imposing the condition that the integration of dN must be equal to the total number of atoms N when θ goes from 0 to π .

The term arising due to the self field can be absorbed into the normalization constant; therefore the \mathbf{B}_{self} seems to play no significant role in the case of paramagnetic substances.

The scope of the magnetic analogue equation is not as wide as that of the Clausius-Mossotti equation. In fact, many seem to be unaware of the very existence of such an equation. However it plays a key role in the study of magnetic fluids [22], permittivity and permeability studies of mixtures [23], and negative effective permeability [24].

Acknowledgement

The authors are thankful to Prof. V.K. Tripathi for useful discussions and comments and to Prof. R. Chatterjee for her valuable remarks.

References

- [1] J.D Jackson, Classical Electrodynamics, 3rd ed. (Wiley, New York, 1975).
- [2] David J. Griffiths, Introduction to Electrodynamics, 3rd ed. (Prentice Hall, Englewood Cliffs, NJ, 1989).
- [3] C. Kittel, introduction to solid state physics, 7th ed. (John Wiley & Sons, Inc., New York, 1996)
- [4] Reference 2, p. 149.
- [5] Reference 2, p. 188.

-
- [6] J.H. Hannay “The Clausius-Mossotti equation: an alternative derivation”, *Eur. J. Phys.* 4, 141-143 (1983).
- [7] D.J. Griffith, “Hyperfine splitting in the ground state of hydrogen”, *Am. J. Phys.* 50, 698-703 (1982)
- [8] Reference 2, p.261
- [9] S.L. O’Dell and R.K.P. Zia, “Classical and semi classical diamagnetism” *Am. J. Phys.* 54 (1), 32-35 (1986)
- [10] Reference 1, p.253
- [11] S.D. Poisson, *Mémoire sur la théorie du magnétisme*, *Mémoires de l’Académie royale de France*, V (1824), 247–338. (copy from “Archives de l’Académie des Sciences, Paris.”)
- [12] J.C. Maxwell, *A Treatise on Electricity and Magnetism* (Vol. 2), Dover, New York (1954). (Republication of 3rd edition of 1891 ed.) p. 57
- [13] *Ibid.*, p. 58
- [14] M. Faraday, *Experimental researches on electricity*, *Phil. Trans. Roy. Soc. London*, II Ser. (1838)
- [15] K. Markov, L. Preziosi, “Heterogeneous media: Micromechanics modelling methods and simulations”, 1st ed. Birkhauser (2000).
- [16] O.F. Mossotti, *Sur les forces qui régissent la constitution intérieure des corps aperçus pour servir à la détermination de la cause et des lois de l’action moléculaire*, *Ac. Sci. Torino*, 22 (1836), 1–36
- [17] Reference 15, p.125-128.
- [18] O.F. Mossotti, *Discussione analitica sull’influenza che l’azione di un mezzo dielettrico ha sulla distribuzione dell’elettricità alla superficie di più corpi elettrici disseminati in esso*, *Mem. Mat. Fis. della Soc. Ital. di Sci. in Modena*, 24 (1850), 49–74.
- [19] Clausius, R., *Die mechanische Behandlung der Elektrizität*, Vieweg, Braunschweig (1879)
- [20] H.A. Lorentz, *The Theory of Electrons* (2nd ed.) Dover publ. (1954), p. 138-145
- [21] L. Lorenz, *Über die Refraktionskonstante*, *Ann. Phys. Chem.* 11 (1880), p.70 .
- [22] I. Brevik, “Fluids in electric and magnetic fields: Pressure variation and stability” *Can. J. Phys.*, 60. 449 (1982).
- [23] L. Lewin, “The electrical constants of a material loaded with spherical particles”, *Proc. Inst. Electr. Eng.* 94, 65 (1947).
- [24] M. S. Wheeler et al., “Three-dimensional array of dielectric spheres with an isotropic negative permeability at infrared frequencies”, *Phys. Rev. B* 72, 193103 (2005).
-