Maxwell's Stress Tensor and Momentum Conservation in Electromagnetic Field

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Abstract

Maxwell's Stress Tensor $\widehat{\boldsymbol{\mathcal{T}}}$ owes its origin to the notion prevailing before the advent of relatvity that 'action at a distance' is actually a mechanical interaction, like push and pull, and is transmitted by an assumed mechanical property of the *Aether* which pervaded all space, in particular vacuum. Even after withrawal of Aether this tensor has a useful role to play not only in formulating conservation of momentum in a time varying electromagnetic field, but also in simplifying several problems in electrostatics and magnetostatics, by removing the distinction between the field caused by 'external sources' and the total field surrounding a distribution of charges and currents. This tensor is to be constructed on the principle that $\mathbf{f}_s(\mathbf{r}) = \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r})$, where $\mathbf{f}_s(\mathbf{r})$ is the force acting on unit volume of a distribution of electric charges and currents. Our derivations of the stress tensors $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}, \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$ and $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}$, corresponding to electrostatic field, magnetostatic field, and time varying electromagnetic field respectively, are based on a single vector identity and application of Maxwell's field equations. We have worked out two examples of how the force on an isolated system can be calculated by surrounding it with a sphere of some radius r and integrating the stress vector over the entire surface, namely, an isolated electric charge in the electrostatic field of another charge, and an isolated magnetic dipole in the magnetostatic field of another magnetic dipole. We have taken the stress tensor to its logical end by writing momentum conservation in a time varying electromagnetic field, and then identifying $-\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}$ as the momentum flux density $\widehat{\boldsymbol{\Phi}}$ of the field. For the special case of a pure radiation field, $\widehat{\boldsymbol{\Phi}} = -\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})} = \boldsymbol{\Pi}\,\mathbf{c}$, where $\boldsymbol{\Pi}$ is the momentum density and $\mathbf{c}$ is the 'velocity' of light. At the beginning of this article we have given a mathematical introduction to tensors, in particular stress tensors.

1 Introduction

'Action at a distance' (AAD) was an enigma to natural philosophers, from Rene Descartes[1] (1596-1650) to James Clerk Maxwell (1831-1879). We find a an account of the evolution of physical concepts in Whittaker [1]. According the Descartes, space was a plenum, a medium called *aether*, capable of transmitting force on material bodies. "It was to be regarded as the solitary tenant of the universe, save for that infinitesimal fraction of space which is occupied by ordinary matter."

Subsequent theoretical physicists and mathematicians, Robert Hooke (1635-1703), Isaac Newton (1642-1727), Reimann (1826-1866), W.Thomson (1824-1907), Maxwell and others lent their support to this view. Implicit in their belief was the assumption that *force cannot be transmitted except by actual pressure or impact.* AAD was a taboo, as abhorrent as witchcraft: I wave my hand here and a fire is ignited there. In order to support their faith in aether they contrived every possible idea, any possible mechanical model, to make aether viable.

According to Newton "All space is pervaded by an elastic medium or aether, which is capable of propagating vibrations in the same way as air propagates the vibrations of sound. This aether pervades the pores of all material bodies, and is the cause of their cohesion; its density varies from one body to another, being greatest in the interplanetary space."

Maxwell inherited this legacy. We shall quote a few passages from his celebrated paper '*A Dynamical Theory of the Electromagnetic Field*' read to the Royal Society of London on December 8, 1964[2].

> "(1) In this way mathematical thories of statical electricity, of magnetism, of the mechanical action between between conductors carrying currents, and of the induction of currents have been formed. *In these theories the force acting between two bodies is treated with reference only to the condition of the bodies and their relative position, and without reference to the surrounding medium.*"

> "(2) The *mechanical* difficulties, however, which are involved in the assumption of particles acting at a distance with forces which depend on their velocities are such as to prevent me from considering this theory as an ultimate one, though it may have been, and may yet be useful to the coordination of phenmena."

> "(3) The theory I propose may therefore be called a theory of the *Electromagnetic Field*, because *it has to do with the space in the neighbourhood of the electric and magnetic bodies*, and it may be called a *Dynamical* Theory, because it assumes that in that space there is matter in motion, by which the observed electromagnetic phenomena are produced."

> "(4) The electromagnetic field is that part of space which contains and surrounds bodies in electric and magnetic

[1] The Cartesian coordinate system is associated with his name

> coditions. ... It may contain any kind of matter, or *we may render it empty of all gross matter*, as in the case of Geissler's Tubes and other so called *vacua*.
>
> *There is always, however, enough matter to receive and transmit the undulations of light and heat*, and it is because of the transmission of these radiations is not greatly altered when transparen bodies of measurable densities are substituted for the so-called vacuum, that we are obliged to admit that the *undulations are those of aetherial substance*, and not of the gross matter, the presence of which merely modifies in some way the motion of the aether.
>
> We have therefore some reason to believe, from the phenomena of light and heat, that *there is an aetherial medium filling space and permeating bodies, capable of being set in motion and of transmitting that motion from one part to another, and communicating that motion to gross matter so as to heat it and affect it in various ways.* "

One aspect of the mechanical model Maxwell built up to present a complete picture of the electromagnetic field was the proposition that space, i.e., aether, can sustain stress, and a force is transmitted from one body (electrified or magnetized) to another by means of stress, in the same way a force is transmitted from one end of a cable to the other by means of tensile stress, and from one part of a beam to another by means of shear stress.

In his two-volume book '*A treatise on Electricity and Magnetism*' Maxwell presents a complete formulation of the Stress in the field (read aether) by constructing the *Stress Tensor* for the *Static Electric Field*[3] and for the *Static Magnetic Field*[4], in terms of the field potentials. The first one is presented in Vol 1 of his book and second one in Vol 2. His derivation of the first tensor (for the Electrostatic field) involves manipulation of Laplace's and Poisson's equations. His derivation of the second tensor (for the Magnetostatic field) involves magnetic poles which are now out of fashion in current physics text books, and may not be of much interest to us.

We have derived the stress tensors for Electrostatic field, Magnetostatic field and time varying Electromagnetic field in terms of the electric field $\mathbf{E}$, magnetic field $\mathbf{B}$ in a unified manner exploiting the useful identity given in Eq. (76).

Einstein's formulation of the Special Theory of Relativity saw the demise of the Luminiferous (i.e., light carrying) Aether. Light travels in empty space, eletric and magnetic forces also propagate from one body to another (with the speed of light) in empty space. Is there then any place for Maxwell's Stress Tensor? Is it only for historical reason that we are writing this long article? We shall attempt to provide the answer in four steps.

First, it is indeed an amazing thing that the force acting on an isolated body A (which may consist of electric charges and currents), due to the presence of charges and currents elsewhere, can be computed *exactly* by drawing a boundary surface $\mathcal{S}$ of our convenience surrounding A, as in Fig.1(a), finding the "*stress*" all over this surface, and by integrating this stress. In other words, there *is* stress even in vacuum. The purpose of this article is to articulate how this stress is to be found out. Also it should be noted with interest that even empty space is not a true vacuum. When loaded with the electric and magnetic fields space comes under stress. Empty space is always buzzing with emission and absorption of virtual particles, with the virtual photons mediating the interaction among electrified and magnetized objects. Aren't these virtual photons the new *Avatar* of the Aaether?

Secondly, calculating the force on an isolated object A requires *exact* knowledge of the $\mathbf{E}$ or $\mathbf{B}$ field in which A is immersed. In recognizing these fields one has to be very careful that these $\mathbf{E}, \mathbf{B}$ fields do not contain any trace of the fields contributed by A itself. This is sometimes a challenging task. Consider for example the force acting on the surface of a conductor carrying a suface charge density σ, as in Fig.1(b). The electric field just outside the surface is $\mathbf{E} = (\sigma/\epsilon_0)\mathbf{n}$ where $\mathbf{n}$ is a unit normal to the surface. One may be tempted to conclude that the force per unit area of the surface is $\mathbf{F}' = \sigma\mathbf{E} = (\sigma^2/\epsilon_0)\mathbf{n}$, forgetting the fact that an infinitesimal area da on the surface contributes the same $\mathbf{E}$ field perpendicular to the surface as the rest of the surface, so that the true force is

$$\mathbf{F} = \frac{1}{2}\mathbf{F}' = (\sigma^2/2\epsilon_0)\mathbf{n} = (\epsilon_0 E^2/2)\mathbf{n}. \quad (1)$$

The stress tensor approach, which uses the total field $\mathbf{E}_{\text{total}}$, making no distinction between the test object and the source object, will give the right result without creating any confusion, as we shall show following Eq. (84).

Thirdly, it is always advisable to arrive at the same answer through several alternative routes, if available, just to make sure that we have not made any mistakes. The stress tensor provides that valuable alternative route.

And fourthly, *Maxwell's Stress Tensor*, which we shall denote by the symbol $\widehat{\boldsymbol{\mathcal{T}}}$, is needed for understanding conservation and flow of *momentum* in the electromagnetic field, which we shall present in Section 6. When one goes deeper into the theory of relativity the same tensor appears as the most important component of the Energy-Momentum tensor required not only for presenting a 4-dimensiinal and unified view of the conservation of energy and momentum, but also for building up the source term in formulating Einstein's field equation for the gravitational field, in his General Theory of Relativity.

Maxwell's stress tensor has been discussed in all standard books on Electrodynamics[5, 6, 7, 8, 9]. However it has received a more detailed treatment in the books by Panofsky and Phillips and Griffiths. Griffiths has worked out a very interesting problem to bring out the meaning of this tensor. In this article (See Secs. 4.3 and 5.2) we have contributed two worked out prblems to illustrate the same concept.

We shall begin this pedagogical article by giving the reader a mathematial introduction to tensor, and then specializing the same to

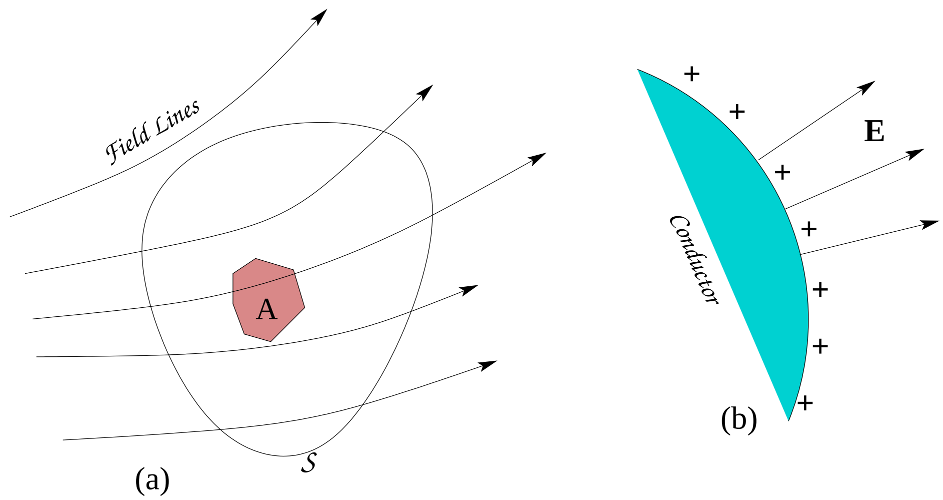

Figure 1: *Electrified object in* **E** *field.*

stress tensor, in particular Maxwell's stress tensor.

2 Introduction to Tensor

2.1 Linear Operator in a Vector Space

What we shall call a *tensor* in this book is actually "a tensor of rank 2". A bold capital letter with a "widehat" on top, e.g., $\widehat{\mathbf{T}}$, will symbolize such a tensor. In fact a scalar, e.g., potential energy V is called a tensor of rank 0, a vector, e.g., momentum $\mathbf{p}$ a tensor of rank 1.

Maxwell's stress tensor $\widehat{\boldsymbol{\mathcal{T}}}$, which is a tensor of rank 2 is needed for understanding conservation and flow of *momentum* in the electromagnetic field. in this esction we shall prepare the ground work for constructing this tensor.

We shall begin by explaining what we mean by *linear operator* in a vector space.

By the 3-dimensional linear vector space $\mathcal{V}$ we mean the set of all vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots$ we can think of and all such vectors we can construct by combining them linearly, e.g., $\eta\mathbf{A} + \lambda\mathbf{B}$ where η, λ are real numbers.

Let us think of two vectors $\mathbf{C}$ and $\mathbf{D}$ having Cartesian components (C_x, C_y, C_z) and (D_x, D_y, D_z) and related to each other in such a way that the values of the former determine the values of the latter. This means that $\mathbf{C}$ is an independent vector and $\mathbf{D}$ is a dependent one. In other words $\mathbf{D}$ is a function of $\mathbf{C}$. Let us further assume that $\mathbf{D}$ is proportional to $\mathbf{C}$. That is, if for example we double $\mathbf{C}$, then $\mathbf{D}$ is doubled. These two vectors, however, may or may not be in the same direction. In that case we say that a *linear operator* $\widehat{\mathcal{O}}$ transforms $\mathbf{C}$ into $\mathbf{D}$. We may like to write this transformation symbolically as

$$\widehat{\mathcal{O}}(\mathbf{C}) = \mathbf{D}. \tag{2}$$

The property of linearity means that

$$\begin{array}{l} \text{If} \quad \widehat{\mathcal{O}}(\mathbf{C}) = \mathbf{D}, \text{ and } \quad \widehat{\mathcal{O}}(\mathbf{E}) = \mathbf{F}, \\ \text{then} \quad \widehat{\mathcal{O}}(a\mathbf{C} + b\mathbf{E}) = a\mathbf{D} + b\mathbf{F}, \end{array} \tag{3}$$

where a, b are two arbitrary scalar constants.

