# Grappling Tensors using Maxwell's equations

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**Abstract**

Physics teachers always face great difficulties in introducing the many component nature of tensors to students. This article is intended to help the instructors with some convincing examples of more than 3 component physical quantities. We will also show that tensor nature of many physical quantities are both medium and interacting physical quantity dependent.

## 1 Introduction

For many physics students physical quantities having $3^0$ components are scalars, $3^1$ components are vectors and physical quantities having $3^2$ or more than 9 components are tensors each obeying different mathematics. In class rooms we could suggest good and convincing examples for scalars and vectors but examples for 9 or more component physical quantities are rarely discussed except moment of inertia or conductivity[1][2]. Many books give only explicit derivation for nine component moment of inertia[1] and some other examples like Maxwell stress tensor[2](where matrix form is not given) among non relativistic systems. In relativistic cases electromagnetic field tensor which is a second rank in 4 dimensional world, is a comfortable physical quantity to be introduced but we deal with it only in postgraduate classes. Tensor form of stress and strain[3] is always beyond the scope of undergraduate classes. Thus the teachers are always forced to say that all anisotropic media are sources of tensorial nature of many physical properties without good and convincing examples. Here in this article we show how anisotropy lead to the tensor nature of many physical quantities by taking the example of the passage of an electromagnetic wave through an anisotropic medium. We also show that the conductivity tensor for an electron passing through a conductor under electric and magnetic field will be different from the conductivity tensor when a plane electromagnetic field is passing through a conducting medium.The present approach is novel by giving a direct technique in deriving the tensor nature of some physical quantities.

## A simple physical tensor

Consider a plane wave and let us find how electric field of the plane electromagnetic wave is related to magnetic field. We usually say they are mutually perpendicular or they propagate with velocity $3 \times 10^8 m/s$ etc. Now let us check how they are related mathematically from Maxwell's equations. Faraday's law gives the relation between changing electric and magnetic field as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

For a plane electromagnetic wave

$$\vec{E} = \vec{E}_0 e^{j(\vec{k}.\vec{r} - \omega t)}$$

and

$$\vec{B} = \vec{B}_0 e^{j(\vec{k}.\vec{r} - \omega t)}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, j represents complex number, $\omega$ is the angular frequency, $\vec{k}$ is the wave vector, t is the time, $\vec{E}_0$ and $\vec{B}_0$ are the complex magnitudes of electric and magnetic fields. Then $\nabla$ operation will give $j\,\vec{k}$ and $\frac{\partial}{\partial t}$ will give $-j\,\omega$. Substituting these operators Faraday's law becomes

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

Expanding component wise we get

$$\omega\, B_x = 0 + -k_z\, E_y + k_y\, E_z$$
$$\omega\, B_y = k_z\, E_x + 0 - k_x\, E_z$$
$$\omega\, B_z = -k_y\, E_x + k_x\, E_y + 0$$

This can be represented in matrix form as

$$\omega \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

This shows that actually $\vec{B}$ is coupled to $\vec{E}$ in a plane electromagnetic wave through a symmetric tensor. If the propagation direction is one dimension say z-direction then we get

$$\omega \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 & -k_z & 0 \\ k_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

and if we have a plane polarized wave with electric field in x-direction we get

$$B_y = \frac{E_x}{c}$$

the well known relation between the magnitude of electric and magnetic fields.

## Tensors

Let us now find the tensor form of various electromagnetic physical quantities. The basic equations where tensors come in electrodynamics are $\vec{J} = \sigma\vec{E}$, $\vec{B} = \mu\vec{H}$, $\vec{P} = \alpha\vec{E}$, $\vec{M} = \chi\vec{H}$ and $\vec{D} = \epsilon\vec{H}$, where $\vec{J}$ is the current density, $\sigma$ is the conductivity $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, $\mu$ is the permeability, $\vec{H}$ is the magnetizing field, $\vec{P}$ is the polarization, $\alpha$ is the polarisability, $\vec{M}$ is the magnetization, $\chi$ is the susceptibility , $\vec{D}$ is the displacement vector and $\epsilon$ is the permittivity.

## Maxwell's equations for plane waves

The Maxwell's equations[ME] for a material medium which is magnetic, dielectric and conducting with sources are

$$\nabla.\vec{D} = \rho$$

$$\nabla.\vec{H} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \left( \vec{J} + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ and $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ and $\rho$ is the charge density. We will require for our purpose only last two equations which can be modified as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

For plane waves the ME's will become

$$\vec{k} \times \vec{E} = \omega\vec{B}$$

$$j\vec{k} \times \vec{H} = \vec{J} - j\omega\vec{D}$$

From these two ME's we can obtain the tensor form of $\mu, \sigma, \epsilon, \chi$ and $\alpha$. The technique is that if we want $\mu$ which is linked with the magnetic property, we will take dielectric and electrical quantities $\vec{P}$ and $\vec{J}$ as zero and rewrite the ME's equations component wise which will naturally yield the tensor form of $\mu$. Similarly we can find the components of all the linking or bridging tensors by suitable elimination of some unrelated quantities. We will do them one by one. Let us first find permeability tensor.

## Permeability tensor

While finding permeability tensor we are only interested in a magnetic medium. So $\vec{P} = 0$ and $\vec{J} = 0$. Then the Maxwell's equations become

$$\vec{k} \times \vec{E} = \omega\vec{B}$$

$$\vec{k} \times \vec{H} = -\omega\vec{D}$$

Here $\vec{D} = \epsilon_0\vec{E}$. Then ME's modify as

$$\vec{k} \times \vec{H} = -\omega\epsilon_0\vec{E}$$

which gives

$$\vec{E} = -\frac{\vec{k} \times \vec{H}}{\omega\epsilon_0}$$

Substituting we get

$$\vec{B} = -\frac{\vec{k} \times \vec{k} \times \vec{H}}{\omega^2 \epsilon_0}$$

Expanding component wise and taking $\mu = \mu_0\,\mu_r$ we get the permeability tensor

$$\mu_r = \frac{1}{\omega^2 \epsilon_0\,\mu_0} \begin{pmatrix} k^2 - k_x^2 & -k_x\,k_y & -k_x\,k_z \\ -k_y\,k_x & k^2 - k_y^2 & -k_y\,k_z \\ -k_z\,k_x & -k_z\,k_y & k^2 - k_z^2 \end{pmatrix}$$

Then each element of the permeability tensor can be written as

$$\mu_{rij} = \frac{1}{\omega^2\,\mu_0\,\epsilon_0}[k^2 \delta_{ij} - k_i\,k_j]$$

But for a magnetic medium the magnitude of wave vector

$$k = \frac{\omega}{c}\sqrt{\mu_r}$$

Then we get

$$\mu_r = \begin{pmatrix} \mu_{rr} - \mu_{r_x} & -\mu_{r_x}\,\mu_{r_y} & -\mu_{r_x}\,\mu_{r_z y} \\ -\mu_{r_y}\,\mu_{r_x} & \mu_{rr} - \mu_{r_y} & -\mu_{r_y}\,\mu_{r_z} \\ -\mu_{r_z}\,\mu_{r_x} & -\mu_{r_z}\,\mu_{r_y} & \mu_{rr} - \mu_{r_z} \end{pmatrix}$$

Thus the nature of permeability tensor as a nine component physical quantity is exhibited naturally. Similarly we can very easily get all other tensors related to electromagnetism.

## Permittivity Tensor

For a pure dielectric medium $\vec{M} = 0$ and $\vec{J} = 0$. Then we get

$$\vec{D} = -\frac{\vec{k} \times \vec{k} \times \vec{E}}{\omega^2 \mu_0}$$

which on expansion will give

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} =$$

$$\frac{1}{\omega^2 \mu_0} \begin{pmatrix} k^2 - k_x^2 & -k_x\,k_y & -k_x\,k_z \\ -k_y\,k_x & k^2 - k_y^2 & -k_y\,k_z \\ -k_z\,k_x & -k_z\,k_y & k^2 - k_z^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Then permittivity tensor will be

$$\epsilon_r = \frac{1}{\omega^2 \epsilon_0\,\mu_0} \begin{pmatrix} k^2 - k_x^2 & -k_x\,k_y & -k_x\,k_z \\ -k_y\,k_x & k^2 - k_y^2 & -k_y\,k_z \\ -k_z\,k_x & -k_z\,k_y & k^2 - k_z^2 \end{pmatrix}$$

In general

$$\epsilon_{rij} = \frac{1}{\omega^2\,\mu_0\,\epsilon_0}[k^2 \delta_{ij} - k_i\,k_j]$$