In Fig.2 we have shown two simple examples of how the operation $\widehat{\mathcal{O}}$ can take place. In Fig.(a) we have shown a particle of constant mass m in abrbtrary motion along some trajectory Γ. At some instant of time t it has velocity $\mathbf{v}$. Therefore its momentum at the same instant is $\mathbf{p} = m\mathbf{v}$. We can therefore think of the operator $\widehat{\mathcal{O}}$ transforming velocity $\mathbf{v}$ into momentum $\mathbf{p}$ by scaling the length of the former by the factor m without changing its direction.

In Fig.(b) we have shown a Rigid Body rotating about some axis pointing in the direction of the unit vector $\mathbf{n}$ with angular speed ω, so that its angular velocity is $\boldsymbol{\omega} = \omega\mathbf{n}$. Its angular momentum is $\mathbf{L}$, which (in general) does not coincide with the direction of $\boldsymbol{\omega}$. In this case the operator $\widehat{\mathcal{O}}$ transforms the angular velocity $\boldsymbol{\omega}$ into angular momentum $\mathbf{L}$ by changing the length as well as the direction. The linear operator $\widehat{\mathcal{O}}$ in this case is the inertia tensor $\widehat{\boldsymbol{\mathcal{I}}}$ about which we shall give some more insight in Sec.2.4.

For our immediate purpose we shall look upon a tensor $\widehat{\mathbf{T}}$ as a linear operator. The linear operation mentioned above suggests that $\widehat{\mathbf{T}}$ can be represented by a matrix, and the "tensor operation" can be represented as a matrix multiplication. This will become evident in the next section.

2.2 Tensor as a Dyadic

Two arbitrary vectors $\mathbf{A}, \mathbf{B}$ can be combined in three types of "multiplication operation", the first two of which the reader is familiar with, namely, (1) the dot product $\mathbf{A} \cdot \mathbf{B}$ which is a scalar; (2) the cross product $\mathbf{A} \times \mathbf{B}$ which is a vector. Now comes (3) the third type, namely the *dyadic product* $\mathbf{AB}$, which is a simple juxtaposition of the vectors, without any dot or cross in between, which we shall call a *dyad.*

We define the *dyad* $\mathbf{AB}$ to be a *linear operator* which converts any vector $\mathbf{C}$ to another vector $\mathbf{D}$ and this conversion can be done in either of the following two ways:

$$\begin{array}{l} (a) \text{ operating } \textit{on the right}: \\ \mathbf{AB} \cdot \mathbf{C} \stackrel{\text{def}}{=} \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) = \eta \mathbf{A} \\ \qquad \text{where } \eta = \mathbf{B} \cdot \mathbf{C} = \text{scalar}. \\ (b) \text{ operating } \textit{on the left}: \\ \mathbf{C} \cdot \mathbf{AB} \stackrel{\text{def}}{=} (\mathbf{C} \cdot \mathbf{A})\mathbf{B} = \lambda \mathbf{B} \\ \qquad \text{where } \lambda = \mathbf{C} \cdot \mathbf{A} = \text{scalar}. \end{array} \tag{4}$$

The linearity property follows from the operation defined in (4). Also note that in general, $\mathbf{AB} \neq \mathbf{BA}$.

We shall write the sum of two dyads $\mathbf{AB}$ and $\mathbf{EF}$ as $\mathbf{AB} + \mathbf{EF}$ and define it by the distributive property:

$$\begin{aligned} (\mathbf{AB} + \mathbf{EF}) \cdot \mathbf{C} &\stackrel{\text{def}}{=} \mathbf{AB} \cdot \mathbf{C} + \mathbf{EF} \cdot \mathbf{C} \\ &= \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{E}(\mathbf{F} \cdot \mathbf{C}). \\ \mathbf{C} \cdot (\mathbf{AB} + \mathbf{EF}) &\stackrel{\text{def}}{=} \mathbf{C} \cdot \mathbf{AB} + \mathbf{C} \cdot \mathbf{EF} \\ &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} + (\mathbf{C} \cdot \mathbf{E})\mathbf{F}. \end{aligned} \tag{5}$$

It should be a simple exercise to show from Eqs. (4) that the dyadic product is distribu-

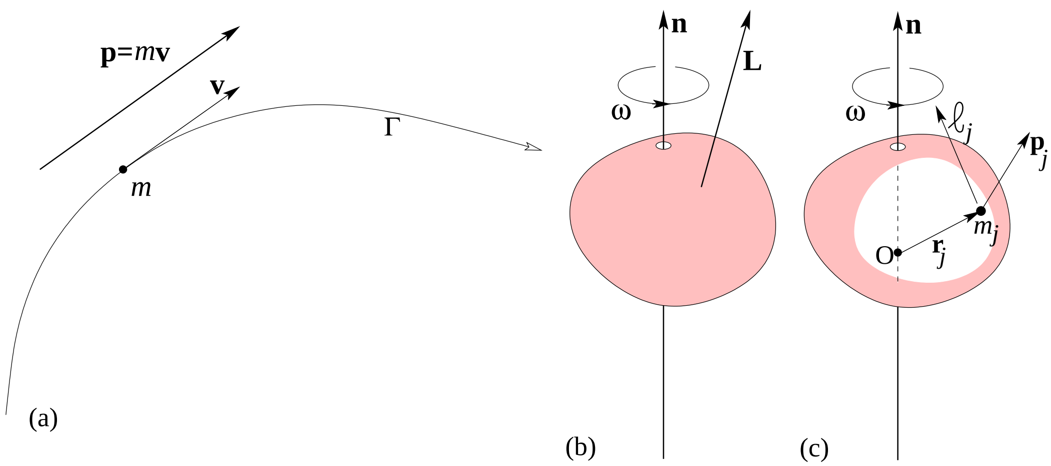

Figure 2: *two examples of how a linear operator $\widehat{\mathcal{O}}$ transforms a vector into another vector: (a) $\widehat{\mathcal{O}}$ acting on* $\mathbf{v}$ *yields* $\mathbf{p}$*; (b,c)* $\widehat{\mathcal{O}}$ *acting on* $\boldsymbol{\omega}$ *yields* $\mathbf{L}$.

tive, i.e., if $\mathbf{E}, \mathbf{F}, \mathbf{C}$ are three arbitrary vectors, then

$$\begin{aligned} (\mathbf{E}+\mathbf{F})\mathbf{C} &= \mathbf{EC}+\mathbf{FC}. \\ \mathbf{C}(\mathbf{E}+\mathbf{F}) &= \mathbf{CE}+\mathbf{CF}. \end{aligned} \quad (6)$$

As a corollary,

$$(\mathbf{A}+\mathbf{B})(\mathbf{E}+\mathbf{F}) = \mathbf{AE}+\mathbf{AF}+\mathbf{BE}+\mathbf{BF}. \quad (7)$$

A sum of dyads can be called a *dyadic*. We shall prefer to use the term "dyadic" as a general name for sums of dyads as well as individual dyads.

We shall frequently use the symbols $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ to represent unit vectors in the directions of the X, Y, Z axes, for which we had used $\mathbf{i}, \mathbf{j}, \mathbf{k}$ earlier in this chapter. As we progress we shall use another set of symbols $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to mean the same unit vectors. This transition $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \to (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \to (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, side by side with $(x, y, z) \to (x_1, x_2, x_3)$ will restore symmetry and help us use Einstein's summation convention (following Eq.9).

Let us now consider the set of 12 dyads: $\{\mathbf{e}_x\mathbf{e}_x,\ \mathbf{e}_x\mathbf{e}_y,\ \mathbf{e}_x\mathbf{e}_z,\ \cdots,\ \mathbf{e}_z\mathbf{e}_z\}$. Using them we can construct the following dyadic

$$\begin{aligned} \widehat{\mathbf{T}} &= T_{xx}\mathbf{e}_x\mathbf{e}_x + T_{yx}\mathbf{e}_y\mathbf{e}_x + \cdots \\ &\quad + T_{yz}\mathbf{e}_y\mathbf{e}_z + T_{zz}\mathbf{e}_z\mathbf{e}_z \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\mathbf{e}_i\mathbf{e}_j \equiv T_{ij}\mathbf{e}_i\mathbf{e}_j, \end{aligned} \quad (8)$$

where the subscripts (1,2,3) represent (x, y, z) respectively. That is

$$\begin{aligned} &\mathbf{e}_1 \equiv \mathbf{e}_x; \quad \mathbf{e}_2 \equiv \mathbf{e}_y; \quad \mathbf{e}_3 \equiv \mathbf{e}_z; \\ \text{and,} \quad &T_{11} \equiv T_{xx};\ T_{12} \equiv T_{xy}; \cdots; \\ &T_{32} \equiv T_{zy};\ T_{33} = T_{zz} \end{aligned} \quad (9)$$

are arbitrary real numbers.

In the second line of Eq. (8) we have introduced *Einstein's summation convention*: *sum over repeated index*, without explicitly inserting the sum symbol $\sum$. The subscript "$_i$" appears twice, implying a sum over i. The subscript "$_j$" appears twice, implying one more sum, this time over j.

The mathematical object $\widehat{\mathbf{T}}$ appearing in Eq. (8) is what we shall call a *tensor* for all purposes in this book. The set of dyads $\{\mathbf{e}_x\mathbf{e}_x, \mathbf{e}_x\mathbf{e}_y, \mathbf{e}_x\mathbf{e}_z, \cdots, \mathbf{e}_z\mathbf{e}_z\}$ can be looked upon as a complete set of base dyads forming a *basis* $\widehat{\mathcal{B}}$ in the tensor space $\mathfrak{T}$ of $\widehat{\mathbf{T}}$. This is analogous to the way that the vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ form a *basis* $\mathcal{B}$ in the vector space $\mathfrak{V}$ of $\mathbf{V}$. Any arbitrary vector $\mathbf{V}$ can be written as a linear superposition of the base vectors as

$$\begin{aligned} &\mathbf{V} = V_x\mathbf{e}_x + V_y\mathbf{e}_y + V_z\mathbf{e}_z, && (a) \\ \text{where}\quad &V_x = \mathbf{V}\cdot\mathbf{e}_x,\ V_y = \mathbf{V}\cdot\mathbf{e}_y, && \\ &V_z = \mathbf{V}\cdot\mathbf{e}_z, && (b) \end{aligned} \tag{10}$$

are the Cartesian (scalar) components of $\mathbf{V}$ in the basis $\mathcal{B}$. In the same way any arbitary tensor $\widehat{\mathbf{T}}$ can be written as a linear superposition of the base dyads, as in Eq. (8), where the nine quantities $\{T_{xx}, T_{xy}, \cdots T_{zy}, T_{zz}\}$ are to be interpreted as the Cartesian (scalar) *components* of $\widehat{\mathbf{T}}$ with respect to this basis $\widehat{\mathcal{B}}$.

From the definition of dyad given in (4), and the orthogonality of the base vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, i.e.,

$$\mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk};\quad j,k = 1,2,3 = x,y,z, \tag{11}$$

it should be apparent that the base dyads operating on any arbitrary vector $\mathbf{V}$ will yield the following vectors:

$$\begin{aligned} &\mathbf{e}_x\mathbf{e}_x\cdot\mathbf{V} = \mathbf{e}_xV_x;\quad \mathbf{e}_x\mathbf{e}_y\cdot\mathbf{V} = \mathbf{e}_xV_y;\ \cdots \\ &\mathbf{e}_z\mathbf{e}_y\cdot\mathbf{V} = \mathbf{e}_zV_y;\quad \mathbf{e}_z\mathbf{e}_z\cdot\mathbf{V} = \mathbf{e}_zV_z. \\ &\mathbf{V}\cdot\mathbf{e}_x\mathbf{e}_x = V_x\mathbf{e}_x;\quad \mathbf{V}\cdot\mathbf{e}_x\mathbf{e}_y = V_x\mathbf{e}_y;\ \cdots \\ &\mathbf{V}\cdot\mathbf{e}_z\mathbf{e}_y = V_z\mathbf{e}_y;\quad \mathbf{V}\cdot\mathbf{e}_z\mathbf{e}_z = V_z\mathbf{e}_z. \end{aligned} \tag{12}$$

Hence, if $\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_x + A_z\mathbf{e}_z$ and $\mathbf{B} = B_x\mathbf{e}_x + B_y\mathbf{e}_x + B_z\mathbf{e}_z$ are two arbitrary vectors, then,

$$\begin{aligned} &\mathbf{A}\cdot\widehat{\mathbf{T}}\cdot\mathbf{B} \overset{\text{def}}{=} \mathbf{A}\cdot(\widehat{\mathbf{T}}\cdot\mathbf{B}) \\ &= A_iT_{ij}B_j = (\mathbf{A}\cdot\widehat{\mathbf{T}})\cdot\mathbf{B}. \quad (a) \\ &\text{As a special case} \\ &\mathbf{e}_i\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_j = T_{ij}. \quad (b) \end{aligned} \tag{13}$$

If the nine components $\{T_{ij}\}$ of a tensor $\widehat{\mathbf{T}}$ are given, the tensor can be constructed using Eq. (8). Conversely, if a tensor $\widehat{\mathbf{T}}$ is given in the form of a mathematical relation, its nine components T_{ij} can be retrieved by means of Eq. (13b).

Using the distributive property given in (7) it is seen that the dyadic product of $\mathbf{A}$ and $\mathbf{B}$ has the following dyadic representation:

$$\begin{aligned} \mathbf{AB} &= A_xB_x\mathbf{e}_x\mathbf{e}_x + A_xB_y\mathbf{e}_x\mathbf{e}_y + \cdots \\ &+ A_zB_y\mathbf{e}_z\mathbf{e}_y + A_zB_z\mathbf{e}_z\mathbf{e}_z \\ &= A_iB_j\mathbf{e}_i\mathbf{e}_j. \end{aligned} \tag{14}$$

Hence, if we write

$$\widehat{\mathbf{T}} = \mathbf{AB},\quad \text{then,}\quad T_{ij} = A_iB_j. \tag{15}$$

Using Eq. ((a)4), the operation of the tensor $\widehat{\mathbf{T}}$ on the vector $\mathbf{C} = C_k\mathbf{e}_k$ placed on the *right* works out as follows.

$$\begin{aligned} \widehat{\mathbf{T}}\cdot\mathbf{C} &= (T_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot(C_k\mathbf{e}_k) \\ &= T_{ij}\,C_k\,\mathbf{e}_i(\mathbf{e}_j\cdot\mathbf{e}_k) \\ &= \mathbf{e}_i\,(T_{ij}\,C_j). \end{aligned} \tag{16}$$

We have used the orthogonality relation (11) to get to the last line.

In a similar way, using Eq. ((b)4), the operation of the tensor $\widehat{\mathbf{T}}$ on the vector $\mathbf{C} = C_k \mathbf{e}_k$ placed on the *left* works out as follows.

$$\begin{aligned} \mathbf{C}\cdot\widehat{\mathbf{T}} &= (C_k\mathbf{e}_k)\cdot(T_{ij}\mathbf{e}_i\mathbf{e}_j) \\ &= C_k\,T_{ij}\,(\mathbf{e}_k\cdot\mathbf{e}_i)\mathbf{e}_j \\ &= (C_k\,T_{kj})\,\mathbf{e}_j. \end{aligned} \tag{17}$$

The above two equations suggest that if we write $\mathbf{D} = \widehat{\mathbf{T}}\cdot\mathbf{C}$ and $\mathbf{F} = \mathbf{C}\cdot\widehat{\mathbf{T}}$, then the Cartesian components (D_1, D_2, D_3) of $\mathbf{D}$ and (F_1, F_2, F_3) of $\mathbf{F}$ can be obtained from matrix multiplications:

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

$$\begin{aligned} &\begin{pmatrix} F_1 & F_2 & F_3 \end{pmatrix} \\ &= \begin{pmatrix} C_1 & C_2 & C_3 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}. \end{aligned} \tag{18}$$

In the above equations, starting from Eq. (4), we have used a dot ($\cdot$) to separate the tensor from the vector on which it is operating. We shall frequently refer to a tensor operation as a *dot product* between the tensor and the vector. Eqs. (18) show that a *dot product actually involves a matrix multiplication.* A tensor is to be represented as a *square matrix*, and a vector either as a *column matrix* or a *row matrix*, depending on whether the tensor operation is on the right or on the left[2].

$$\begin{aligned} \widehat{\mathbf{T}} &= \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = [T]; \\ \mathbf{C} &= \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \{C\}; \\ \mathbf{F} &= \begin{pmatrix} F_1 & F_2 & F_3 \end{pmatrix} = (F). \end{aligned} \tag{19}$$

In the above we have adopted the convention of indicating a 3×3 square matrix by [ ], a 3×1 column matrix by { }, and a 1×3 row matrix by ( ). Hence, Eqs.(18) can be written as

$$\{D\} = [T]\{C\}; \quad (F) = (C)[T]. \tag{20}$$

It follows from Eq. (14) that the matrix representation of the dyadic $\mathbf{AB}$ is

$$\mathbf{AB} = \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{pmatrix}. \tag{21}$$

We shall define the dot product of two tensors $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{T}}$ as the tensor $\widehat{\mathbf{R}} = \widehat{\mathbf{S}}\cdot\widehat{\mathbf{T}}$ by its operation on an arbitrary vector $\mathbf{C}$ *on the right* in the following way.

$$(\widehat{\mathbf{S}}\cdot\widehat{\mathbf{T}})\cdot\mathbf{C} \stackrel{\text{def}}{=} \widehat{\mathbf{S}}\cdot(\widehat{\mathbf{T}}\cdot\mathbf{C}). \tag{22}$$

[2] In Quantum Mechanics (QM) a clear distinction is made between a vector $\mathbf{A}$ on left and a vector $\mathbf{B}$ on right, as in the scalar product $\mathbf{A}\cdot\mathbf{B}$. The former is called a *bra* vector and the latter a *ket* vector, and together, in the scalar product, they constitute a *bra-ket*: $\mathbf{A} \rightarrow < A|$; $\mathbf{B} \rightarrow |\, B >$; $\mathbf{A}\cdot\mathbf{B} \rightarrow < A|B >$. However, these vectors are in genereral infinite dimensional, their components are complex numbers, and the components of the bra vector $< A|$ are complex conjugates of the respective components of the ket vector $|A >$.

From this it follows that the matrix representing $\widehat{\mathbf{R}}$ is given by the product of the matrices representing $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{T}}$. That is,

$$[R] = [S]\,[T], \text{ implying: } R_{ij} = S_{ik}T_{kj}. \quad (23)$$

It is then obvious that, in general, $\widehat{\mathbf{S}} \cdot \widehat{\mathbf{T}} \neq \widehat{\mathbf{T}} \cdot \widehat{\mathbf{S}}$.

Using the matrix representation as given in Eq. (23), and the tensor operation on the left as found out in (17), we can now see how the product tensor $\widehat{\mathbf{R}} = \widehat{\mathbf{S}} \cdot \widehat{\mathbf{T}}$ will act *on the left.*

$$\begin{aligned} \mathbf{C} \cdot \widehat{\mathbf{R}} &= (C_k R_{kj})\mathbf{e}_j = (C_k S_{km} T_{mj})\mathbf{e}_j \\ &= (C_k S_{km})(T_{mj}\mathbf{e}_j). \quad (24) \\ \text{Or,} \quad \mathbf{C} \cdot (\widehat{\mathbf{S}} \cdot \widehat{\mathbf{T}}) &= (\mathbf{C} \cdot \widehat{\mathbf{S}}) \cdot \widehat{\mathbf{T}}. \end{aligned}$$

We can extend the definition of matrix product to any number of tensors, by writing the matrix representation of the product tensor as the product of the representative matrices of the component tensors. For example

$$\text{If } \widehat{\mathbf{R}} = \widehat{\mathbf{A}} \cdot \widehat{\mathbf{B}} \cdot \widehat{\mathbf{C}}, \text{ then } [R] = [A]\,[B]\,[C]. \quad (25)$$

At this point we shall add a word of caution. A tensor is not the same as a square matrix, just as a vector is not the same as a column matrix or a row matrix. The row matrix shown in Eq. (19), for example, gives the components of the vector $\mathbf{F}$ in a given coordinate system XYZ. As the coordinates are changed from (x, y, z) to (x', y', z'), the components will transform from (F_1, F_2, F_3) to (F_1', F_2', F_3'). However, the vector $\mathbf{F}$ itself is a “geometrical object” (a straight line of measured length pointing in an assigned direction) which remains invariant under all coordinate transformations. In the same way the tensor $\widehat{\mathbf{T}}$ is a geometrical object, which remains invariant under all coordinate transformations, even though its components will change from the square matrix $[T_{ij}]$ to another square matrix $[T'_{ij}]$ under the same coordinate transformation.

Yes, the components of all tensors will transform, except the components of the identity tensor which we shall introduce in the next section. They will remain the same, the same as in (27), following any coordinte transformations.

2.3 Identity Tensor, Completeness Relation, Components of a Tensor in the Spherical coordinate system

In matrix multiplication one needs the *identity matrix* $\widehat{1}$ which in the present context, is the matrix representation of the *identity tensor*, also known by the alternative name *idemfactor*. It will be recognized by the symbol $\widehat{\mathbf{1}}$. Its sole property is that when it operates on any vector $\mathbf{V}$, either on the right, or on the left, it gives back the same vector.

$$\widehat{\mathbf{1}} \cdot \mathbf{V} \stackrel{\text{def}}{=} \mathbf{V}; \qquad \mathbf{V} \cdot \widehat{\mathbf{1}} \stackrel{\text{def}}{=} \mathbf{V}. \quad (26)$$

Such a tensor must have $\widehat{1}$ for its matrix representation. The dyadic representation (shown below) follows from the above property and the orthogonality relation (11).

$$
\begin{aligned}
\widehat{\mathbf{1}} &= \widehat{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. && (a) \\
\widehat{\mathbf{1}} &= \mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z = \mathbf{e}_i\mathbf{e}_i. && (b)
\end{aligned}
\tag{27}
$$

Eq.(a) gives the Matrix representation, and Eq.(b) the Dyadic representation.

It will be advantageous to write the tensor $\widehat{\mathbf{T}}$ in a curvilinear coordinate systen, in particular, spherical coordinate system. The reader must be familar with the following transformation equations for the coordinates and the base vectors.

$$
\begin{aligned}
&(x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta). \\
&\mathbf{e}_r = \sin\theta\,(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) + \cos\theta\,\mathbf{e}_z; \\
&\mathbf{e}_\theta = \cos\theta\,(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) - \sin\theta\,\mathbf{e}_z; \\
&\mathbf{e}_\phi = -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y.
\end{aligned}
\tag{28}
$$

Using these equations (and remembering that $\mathbf{e}_r\mathbf{e}_\theta \neq \mathbf{e}_\theta\mathbf{e}_r$, for example) it should be a simple exercise to show that

$$
\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_\phi\mathbf{e}_\phi = \mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z = \widehat{\mathbf{1}}. \tag{29}
$$

If we have three unit vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ which are mutually orthogonal at every point in space and such that

$$
\boldsymbol{a}\boldsymbol{a} + \boldsymbol{b}\boldsymbol{b} + \boldsymbol{c}\boldsymbol{c} = \widehat{\mathbf{1}}, \tag{30}
$$

then we say that these three vectors form a *complete orthogonal set*, and hence a basis, so that any arbitrary vector $\mathbf{V}$ can be represented as a linear superposition of these three vectors [3]. This should be clear from the following.

$$
\begin{aligned}
&\mathbf{V} = \mathbf{V}\cdot\widehat{\mathbf{1}} = \mathbf{V}\cdot(\boldsymbol{a}\boldsymbol{a} + \boldsymbol{b}\boldsymbol{b} + \boldsymbol{c}\boldsymbol{c}) \\
&= V_a\boldsymbol{a} + V_b\boldsymbol{b} + V_c\boldsymbol{c}, \\
\text{where}\quad &V_a = \mathbf{V}\cdot\boldsymbol{a},\ V_b = \mathbf{V}\cdot\boldsymbol{b},\ V_c = \mathbf{V}\cdot\boldsymbol{c},
\end{aligned}
\tag{31}
$$

are the components of $\mathbf{V}$ in the directions of $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ respectively. Using the completeness property it can be advantageous to write a tensor in the following style.

$$
\begin{aligned}
&\widehat{\mathbf{T}} = \widehat{\mathbf{1}}\cdot\widehat{\mathbf{T}}\cdot\widehat{\mathbf{1}} \\
&= (\boldsymbol{a}\boldsymbol{a} + \boldsymbol{b}\boldsymbol{b} + \boldsymbol{c}\boldsymbol{c})\cdot\widehat{\mathbf{T}}\cdot(\boldsymbol{a}\boldsymbol{a} + \boldsymbol{b}\boldsymbol{b} + \boldsymbol{c}\boldsymbol{c}) \\
&= T_{aa}\boldsymbol{a}\boldsymbol{a} + T_{ab}\boldsymbol{a}\boldsymbol{b} + T_{ac}\boldsymbol{a}\boldsymbol{c}\cdots \\
&+ T_{cb}\boldsymbol{c}\boldsymbol{b} + T_{cc}\boldsymbol{c}\boldsymbol{c}, \quad \text{where} \\
&T_{aa} = \boldsymbol{a}\cdot\widehat{\mathbf{T}}\cdot\boldsymbol{a},\ T_{ab} = \boldsymbol{a}\cdot\widehat{\mathbf{T}}\cdot\boldsymbol{b}, \cdots, \\
&T_{cb} = \boldsymbol{c}\cdot\widehat{\mathbf{T}}\cdot\boldsymbol{b},\ T_{cc} = \boldsymbol{c}\cdot\widehat{\mathbf{T}}\cdot\boldsymbol{c}
\end{aligned}
\tag{32}
$$

are the components of $\widehat{\mathbf{T}}$ with respect to the basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$.