But for a pure dielectric medium $k = \frac{\omega}{c}\sqrt{\epsilon_r}$ and we will get

$$\epsilon_r = \begin{pmatrix} \epsilon_{rr} - \epsilon_{r_x} & -\epsilon_{r_x}\,\epsilon_{r_y} & -\epsilon_{r_x}\,\epsilon_{r_z} \\ -\epsilon_{r_y}\,\epsilon_{r_x} & \epsilon_{rr} - \epsilon_{r_y} & -\epsilon_{r_y}\,\epsilon_{r_z} \\ -\epsilon_{r_z}\,\epsilon_{r_x} & -\epsilon_{r_z}\,\epsilon_{r_y} & \epsilon_{rr} - \epsilon_{r_z} \end{pmatrix}$$

## Polarisability Tensor

Here $\vec{M} = 0$ and $\vec{J} = 0$. Taking $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$$\vec{k} \times \vec{H} + \omega(\epsilon_0\,\vec{E} + \vec{P}) = 0$$

and substituting for $\vec{H}$

$$\vec{P} = -\frac{\vec{k} \times \vec{k} \times \vec{E}}{\omega^2 \mu_0} - \epsilon_0 \vec{E}$$

This gives the polarisability components as

$$\alpha_{ij} = \frac{1}{\omega^2\,\mu_0}[(k^2 - \epsilon_0 \omega^2 \mu_0)\delta_{ij} - k_i\,k_j]$$

## Conductivity Tensor

Taking $\vec{M} = 0$ and $\vec{P} = 0$ we get

$$\vec{J} = j\frac{\vec{k}\times\vec{k}\times\vec{E}}{\omega^2\mu_0} + \omega\epsilon_0\vec{E}$$

which on expansion will give

$$\sigma_{ij} = \frac{j}{\omega^2\,\mu_0}[(\omega^2\epsilon_0\mu_0 - k^2)\delta_{ij} - k_i\,k_j]$$

Here conductivity tensor components will be complex which accounts for the damping of the plane wave inside the conductor[4]

## Magnetic susceptibility Tensor

As done earlier taking $\vec{P} = 0$ and $\vec{J} = 0$ we can find susceptibility tensor which will be given by

$$\chi_{ij} = \frac{1}{\omega^2\,\mu_0\epsilon_0}[(k^2 - \omega^2\epsilon_0\mu_0)\delta_{ij} - k_i\,k_j]$$

## Discussion

Thus we could obtain the tensor form of some electrodynamic quantities. But these tensor forms are not only medium dependent but also depend on the interacting quantities and external fields acting on the medium. For tensors like moment of inertia, stress or strain they are medium dependent. We will prove this with an example given below. Consider an electron traveling in a conductor under the action of an electric and magnetic field. When there is equilibrium electron will be having a drift velocity $\mathbf{v}$ and the force acting on the electron given by Lorentz force is

$$q\mathbf{E} + q\mathbf{v}\times\mathbf{B} = \frac{m\mathbf{v}}{\tau}$$

where q is the charge of the electron and $\tau$ is the time between two consecutive collisions. Let $\frac{ne^2\tau}{m} = C$ where C is a constant. Replacing $\mathbf{v}$ by $\frac{\mathbf{J}}{-ne}$ and substituting for $\mu = \frac{e\tau}{m}$ we get

$$\mathbf{E} = C\,(\mathbf{J} - \mu\,\mathbf{B}\times\mathbf{J})$$

Let us assume that magnetic field is applied in the z direction and putting $B_z = B$ we get

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = C\begin{pmatrix} 1 & \mu B & 0 \\ -\mu B & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}$$

This is

$$\mathbf{E} = \rho\mathbf{J}$$

where $\rho$ is the resistivity tensor with nine components. In the absence of magnetic field

$$\rho = C$$

the scalar conductivity. The conductivity tensor is given by the inverse of this and is

$$\sigma = \frac{1}{C}\begin{pmatrix} \frac{1}{1+\mu^2B^2} & \frac{-\mu B}{1+\mu^2B^2} & 0 \\ \frac{\mu B}{1+\mu^2B^2} & \frac{1}{1+\mu^2B^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Comparing the two expressions for conductivity we find that the tensor components are different and they dependent on both material and the interacting quantity. To conclude we wish to point out that there are books[5],[6], [7], [8],[9] to understand and study tensors but books which give good examples of tensors from physics point of few are very rare. This article in an attempt bridge the gap in that direction.

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