We shall illustrate the operation shown in Eq. (32) by writing the tensor $\widehat{\mathbf{T}}$ in Cartesian and spherical coordinate systems.

$$
\begin{aligned}
\widehat{\mathbf{T}} &= (\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z)\cdot\widehat{\mathbf{T}}\cdot(\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z) && (a) \\
&= T_{xx}\mathbf{e}_x\mathbf{e}_x + T_{xy}\mathbf{e}_x\mathbf{e}_y + T_{xz}\mathbf{e}_x\mathbf{e}_z + \cdots + T_{zx}\mathbf{e}_z\mathbf{e}_y + T_{zz}\mathbf{e}_z\mathbf{e}_z, \quad \text{where} && (b) \\
T_{xx} &= \mathbf{e}_x\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_x,\ T_{xy} = \mathbf{e}_x\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_y, \cdots, T_{zy} = \mathbf{e}_z\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_y,\ T_{zz} = \mathbf{e}_z\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_z. && (c) \\
\widehat{\mathbf{T}} &= (\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_\phi\mathbf{e}_\phi)\cdot\widehat{\mathbf{T}}\cdot(\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_\phi\mathbf{e}_\phi) && (d) \\
&= T_{rr}\mathbf{e}_r\mathbf{e}_r + T_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + T_{r\phi}\mathbf{e}_r\mathbf{e}_\phi + \cdots + T_{\phi\theta}\mathbf{e}_\phi\mathbf{e}_\theta + T_{\phi\phi}\mathbf{e}_\phi\mathbf{e}_\phi, \quad \text{where} && (e) \\
T_{rr} &= \mathbf{e}_r\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_r,\ T_{r\theta} = \mathbf{e}_r\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_\theta, \cdots, T_{\phi\theta} = \mathbf{e}_\phi\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_\theta,\ T_{\phi\phi} = \mathbf{e}_\phi\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_\phi. && (f)
\end{aligned}
\tag{33}
$$

[3] In QM the completeness of a set of orthonormal vectors $\{|u_i >; i = 1, 2, \ldots, \infty\}$ is expressed through the statement $\sum_i |u_i >< u_i| = 1$. This relation is used to change the representation of a Hermitean operator $\widehat{\mathbf{T}}$, the equivalent of the tensors we are con-

Lines (a)-(c) represent the tensor $\widehat{\mathbf{T}}$ in a Cartesian coordinate system, and lines (d)-(f) in a spherical coordinate system

We can then write the components of $\widehat{\mathbf{T}}$ in the following matrix forms

$$\begin{array}{l} \widehat{\mathbf{T}} \overset{(\text{ Cart})}{\longrightarrow} \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}. \\ \widehat{\mathbf{T}} \overset{(\text{sphr})}{\longrightarrow} \begin{pmatrix} T_{rr} & T_{r\theta} & T_{r\phi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\phi} \\ T_{\phi r} & T_{\phi\theta} & T_{\phi\phi} \end{pmatrix}. \end{array} \quad (34)$$

The first matrix gives the Cartesian components, and the second one the spherical components.

Using the transformation of the base vectors (Eq. 28), and the completeness relations (29), one can transform the Cartesian components to spherical components, for both vectors and tensors, as we shall show. For this purpose we shall temporarily denote the spherical base vectors with a prime, i.e., $\{\mathbf{e}'_i : i = r, \theta, \phi\}$ and make a table of *transformation coefficients* $\{c_{ij}\}$:

$$\begin{array}{c} \mathbf{e}'_i = \mathbf{e}'_i \cdot \mathbf{e}_j \mathbf{e}_j = c_{ij} \mathbf{e}_j, \\ \text{where} \quad c_{ij} \equiv \mathbf{e}'_i \cdot \mathbf{e}_j : \; i = r, \theta, \phi; \; j = x, y, z. \\ = \begin{pmatrix} \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \\ \cos\theta\,\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \end{array} \quad (35)$$

Now, let $\mathbf{V}$ be a vector and $\widehat{\mathbf{T}}$ a tensor with Cartesian components $[\{V_j\}, \; \{T_{ij}\}, \; i, j = x, y, z\,]$ respectively. Then the spherical components of the same vector and tensor, namely, $[\{V'_j\}, \; \{T'_{ij}\}, \; i, j = r, \theta, \phi]$ will be obtained in the following ways[4] :

$$\begin{aligned} V'_j &= \mathbf{V} \cdot \mathbf{e}'_j = \mathbf{V} \cdot \mathbf{e}_k \mathbf{e}_k \cdot \mathbf{e}'_j = c_{jk} V_k. \\ T'_{ij} &= \mathbf{e}'_i \cdot \widehat{\mathbf{T}} \cdot \mathbf{e}'_j = \mathbf{e}'_i \cdot \mathbf{e}_k \mathbf{e}_k \cdot \widehat{\mathbf{T}} \cdot \mathbf{e}_l \mathbf{e}_l \cdot \mathbf{e}'_j \\ &= c_{ik} c_{jl} T_{kl}. \end{aligned} \quad (36)$$

Note that we have used the summation convention: sum over k in line (a), sum over k, l in line (b).

We shall illustrate the transformation formulas (36) with two examples, i.e,$V_r \equiv V'_1$ and $T_{r\theta} \equiv T'_{12}$.

$$\begin{array}{l} V_r = \sin\theta\cos\phi\, V_x + \sin\theta\sin\phi\, V_y + \cos\theta\, V_z. \\ T_{r\theta} = \sin\theta\cos\phi\,(\cos\theta\cos\phi\, T_{xx} \\ + \cos\theta\sin\phi\, T_{xy} - \sin\theta\, T_{xz}) \\ + \sin\theta\sin\phi\,(\cos\theta\cos\phi\, T_{yx} + \cos\theta\sin\phi\, T_{yy} \\ - \sin\theta\, T_{yz}) \\ + \cos\theta\,(\cos\theta\cos\phi\, T_{zx} + \cos\theta\sin\phi\, T_{zy} \\ - \sin\theta\, T_{zz}). \end{array} \quad (37)$$

2.4 Example: Inertia Tensor

We shall illustrate the tensor concept by showing two important examples, namely (1)

[4] In Tensor analysis, the primary language of the theory of relativity, the rule of transformation has different forms for contravariant and covariant vectors, and for contravariant, covariant and mixed tensors. The rules we are establishing here are different from them. The components of vectors, tensors we are using may be called *physical components*, in contrast to their contravariant and covariant components for which a more elegant transformation rule is used.

the inertia tensor and (2) the stress tensor. We shall take up a short discussion of the first example in this section leaving the second example, which needs a more detailed coverage, to the next section.

In Sec. 2.1 we talked about the tensor operation converting the angular velocity $\boldsymbol{\omega}$ into angular momentum $\mathbf{L}$. The corresponding operator is the *inertia tensor* $\widehat{\boldsymbol{\mathcal{I}}}$ of the rigid body. Its dot product with the angular velocity $\boldsymbol{\omega}$ gives the *angular momentum* $\mathbf{L}$ of the rigid body. That is,

$$\mathbf{L} = \widehat{\boldsymbol{\mathcal{I}}} \cdot \boldsymbol{\omega}. \tag{38}$$

We shall find an expression for the vector angular momentum $\mathbf{L}$ of a rigid body which is rotating about a point O (which can be a moving point, e.g., the CM) with angular velocity $\boldsymbol{\omega} = \omega\mathbf{n}$ about the axis pointing in the direction of the unit vector $\mathbf{n}$. Let j be one of the constituent particles, having mass m_j, and located at the radius vector $\mathbf{r}_j$ with respect to O, as shown in Fig. 2(c). The velocity of this point is $\mathbf{v}_j = \boldsymbol{\omega} \times \mathbf{r}_j$. Therefore this particle has an angular momentum with respect to the point O, equal to

$$\begin{aligned}\boldsymbol{\ell}_j = \mathbf{r}_j \times \mathbf{p}_j = \mathbf{r}_j \times m_j\mathbf{v}_j = m_j\mathbf{r}_j \times (\boldsymbol{\omega} \times \mathbf{r}_j) \\ = m_j[r_j^2\boldsymbol{\omega} - (\mathbf{r}_j \cdot \boldsymbol{\omega})\mathbf{r}_j]. \quad (39)\end{aligned}$$

Assuming that the rigid body is made of N particles (which is a very large number), we add the angular momentum of each particle to obtain the angular momentum of the rigid body about the point O, given as

$$\mathbf{L}_O = \sum_{j=0}^{N} m_j[r_j^2\boldsymbol{\omega} - (\mathbf{r}_j \cdot \boldsymbol{\omega})\mathbf{r}_j]. \tag{40}$$

We can write the quantity within square brackets as

$$[r_j^2\boldsymbol{\omega} - (\mathbf{r}_j \cdot \boldsymbol{\omega})\mathbf{r}_j] = [r_j^2\widehat{\mathbf{1}} - \mathbf{r}_j\mathbf{r}_j] \cdot \boldsymbol{\omega}, \tag{41}$$

and construct the Inertia tensor as the dyadic (sum of infinitely small dyads)

$$\widehat{\boldsymbol{\mathcal{I}}} = \sum_{j=0}^{N} m_j[r_j^2\widehat{\mathbf{1}} - \mathbf{r}_j\mathbf{r}_j]. \tag{42}$$

Then we get the angular momentum as the dot product

$$\mathbf{L}_O = \widehat{\boldsymbol{\mathcal{I}}} \cdot \boldsymbol{\omega}. \tag{43}$$

We have thus derived Eq. (38), and along with it have found an expression for the inertia tensor in Eq. (42). Note that the expression within the square brackets is the difference of two dyadics, namely, the identity dyadic $\widehat{\mathbf{1}}$ multiplied by the scalar r_j^2, and the dyadic product of $\mathbf{r}_j$ with itself.

For further clarification we shall write down the components of the tensor. Assumimg that the rigid body has uniform mass density ρ distributed over its volume V, the sum in Eq. (42) becomes the integral:

$$\widehat{\boldsymbol{\mathcal{I}}} = \rho \iiint_V [r^2\widehat{\mathbf{1}} - \mathbf{r}\mathbf{r}] d^3r. \tag{44}$$

Some of its components are

$$\begin{aligned}&\mathcal{I}_{xx} = \rho \iiint_V [r^2 - x^2] d^3r = \rho \iiint_V (y^2 + z^2) d^3r; \\ &\mathcal{I}_{xy} = -\rho \iiint_V (xy)\, d^3r; \quad \text{etc.}\end{aligned} \tag{45}$$

It is now seen that the *inertia tensor is a symmetric tensor*, i.e.,

$$\mathcal{I}_{xy} = \mathcal{I}_{yx}; \quad \mathcal{I}_{yz} = \mathcal{I}_{zy}; \quad \mathcal{I}_{zx} = \mathcal{I}_{xz}. \tag{46}$$

This symmetry property is preserved under all coordinate transformations.

3 Stress in a Medium

Stress and Stress Tensor are discussed in engineering books on Fluid Mechanics[10] and Strength of Materials[11]. However, physics students may find useful the introductory lessons on these concepts by Symon[12] and Feynman[13].

3.1 Stress Vector

By (mechanical) stress we mean *internal forces* (in the form of intermolecular interactions) called into play when bulk matter, either in the form of solid, liquid or gas, is subjected to external forces. These internal forces exist throughout the bulk matter and its mathematical expression is given by a stress tensor field $\widehat{\boldsymbol{\mathcal{T}}}(x, y, z)$.

For simplicity we shall consider a solid block in Fig. 3(a). It has been cut into two parts, the upper block $\mathcal{U}$ and the lower block $\mathcal{L}$, by an imaginary plane Σ, leaving a trace Γ of its boundary. This plane is identified by the unit normal vector $\mathbf{n}$ pointing from the lower block to the upper block.

In Fig. 3(b) we have shown the lower block $\mathcal{L}$ with the plane of separation Σ exposed. Let us consider a small area da at the point $\mathrm{P}(x, y, z)$ inside the solid, but lying on this plane. Then the *stress vector* $\boldsymbol{\mathcal{T}}^{(n)}(x, y, z)$ is defined to be the force per unit area at $\mathrm{P}(x, y, z)$, exerted by the atoms of the upper block $\mathcal{U}$ on the atoms of the lower block $\mathcal{L}$ across the plane $\mathbf{n}$. The infinitesimal force acting on the area da is then

$$d\mathbf{F}^{(n)} = \boldsymbol{\mathcal{T}}^{(n)}(x, y, z)\, da. \tag{47}$$

Note that in general the direction of the stress vector $\boldsymbol{\mathcal{T}}^{(n)}(x, y, z)$ is different from the direction of the normal $\mathbf{n}$. If, however, $\boldsymbol{\mathcal{T}}^{(n)}(x, y, z) \parallel \mathbf{n}$ (i.e., perpendicular to the plane), the stress (vector) is called *normal stress*. If $\boldsymbol{\mathcal{T}}^{(n)}(x, y, z) \perp \mathbf{n}$ (i.e., parallel to the plane), it is called *shear stress*.

3.2 Stress Tensor

In Fig. 3(c) we have shown the stress vectors $\boldsymbol{\mathcal{T}}^{(x)}, \boldsymbol{\mathcal{T}}^{(y)}, \boldsymbol{\mathcal{T}}^{(z)}$ on three perpendicular faces of a tiny rectangular block, identified by the normal vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. Let $\{n_x, n_y, n_z\}$ be the direction cosines of $\mathbf{n}$ so that

$$\mathbf{n} = n_x\mathbf{e}_x + n_x\mathbf{e}_x + n_x\mathbf{e}_z. \tag{48}$$

It can be shown, using the equation of motion of the prism shown in Fig. 3(d) that

$$\boldsymbol{\mathcal{T}}^{(n)} = \boldsymbol{\mathcal{T}}^{(x)} n_x + \boldsymbol{\mathcal{T}}^{(y)} n_y + \boldsymbol{\mathcal{T}}^{(z)} n_z. \tag{49}$$

Eq. (49) can be given an elegant form if we write the stress vectors as column matrices

$$\boldsymbol{\mathcal{T}}^{(n)} = \begin{pmatrix} \mathcal{T}_x^{(n)} \\ \mathcal{T}_y^{(n)} \\ \mathcal{T}_z^{(n)} \end{pmatrix}; \ \boldsymbol{\mathcal{T}}^{(x)} = \begin{pmatrix} \mathcal{T}_{xx} \\ \mathcal{T}_{yx} \\ \mathcal{T}_{zx} \end{pmatrix}; \quad \boldsymbol{\mathcal{T}}^{(y)} = \begin{pmatrix} \mathcal{T}_{xy} \\ \mathcal{T}_{yy} \\ \mathcal{T}_{zy} \end{pmatrix}; \ \boldsymbol{\mathcal{T}}^{(z)} = \begin{pmatrix} \mathcal{T}_{xz} \\ \mathcal{T}_{yz} \\ \mathcal{T}_{zz} \end{pmatrix}; \tag{50}$$

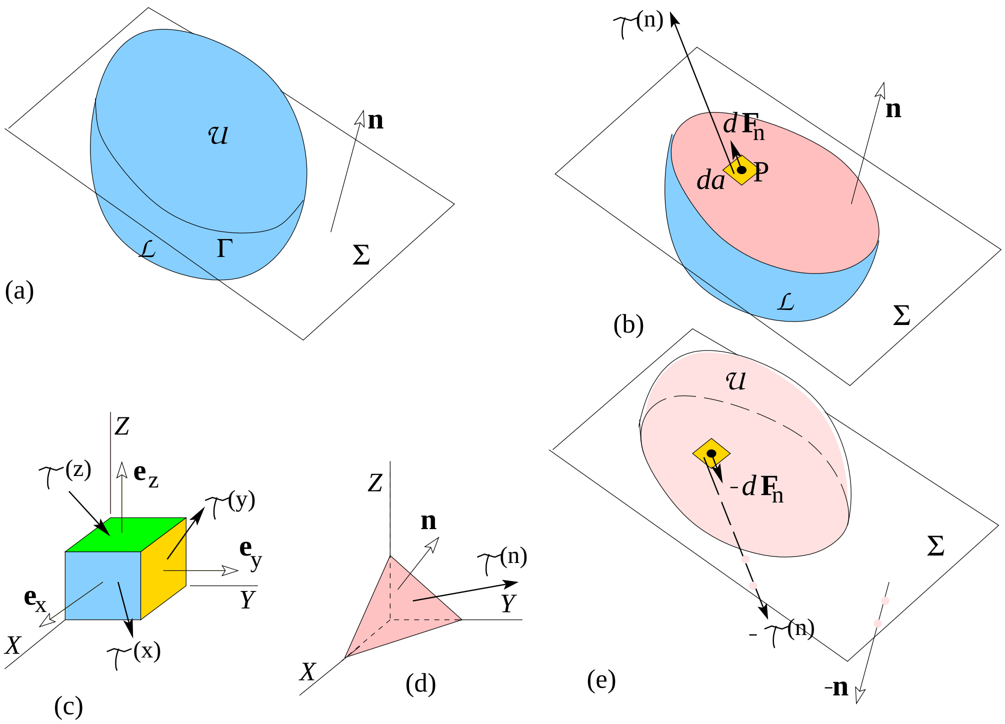

Figure 3: *Explaining the Stress Tensor*

invoke a *Stress Tensor* $\widehat{\boldsymbol{\mathcal{T}}}$ having the *matrix representation*

$$\widehat{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} \mathcal{T}_{xx} & \mathcal{T}_{xy} & \mathcal{T}_{xz} \\ \mathcal{T}_{yx} & \mathcal{T}_{yy} & \mathcal{T}_{yz} \\ \mathcal{T}_{zx} & \mathcal{T}_{zy} & \mathcal{T}_{zz} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{T}}^{(x)} \ \boldsymbol{\mathcal{T}}^{(y)} \ \boldsymbol{\mathcal{T}}^{(z)} \end{pmatrix} \quad (51)$$

so that Eq. (49) can be represented by the following matrix multiplication.

$$\begin{pmatrix} \mathcal{T}_x^{(n)} \\ \mathcal{T}_y^{(n)} \\ \mathcal{T}_z^{(n)} \end{pmatrix} = \begin{pmatrix} \mathcal{T}_{xx} & \mathcal{T}_{xy} & \mathcal{T}_{xz} \\ \mathcal{T}_{yx} & \mathcal{T}_{yy} & \mathcal{T}_{yz} \\ \mathcal{T}_{zx} & \mathcal{T}_{zy} & \mathcal{T}_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}. \quad (52)$$

Alternatively, we can write the stress tensor in the *dyadic representation*

$$\widehat{\boldsymbol{\mathcal{T}}} = \boldsymbol{\mathcal{T}}^{(x)}\mathbf{e}_x + \boldsymbol{\mathcal{T}}^{(y)}\mathbf{e}_y + \boldsymbol{\mathcal{T}}^{(z)}\mathbf{e}_z, \quad (53)$$

so that Eq. (49) can be retrieved from the dot product of the above dyadic with the unit vector $\mathbf{n}$ placed on the right, i.e.,.

$$\boldsymbol{\mathcal{T}}^{(n)} = \widehat{\boldsymbol{\mathcal{T}}} \cdot \mathbf{n}. \quad (54)$$

Note from (51) that in $\mathcal{T}_{ij}$ the second index j is the "surface index" (indicating the direction of the surface on which stands the stress vector $\boldsymbol{\mathcal{T}}^{(j)}$) and the first index i the "component index" (indicating x, y, z component of $\boldsymbol{\mathcal{T}}^{(j)}$).

In Fig. 3(e) we have shown the upper part of the solid of Fig.(a), and the same area da as in Fig.(b), but now on the upper block $\mathcal{U}$. The normal vector now is $-\mathbf{n}$, and the stress vector is

$$\begin{aligned} \boldsymbol{\mathcal{T}}^{(-n)}(x,y,z) &= \widehat{\boldsymbol{\mathcal{T}}}(x,y,z) \cdot (-\mathbf{n}) \\ &= -\boldsymbol{\mathcal{T}}^{(n)}(x,y,z), \quad (55) \end{aligned}$$

so that the force exerted by the atoms of the lower block $\mathcal{L}$ on the atoms of the upper block $\mathcal{U}$ across the same area da is $d\mathbf{F}'^{(n)} = -\boldsymbol{\mathcal{T}}^{(n)}da = -d\mathbf{F}^{(n)}$. Which is in conformity with Newton's Third Law of Motion.

In obtaining the last equality in Eq. (55) we have used the linearity property of the tensor as stipulated in (3). In this case $\widehat{\boldsymbol{\mathcal{T}}} \cdot (a\mathbf{n}) = a\widehat{\boldsymbol{\mathcal{T}}} \cdot \mathbf{n}$ where $a = -1$.

Like the inertia tensor, the stress tensor is a *symmetric tensor*, i.e.,

$$\mathcal{T}_{xy} = \mathcal{T}_{yx}; \quad \mathcal{T}_{yz} = \mathcal{T}_{zy}; \quad \mathcal{T}_{zx} = \mathcal{T}_{xz}. \quad (56)$$

which can be proved using the equation of motion of the angular momentum.

3.3 Gauss's Divergence Theorem for a Tensor Field

When we say tensor field, we mean a physical quantity represented by a tensor $\widehat{\mathbf{T}}(x,y,z)$ whose nine components $T_{xx}(x,y,z), T_{xy}(x,y,z), \cdots T_{zz}(x,y,z)$ are defined at every coordinate point (x,y,z). We assume that these nine components are all differentiable functions of the coordinates x, y, z. For such a tensor field we define its *divergence* to be the formal dot product of the grad operator $\boldsymbol{\nabla}$ with the tensor $\widehat{\mathbf{T}}(x,y,z)$, it being assumed that $\boldsymbol{\nabla}$ will appear on the left.

Let us write the tensor $\widehat{\mathbf{T}}$ by the dyadic representation

$$\widehat{\mathbf{T}} = \mathbf{T}^{(x)}\mathbf{e}_x + \mathbf{T}^{(y)}\mathbf{e}_y + \mathbf{T}^{(z)}\mathbf{e}_z, \quad (57)$$

as in Eq. (53). Then

$$\operatorname{div}\widehat{\mathbf{T}} \equiv \boldsymbol{\nabla}\cdot\widehat{\mathbf{T}} = \boldsymbol{\nabla}\cdot\left(\mathbf{T}^{(x)}\mathbf{e}_x + \mathbf{T}^{(y)}\mathbf{e}_y + \mathbf{T}^{(z)}\mathbf{e}_z\right)$$
$$\stackrel{\text{def}}{=} \left(\boldsymbol{\nabla}\cdot\mathbf{T}^{(x)}\right)\mathbf{e}_x + \left(\boldsymbol{\nabla}\cdot\mathbf{T}^{(y)}\right)\mathbf{e}_y + \left(\boldsymbol{\nabla}\cdot\mathbf{T}^{(z)}\right)\mathbf{e}_z. \tag{58}$$

Note that $\boldsymbol{\nabla}\cdot\mathbf{T}^{(x)}, \boldsymbol{\nabla}\cdot\mathbf{T}^{(y)}, \boldsymbol{\nabla}\cdot\mathbf{T}^{(z)}$ are the familiar scalar divergences of the vector fields $\mathbf{T}^{(x)}, \mathbf{T}^{(y)}, \mathbf{T}^{(z)}$ respectively,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{T}^{(x)} &= \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}, \\
\boldsymbol{\nabla}\cdot\mathbf{T}^{(y)} &= \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zx}}{\partial z}, \\
\boldsymbol{\nabla}\cdot\mathbf{T}^{(z)} &= \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}.
\end{aligned} \tag{59}$$

and constitute three (scalar) components of the vector $\boldsymbol{\nabla}\cdot\widehat{\mathbf{T}}$ along the X, Y and Z axes respectively. Combining (58) and (59) we get

$$\boldsymbol{\nabla}\cdot\widehat{\mathbf{T}} = \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_i}\,\mathbf{e}_j \equiv \frac{\partial T_{ij}}{\partial x_i}\,\mathbf{e}_j. \tag{60}$$

In the second equality we have employed Einestein's summation convention (introduced on page 8.)

The divergence of a vector field is sometimes interpreted as "outflux per unit volume". This association of divergence with outflux is due to Gauss's divergence theorem. Applying the divergence theorem to the three vector fields $\mathbf{T}^{(x)}, \mathbf{T}^{(y)}, \mathbf{T}^{(z)}$ separately, we get the following three equivalence relations.

$$\begin{aligned}
\iiint_V \boldsymbol{\nabla}\cdot\mathbf{T}^{(x)}(\mathbf{r})\,d^3r &= \iint_S \mathbf{n}(\mathbf{r})\cdot\mathbf{T}^{(x)}(\mathbf{r})\,da. \\
\iiint_V \boldsymbol{\nabla}\cdot\mathbf{T}^{(y)}(\mathbf{r})\,d^3r &= \iint_S \mathbf{n}(\mathbf{r})\cdot\mathbf{T}^{(y)}(\mathbf{r})\,da. \\
\iiint_V \boldsymbol{\nabla}\cdot\mathbf{T}^{(z)}(\mathbf{r})\,d^3r &= \iint_S \mathbf{n}(\mathbf{r})\cdot\mathbf{T}^{(z)}(\mathbf{r})\,da.
\end{aligned} \tag{61}$$

Multiplying either side of the first, second and third lines with $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ respectively, and adding, we get

$$\begin{aligned}
&\iiint_V \boldsymbol{\nabla}\cdot\left(\mathbf{T}^{(x)}\mathbf{e}_x + \mathbf{T}^{(y)}\mathbf{e}_y + \mathbf{T}^{(z)}\mathbf{e}_z\right), d^3r \\
&= \iint_S \mathbf{n}(\mathbf{r})\cdot\left(\mathbf{T}^{(x)}\mathbf{e}_x + \mathbf{T}^{(y)}\mathbf{e}_y + \mathbf{T}^{(z)}\mathbf{e}_z\right)\,da.
\end{aligned} \tag{62}$$

Identifying the dyadic within the parantheses as the tensor $\widehat{\mathbf{T}}$, we obtain the divergence theorem for the tensor field.

$$\iiint_V \boldsymbol{\nabla}\cdot\widehat{\mathbf{T}}(\mathbf{r})\,d^3r = \iint_S \mathbf{n}(\mathbf{r})\cdot\widehat{\mathbf{T}}(\mathbf{r})\,da. \tag{63}$$

Specializing the above theorem to stress tensor, using its symmetry property, we can write the integrand on the right side as

$$\begin{aligned}
\mathbf{n}\cdot\widehat{\boldsymbol{\mathcal{T}}} &= n_j\boldsymbol{\mathcal{T}}_{jk}\mathbf{e}_k = \mathbf{e}_k\boldsymbol{\mathcal{T}}_{kj}n_j \\
&= \mathbf{e}_k(\boldsymbol{\mathcal{T}}\cdot\mathbf{n})_k = \widehat{\boldsymbol{\mathcal{T}}}\cdot\mathbf{n}.
\end{aligned} \tag{64}$$

We shall write the divergence theorem for stress tensor in the following form

$$\boxed{\iiint_V \boldsymbol{\nabla}\cdot\widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r})\,d^3r = \iint_S \cdot\widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r})\cdot\mathbf{n}\,da.} \tag{65}$$

We shall find Eq. (65) to be crucial for constructing Maxwell's stress tensor in the following sections.

3.4 Volume force density in a stress tensor field

Fig. 4 shows an imaginary rectangular box *abcdefgh* of infinitesimal dimensions

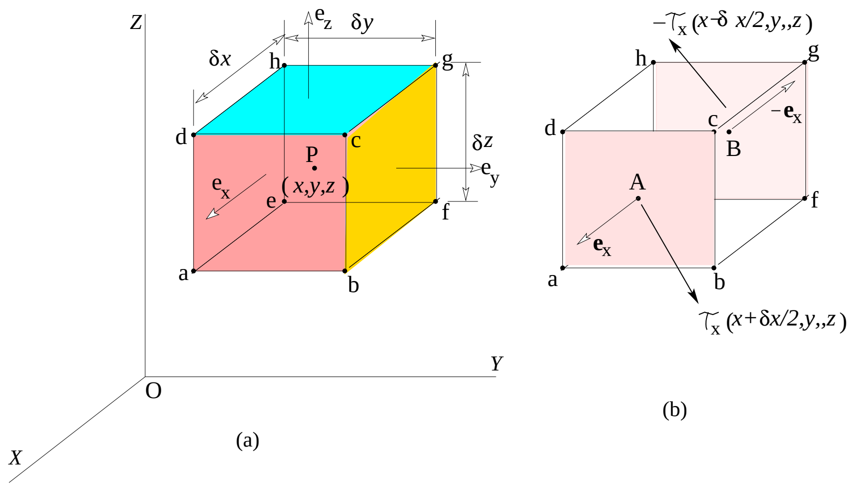

Figure 4: *Stress Force on a Volume Element*

$\delta x, \delta y, \delta z$ inside a medium under stress (which may be matter, or field). The centre P of this box is located at the coordinates (x, y, z). Let us assume that the stress in the medium is given by the tensor field $\widehat{\boldsymbol{\mathcal{T}}}(x, y, z)$, whose components are differentiable functions of the coordinates. We shall find the total force on this box due to this stress.

We have shown in Fig.(a) the outward normal vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ on the three faces of the box that are exposed to our view. The outward normals on the other faces which are hidden from our view are $(-\mathbf{e}_x, -\mathbf{e}_y, -\mathbf{e}_z)$. We shall identify each one of the six surfaces of the box by their outward normal vectors.

Let us consider the opposite faces *abcd* and *efgh*, recognized by the normals $(\mathbf{e}_x)$ and $(-\mathbf{e}_x)$. The locations of their centres are $(x + \frac{\delta x}{2}, y, z)$ and $(x - \frac{\delta x}{2}, y, z)$ respectively. The stress forces on these two faces are

$$\begin{aligned}
&\delta\mathbf{F}_{+x} = \widehat{\boldsymbol{\mathcal{T}}}(x + \tfrac{\delta x}{2}, y, z) \cdot (+\mathbf{e}_x)\, \delta y\, \delta z \\
&= \boldsymbol{\mathcal{T}}^{(x)}(x + \tfrac{\delta x}{2}, y, z)\, \delta y\, \delta z \\
&= [\boldsymbol{\mathcal{T}}^{(x)}(x, y, z) + \tfrac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} \tfrac{\delta x}{2}]\, \delta y\, \delta z \\
&\delta\mathbf{F}_{-x} = \widehat{\boldsymbol{\mathcal{T}}}(x - \tfrac{\delta x}{2}, y, z) \cdot (-\mathbf{e}_x)\, \delta y\, \delta z \\
&= -\boldsymbol{\mathcal{T}}^{(x)}(x - \tfrac{\delta x}{2}, y, z)\, \delta y\, \delta z \\
&= -[\boldsymbol{\mathcal{T}}^{(x)}(x, y, z) - \tfrac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} \tfrac{\delta x}{2}]\, \delta y\, \delta z
\end{aligned}$$

$$\delta\mathbf{F}_{+x} + \delta\mathbf{F}_{-x} = \frac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} \delta x\, \delta y\, \delta z = \frac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} \delta V. \quad (66)$$

where $\delta V = \delta x\, \delta y\, \delta z$ is the volume of the infinitesimal box. In the same way we find the forces on the other four faces of the block. Adding the stress forces on all the six surfaces we get

$$\delta\mathbf{F}_s = \left[\frac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} + \frac{\partial \boldsymbol{\mathcal{T}}^{(y)}}{\partial y} + \frac{\partial \boldsymbol{\mathcal{T}}^{(z)}}{\partial z} \right] \delta V \quad (67)$$

as the total stress force on the box. The volume force density $\mathbf{f}_s$, which gives the stress

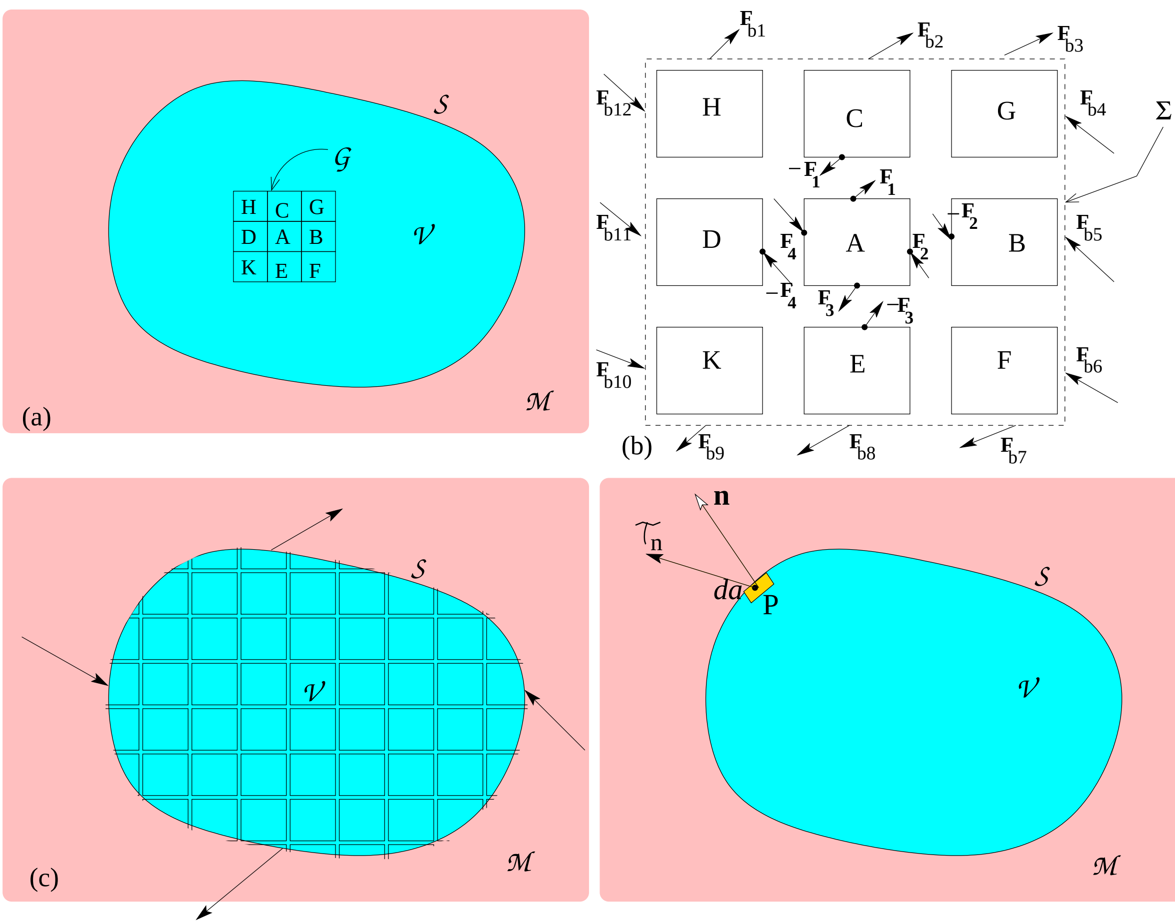

Figure 5: *Stress Forces on a Bulk Volume*

force acting per unit volume of the media under stress, is then given as

$$\begin{aligned}\mathbf{f}_s &= \frac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} + \frac{\partial \boldsymbol{\mathcal{T}}^{(y)}}{\partial y} + \frac{\partial \boldsymbol{\mathcal{T}}^{(z)}}{\partial z} \\ &= \frac{\partial(\mathbf{e}_x \cdot \widehat{\boldsymbol{\mathcal{T}}})}{\partial x} + \frac{\partial(\mathbf{e}_y \cdot \widehat{\boldsymbol{\mathcal{T}}})}{\partial y} + \frac{\partial(\mathbf{e}_z \cdot \widehat{\boldsymbol{\mathcal{T}}})}{\partial z} \\ &= \left[\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_x \frac{\partial}{\partial x}\right] \cdot \widehat{\boldsymbol{\mathcal{T}}} \\ &= \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}. \end{aligned} \tag{68}$$

One may conclude that total stress force $\mathbf{F}_s$ on a bulk volume $\mathcal{V}$ carved out inside a medium $\mathcal{M}$, as shown in Fig.5(a), is the volume integral of the force density $\mathbf{f}_s$ carried out over the entire volume $\mathcal{V}$. We shall carefully analyze the forces inside the medium before jumping into this conclusion.

Let us consider a two-diemnsional view of nine tiny, imaginary neighbouring blocks lying inside the medium and forming a group $\mathcal{G}$. We have marked the blocks as A,B,C,D,E,F,G,H,K, with A at the centre. In Fig.(b) we have shown the forces on the four sides of A as $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4$. The force $\mathbf{F}_1$ comes from the neighbour B, and by Newton's third law of motion, A applies an equal and opposite force $-\mathbf{F}_1$ on B. Similarly, the forces $\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4$ come from the neighbours C, D, E . And A applies equal and opposite forces $-\mathbf{F}_2, -\mathbf{F}_3, -\mathbf{F}_4$ on them. It *may* then appear that these internal forces, when added together, get cancelled out and there should not be any stress force on the group $\mathcal{G}$ at all.

A close examination will disprove this judgement. We have surrounded $\mathcal{G}$ by an imaginary boundary surface Σ. It is now seen that even though the action-reaction forces cancel out in the interior of the group $\mathcal{G}$, they survive on the boundary surface Σ. These surface forces $\mathbf{F}_{b1}, \mathbf{F}_{b2}, \cdots \mathbf{F}_{b12}$, when added together constitute the total force $\mathbf{F}_s$ on the group $\mathcal{G}$.

In Fig.(c) we have divided the volume $\mathcal{V}$ into an infinite number of infinitesimal blocks. The interior stress forces between adjoining blocks will cancel out. However, the forces on the bounadry surface, some of which we have shown as $\mathbf{F}_{b1}, \mathbf{F}_{b2}, \mathbf{F}_{b3}, \mathbf{F}_{b4}$, will survive and add together to constitute the net stress force $\mathbf{F}_s$ on the volume $\mathcal{V}$.

We now get a clue of how to find the net stress force $\mathbf{F}_s$ on the volume $\mathcal{V}$. In Fig.(d) we have shown the volume $\mathcal{V}$ once again. At a certain point P on this surface we have pictured a tiny patch of area da, on which we have drawn a unit outward normal $\mathbf{n}$. The stress force on this patch is $d\mathbf{f}_s = \boldsymbol{\mathcal{T}}^{(n)}\, da = \widehat{\boldsymbol{\mathcal{T}}} \cdot \mathbf{n}\, da$. Integrate this force over the entire boundary to get $\mathbf{F}_s$. We shall perform this integration and convert the surface integral into volume integral by applying Gauss's Divergence Theorem as derived in Eq. (63).

$$\mathbf{F}_s = \iint_S \widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r}) \cdot \mathbf{n}\, da = \iiint_V \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r})\, d^3 r. \tag{69}$$

We have thus confirmed our guess following Eq. (68). We shall rewrite the same equation with emphasis, as this equation will serve as the cornerstone for the constructon of Maxwell's Stress tensor.

$$\boxed{\mathbf{f}_s(\mathbf{r}) = \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r}).} \tag{70}$$

4 Maxwell's Stress Tensor for the Electrostatic Field

4.1 Volume force density in terms of the field

We shall now construct the stress tensor for the electrostatic filed. We shall call this tensor *Maxwell's Stress Tensor* and represent it by the symbol $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}$, where the superscript $^{(\mathrm{E})}$ implies Electric field.

Fig. 6 shows a system of electric charges $\mathcal{S}$ placed in an Electric field $\mathbf{E}(\mathbf{r})$. In Fig.(a) the system consists of discrete charges $q_1, q_2, q_3, \cdots$ placed at the radius vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots$. In Fig.(b) the system is a continuous distribution characterized by a smooth charge density function $\rho(\mathbf{r})$ confined within a volume. Our intention is to write the total electric force $\mathbf{F}$ on this system.

The force on the discrete system shown in Fig.(a) is given as

$$\mathbf{F} = \sum_j q_j \mathbf{E}^{(\mathrm{ext})}(\mathbf{r}_j). \tag{71}$$

Here the sum is over all the charges in the system, and $\mathbf{E}^{(\mathrm{ext})}(\mathbf{r}_j)$ is the *external electric field* at the radius vector $\mathbf{r}_j$ caused by the presence of all *other* charges lying outside the system $\mathcal{S}$.

For the case of continuous distribution, shown in Fig.(b), the individual charges become infinitesimal elementary charges i.e., $q_j \to \rho(\mathbf{r}) d^3r$, and the sum becomes the integral

$$\mathbf{F} = \iiint\limits_V \rho(\mathbf{r}) \mathbf{E}^{(\mathrm{ext})}(\mathbf{r})\, d^3r. \tag{72}$$

What about the force from the charges inside the system $\mathcal{S}$. They are *internal* forces, and cancel due to Newton's third law of motion.

Let $\mathbf{E}_i^{(\mathrm{int})}(\mathbf{r}_j)$ be the "internal" field caused at $\mathbf{r}_j$ by a member particle i lying within the sytem $\mathcal{S}$. Then $\mathbf{F}_{ij} = q_j \mathbf{E}_i^{(\mathrm{int})}(\mathbf{r}_j)$ is the force that the member particle i exerts on the member particle j. By Newton's third law of motion, $q_j \mathbf{E}_i^{(\mathrm{int})}(\mathbf{r}_j) + q_i \mathbf{E}_j^{(\mathrm{int})}(\mathbf{r}_i) = \mathbf{0}$. Adding together over all pairs for the discrete distribution, and integrating over the entire distribution for the continuous distribution we get

$$\begin{aligned} &\text{For discrete: } \textstyle\sum_{j=1}^N q_j \sum_{i=1}^{\prime\, N} \mathbf{E}_i^{(\mathrm{int})}(\mathbf{r}_j) \\ &\quad = \textstyle\sum_{j=1}^N q_j \mathbf{E}^{(\mathrm{int})}(\mathbf{r}_j) = \mathbf{0}. \\ &\text{For continuous: } \iiint\limits_V \rho(\mathbf{r})\, \mathbf{E}^{(\mathrm{int})}(\mathbf{r}) = \mathbf{0}. \end{aligned} \tag{73}$$

In the first equation the sum symbol $\sum'$ means that while summing over i, the term $i = j$ (corresponding to the "self field" of the member j) is to be avoided. The "internal field" $\mathbf{E}^{(\mathrm{int})}(\mathbf{r}_j)$ is the field at the location of the member j caused by "all other members" in the system $\mathcal{S}$. In the second equation $\mathbf{E}^{(\mathrm{int})}(\mathbf{r})$ is the "internal field" at the radius vector $\mathbf{r}$, as sensed by a tiny volume element d^3r at this point.

We shall add the null contribution shown in the second line of Eq. (73) to the right side of Eq. (72) and write

$$\mathbf{F} = \iiint\limits_V \rho(\mathbf{r}) \mathbf{E}\,(\mathbf{r})\, d^3r. \tag{74}$$

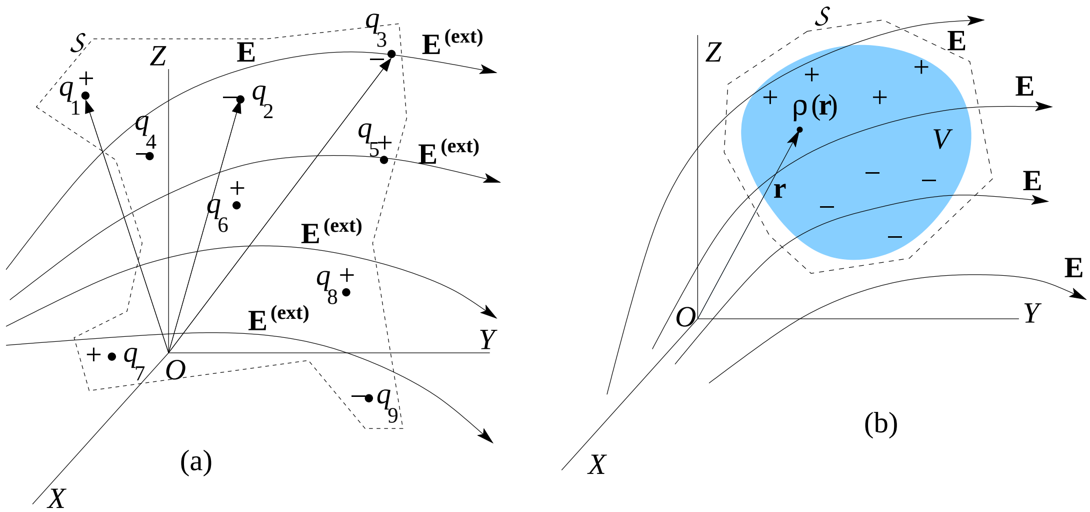

Figure 6: *Forces on Charges in an Electric Field*

Here $\mathbf{E}(\mathbf{r})$ is the actual field at the point $\mathbf{r}$, being the sum of two contributions, from the (i) external sources, and (ii) the internal sources of the system $\mathcal{S}$.

The purpose of adding the null integral of Eq. (73b) to Eq. (72) is that when we write the force density $\mathbf{f}$, the internal forces need to be added. That is,

$$\mathbf{f}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) \tag{75}$$

is the force on unit volume of the charge distribution at $\mathbf{r}$, in which $\mathbf{E}(\mathbf{r})$ is necessarily the *total* field at this location, caused by *both* external and internal sources. Now we manipulate the right hand side of Eq. (75) so as to convert $\rho\mathbf{E} \to \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}$, as suggested in Eq. (70). This new tensor field $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}(\mathbf{r})$ would represent "stress" in the electrostatic field.

Construction of the stress tensor for electrostatic field, magnetostatic field and time varying electromagnetic field will be facilitated by the following identity[14]

$$\boldsymbol{\nabla} \cdot \left[\mathbf{A}\mathbf{A} - \frac{1}{2}A^2\widehat{\mathbf{1}}\right] = (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{A} - \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}). \tag{76}$$

Before establishing the above identity we shall need a standard formula (See for example, *Vector Formulas* compiled in Grifffiths, 3rd Ed)

$$\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A}. \tag{77}$$

By setting $\mathbf{B} = \mathbf{A}$ in the above formula and get

$$\boldsymbol{\nabla}\left(\frac{1}{2}A^2\right) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{A}. \tag{78}$$

We shall now prove the identity (76).

Proof:

$$\begin{aligned}
&\boldsymbol{\nabla}\cdot(\mathbf{A}\mathbf{A}) = \left(\mathbf{e}_l\frac{\partial}{\partial x_l}\right)\cdot(\mathbf{e}_i\mathbf{e}_j A_i A_j)\\
&= \frac{\partial}{\partial x_i}(A_iA_j)\mathbf{e}_j\\
&= \left\{\left(\frac{\partial A_i}{\partial x_i}\right)A_j + \left(A_i\frac{\partial}{\partial x_i}\right)A_j\right\}\mathbf{e}_j\\
&= (\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{A} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}. \qquad (a)\\
&\boldsymbol{\nabla}\cdot\left(\frac{1}{2}A^2\widehat{\mathbf{1}}\right) = \left(\mathbf{e}_l\frac{\partial}{\partial x_l}\right)\cdot\left(\frac{1}{2}\mathbf{e}_i\mathbf{e}_iA^2\right)\\
&= \frac{1}{2}\mathbf{e}_i\frac{\partial A^2}{\partial x_i} = \boldsymbol{\nabla}\left(\frac{1}{2}A^2\right)\\
&= \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}, \text{ by (78).} \qquad (b)
\end{aligned}$$

The identity (76) follows when we subtract line (b) from line (a).

Q.E.D.

Note that we have used Einstein's summation convention introduced on page 8. That is, $\mathbf{e}_l\frac{\partial}{\partial x_l} \equiv \sum_{l=1}^{3}\mathbf{e}_l\frac{\partial}{\partial x_l}$; $\mathbf{e}_i\mathbf{e}_jA_iA_j \equiv \sum_{i=1}^{3}\sum_{j=1}^{3}\mathbf{e}_i\mathbf{e}_jA_iA_j$, etc.

The stress tensor for the electrostatic field follows when we set $\mathbf{E}$ for $\mathbf{A}$ in (76), and use the field equations: $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$; $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$:

$$\begin{aligned}
\mathbf{f}^{(\mathrm{E})} &= \rho\mathbf{E} = \boldsymbol{\nabla}\cdot\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}, && (a)\\
\text{where } \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \epsilon_0\left[\mathbf{E}\mathbf{E} - \frac{1}{2}E^2\widehat{\mathbf{1}}\right]. && (b)
\end{aligned} \tag{79}$$

It will be a simple exercise to write the Cartesian components of this tensor:

$$\begin{aligned}
\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \left(\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}\cdot\mathbf{e}_x \quad \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}\cdot\mathbf{e}_y \quad \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}\cdot\mathbf{e}_z\right)\\
&= \epsilon_0\begin{pmatrix}
\frac{1}{2}(E_x^2 - E_y^2 - E_z^2) & E_xE_y & E_xE_z\\
E_yE_x & \frac{1}{2}(E_y^2 - E_z^2 - E_x^2) & E_yE_z\\
E_zE_x & E_zE_y & \frac{1}{2}(E_z^2 - E_y^2 - E_z^2)
\end{pmatrix}.
\end{aligned} \tag{80}$$

4.2 Example: Stress vector on a plane as a function of the angle of inclination

The stress tensor (79) will remain abstract and obscure unless the reader works out a few examples. Griffiths has shown a beautiful example: the force on the upper half of a uniformly charged sphere using the stress tensor as given in formula (79b). However, he has worked in the Cartesian coordinate system. The reader should work out the same problem using the spherical coordinate system, spending much less time in getting the answer.

We shall provide two examples of which the first one is depicted in Fig. 7. A *uniform* electric field $\mathbf{E} = E\mathbf{e}_x$ exists in a certain region of space. The stress tensor is then given by the following expression.

$$\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} = \frac{\epsilon_0}{2}E^2(\mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y - \mathbf{e}_z\mathbf{e}_z) = \frac{\epsilon_0}{2}\begin{pmatrix} E^2 & 0 & 0 \\ 0 & -E^2 & 0 \\ 0 & 0 & -E^2 \end{pmatrix}. \quad (81)$$

Imagine a plane running parallel to the Z axis, but inclined to the X axis by an angle θ (Fig a). The normal vector is then given as

$$\mathbf{n} = \mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}. \quad (82)$$

The stress vector $\boldsymbol{\mathcal{T}}^{(n)}$ on this plane is then

$$\boldsymbol{\mathcal{T}}^{(n)} = \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} \cdot \mathbf{n} = \frac{\epsilon_0}{2}E^2(\mathbf{e}_x \sin\theta - \mathbf{e}_y \cos\theta) = \frac{\epsilon_0}{2}E^2 \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}. \quad (83)$$

Let us consider some special cases.

$$\begin{array}{rcll} \boldsymbol{\mathcal{T}}^{(x)} & = & \frac{\epsilon_0}{2}E^2\mathbf{e}_x, & (\text{by setting}\, \theta = \pi/2) \quad (a) \\ \boldsymbol{\mathcal{T}}^{(y)} & = & -\frac{\epsilon_0}{2}E^2\mathbf{e}_y, & (\text{by setting}\, \theta = 0) \quad (b) \\ \boldsymbol{\mathcal{T}}^{(z)} & = & -\frac{\epsilon_0}{2}E^2\mathbf{e}_z, & (\text{same as}\, \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} \cdot \mathbf{e}_z) \quad (c) \\ \boldsymbol{\mathcal{T}}^{(45^o)} & = & \frac{\epsilon_0}{2}E^2\frac{1}{\sqrt{2}}(\mathbf{e}_x - \mathbf{e}_y). & \quad (d) \end{array} \quad (84)$$

Lines (a) - (c) give the stress vectors on the planes identified by the normal vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, and line (d) gives the stress vector on a plane making an angle of 45^o with X axis. We have illustrated these points in Figs. (b) and (c). We have shown the stress vectors with thick arrows, and labelled them with the bold Greek letter $\boldsymbol{\mathcal{T}}$. We draw the following conclusion.

Conclusion:
(a) If the field is *perpendicular* to the plane, the stress vector is *normal* and *outward* (tensile stress), and equal to $\frac{\epsilon_0}{2}E^2$.
(b) If the field is *tangential* to the plane, the stress vector is *normal* and *inward* (compressive stress), and equal to $\frac{\epsilon_0}{2}E^2$.
(c) If the field makes angle 45^o to the plane, the stress vector is *tangential* (shear stress), and equal to $\frac{\epsilon_0}{2}E^2$.

Case (a) applies to a conductor in an electric field $\mathbf{E}$. The field is perpendicular to the surface. The surface force density is the same as the stress vector. We get back the same answer as in Eq. (1) using the stress tensor, without labouring to find out what is the "external field".

4.3 Example: Force transmitted between two charged particles across a spherical boundary

We shall first obtain an expression for the $\mathbf{E}$ field at any arbitrary point P (r, θ, ϕ) located on the spherical surface Σ. The point P is at the displacement vector $\boldsymbol{\eta}$ from A and $\mathbf{r}$ from O. In order to avoid repeated appearance of the constant $\frac{1}{4\pi\epsilon_0}$, we shall set $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\boldsymbol{\mathcal{E}}$. Note that

$$\begin{array}{rrcll} & \boldsymbol{\eta} & = & \mathbf{r} - \boldsymbol{a} = \mathbf{r} - a\mathbf{e}_z, & (a) \\ \text{so that} & \eta^2 & = & r^2 + a^2 - 2ra\cos\theta, & (b) \\ \text{and} & \mathbf{e}_z & = & \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta. & (c) \end{array} \quad (85)$$

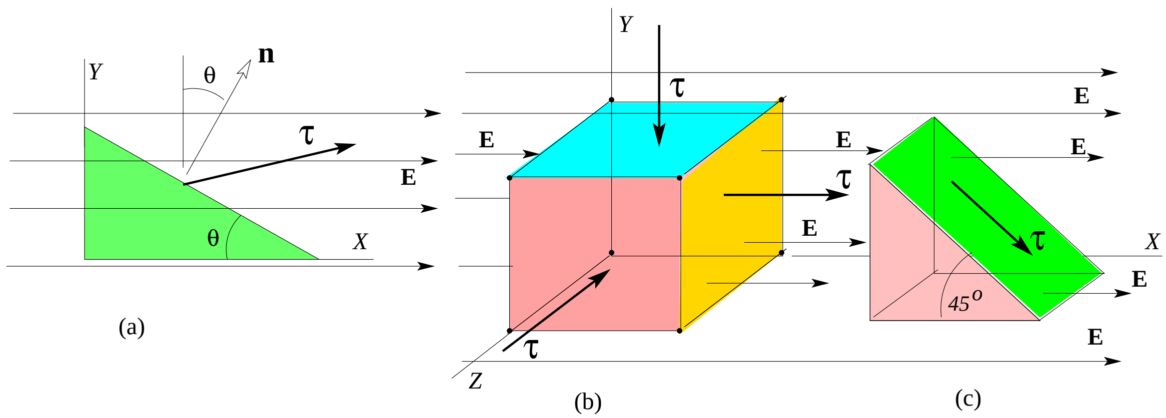

Figure 7: *Stress vector on an inclined plane placed in a uniform electric field.*

Then

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \frac{Q\mathbf{r}}{r^3} + \frac{q\boldsymbol{\eta}}{\eta^3}. & (a)\\ &= \frac{Q\mathbf{e}_r}{r^2} + \frac{q(\mathbf{r}-a\mathbf{e}_z)}{(r^2+a^2-2ra\,\cos\theta)^{3/2}}. & (b) \end{aligned} \quad (86)$$

Therefore,

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \mathcal{E}_r\mathbf{e}_r + \mathcal{E}_\theta\mathbf{e}_\theta, & (a)\\ \text{where}\quad \mathcal{E}_r &= \frac{Q}{r^2} + \frac{q(r-a\cos\theta)}{(r^2+a^2-2ra\,\cos\theta)^{3/2}}. & (b)\\ \mathcal{E}_\theta &= \frac{qa\sin\theta}{(r^2+a^2-2ra\,\cos\theta)^{3/2}}. & (c) \end{aligned} \quad (87)$$

From Eq. (79) the stress tensor is

$$\begin{aligned} \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \epsilon_0\left(\mathbf{EE} - \frac{1}{2}E^2\widehat{\mathbf{1}}\right)\\ &= \frac{1}{16\pi^2\epsilon_0}\left(\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}} - \frac{1}{2}\mathcal{E}^2\widehat{\mathbf{1}}\right) = \frac{1}{16\pi^2\epsilon_0}\tilde{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}\\ \text{where}\quad \tilde{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}} - \frac{1}{2}\mathcal{E}^2\widehat{\mathbf{1}}, \end{aligned} \quad (88)$$

which we may refer to as the "reduced stress tensor".

Since we have invoked the spherical coordinate system to write the expression for the $\boldsymbol{\mathcal{E}}$ field, the components of the tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}$ will have to be written in this coordinate system. Since only r and θ components of $\boldsymbol{\mathcal{E}}$ are non-zero, the non-zero components of this tensor are $\mathcal{T}_{rr}, \mathcal{T}_{r\theta}, \mathcal{T}_{\theta r}, \mathcal{T}_{\theta\theta}$, as seen from (88). Therefore $\mathcal{E}^2 = \mathcal{E}_r^2 + \mathcal{E}_\theta^2$, and we write this tensor as

$$\begin{aligned} \tilde{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \begin{pmatrix} \mathcal{T}_{rr} & \mathcal{T}_{r\theta} & 0\\ \mathcal{T}_{\theta r} & \mathcal{T}_{\theta\theta} & 0\\ 0 & 0 & 0 \end{pmatrix}, \text{ where}\\ \mathcal{T}_{rr} &= \mathcal{E}_r^2 - \tfrac{1}{2}\mathcal{E}^2 = \tfrac{1}{2}(\mathcal{E}_r^2 - \mathcal{E}_\theta^2).\\ \mathcal{T}_{r\theta} &= \mathcal{T}_{\theta r} = \mathcal{E}_r\mathcal{E}_\theta.\\ \mathcal{T}_{\theta\theta} &= \mathcal{E}_\theta^2 - \tfrac{1}{2}\mathcal{E}^2 = \tfrac{1}{2}(\mathcal{E}_\theta^2 - \mathcal{E}_r^2). \end{aligned} \quad (89)$$

The first column in the square matrix on the left represents the stress vector $\boldsymbol{\mathcal{T}}_r$ on the spherical surface Σ (corresponding to $\mathbf{n} = \mathbf{e}_r$, analogous to the first column in Eq. 50). Using the expressions for $\mathcal{E}_r, \mathcal{E}_\theta$ given in (87) we shall work out the components of $\boldsymbol{\mathcal{T}}_r$ explic-

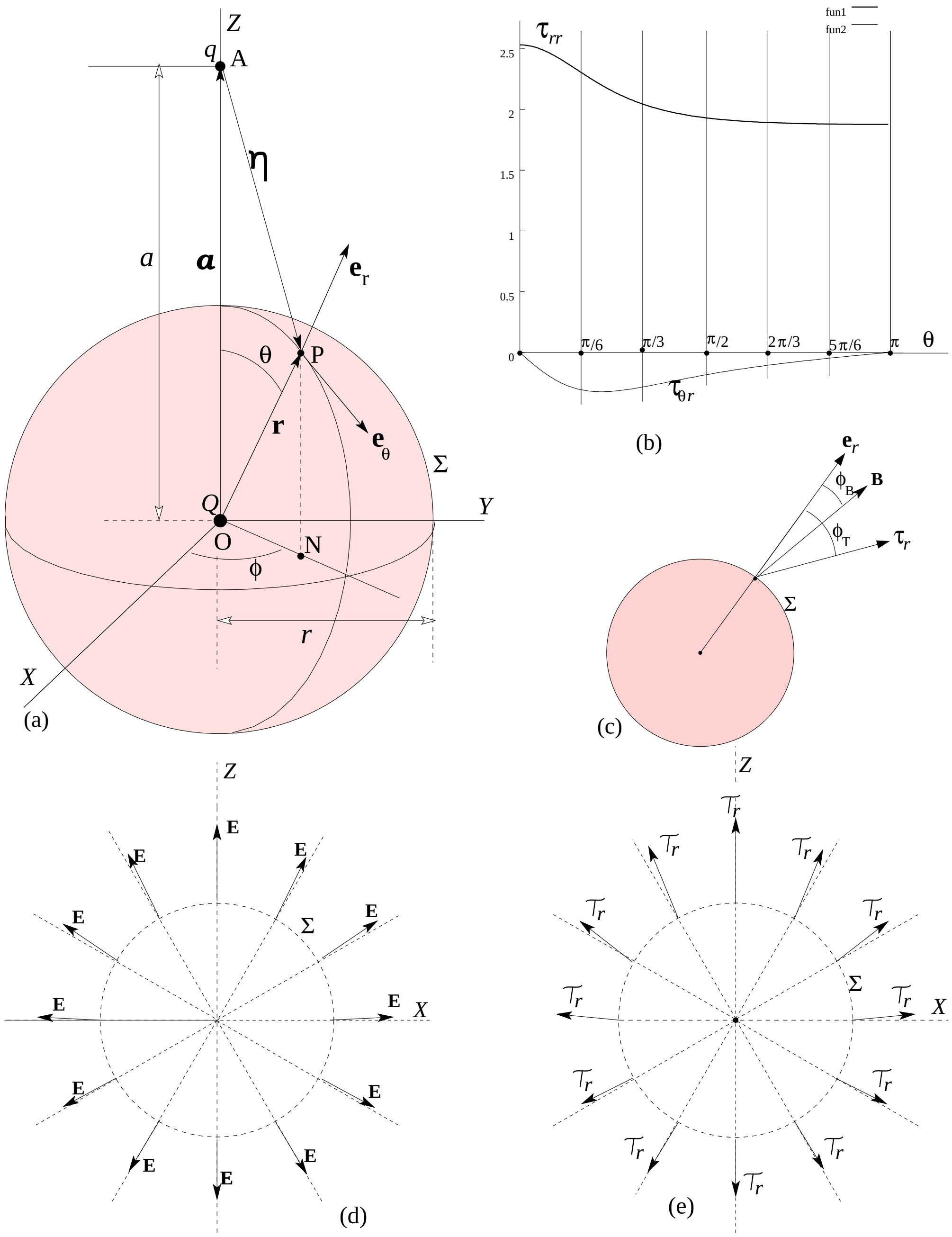

Figure 8: *Stress on a spherical surface.*

itly as follows

$$\begin{aligned}&\boldsymbol{\mathcal{T}}_r = \mathbf{e}_r\mathcal{T}_{rr} + \mathbf{e}_\theta\mathcal{T}_{\theta r}\\&\mathcal{T}_{rr} = \tfrac{1}{2}(\mathcal{E}_r^2 - \mathcal{E}_\theta^2) = \tfrac{1}{2}\left[\tfrac{Q^2}{r^4} + \right.\\&\left. + \tfrac{q^2[(r-a\cos\theta)^2-(a\sin\theta)^2]}{(r^2+a^2-2ra\,\cos\theta)^3} + \tfrac{2Qq(r-\cos\theta)}{r^2(r^2+a^2-2ra\,\cos\theta)^{3/2}}\right].\\&\mathcal{T}_{\theta r} = \mathcal{E}_r\mathcal{E}_\theta = \tfrac{Qqa\sin\theta}{r^2(r^2+a^2-2ra\,\cos\theta)^{3/2}}\\&+\tfrac{q^2a\sin\theta(r-a\cos\theta)}{(r^2+a^2-2ra\,\cos\theta)^3}.\end{aligned} \tag{90}$$

The first component $\mathcal{T}_{rr}$ is the normal stress on the surface Σ and the second one $\mathcal{T}_{\theta r}$ the tangential (or, the shear) stress.

In order to illustrate the above equations, and to see how the electric field vector **E** and the Maxwell's stress vector $\boldsymbol{\mathcal{E}}$ vary on the surface of the imaginary sphere Σ, we shall make a numerical example, setting $Q = 2, q = -1, a = 3, r = 1$ in Eqs. (87) and (90). The expressions we now get are functions of the polar angle θ only. We have plotted $\mathcal{T}_{rr}, \mathcal{T}_{\theta r}$ in Fig. 8(b), using Maxima.

In order to show how the field vector $\boldsymbol{\mathcal{E}}$ and the stress vector $\boldsymbol{\mathcal{T}}_r$ vary on the surface of the sphere Σ we have prepared the Table 3.1 after evaluating the corresponding quantities in the columns 1-9, using Maxima. The angles $\phi_{\text{E}}, \phi_{\text{T}}$ appearing in columns 5 and 9 have been explained in Fig. 8(c). The first one is the angle between the normal $\mathbf{e}_r$ to the surface Σ and the electric field $\boldsymbol{\mathcal{E}}$ at the surface, and the second one is the angle between $\mathbf{e}_r$ and the stress vector $\boldsymbol{\mathcal{T}}_r$ on the surface.

$$\mathcal{E} = \sqrt{\mathcal{E}_r^2 + \mathcal{E}_\theta^2}; \quad \tan\phi_{\text{E}} = \frac{\mathcal{E}_\theta}{\mathcal{E}_r};$$
$$\mathcal{T}_r = \sqrt{\mathcal{T}_{rr}^2 + \mathcal{T}_{\theta r}^2}; \quad \tan\phi_{\text{T}} = \frac{\mathcal{T}_{\theta r}}{\mathcal{T}_{rr}}. \tag{91}$$

We have drawn the field vectors **E** and the stress vectors $\boldsymbol{\mathcal{T}}_r$ on the sphere Σ in Fig. 8(d) and (e) (using two diffferent scales for the two sets of vectors.)

TABLE 3.1: $\boldsymbol{\mathcal{E}}$ AND $\boldsymbol{\mathcal{T}}_r$ VECTORS ON THE SURFACE OF THE SPHERE

1	2	3	4	5	6	7	8	9
θ	$\mathcal{E}_r$	$\mathcal{E}_\theta$	$\mathcal{E}$	ϕ_{E}	$\mathcal{T}_{rr}$	$\mathcal{T}_{\theta r}$	$\mathcal{T}_r$	ϕ_{T}
0^o	2.25	0	2.25	0^o	2.53	0	2.53	0^o
30^o	2.15	-0.14	2.16	-3.8^o	2.30	-0.31	2.33	-7.6^o
60^o	2.03	-0.14	2.03	-4^o	2.04	-0.28	2.06	-7.9^o
90^o	1.97	-0.10	1.97	-2.8^o	1.93	-0.19	1.94	-5.5^o
120^o	1.95	-0.05	1.95	-1.6^o	1.89	-0.11	1.90	-3.3^o
150^o	1.94	-0.02	1.94	-0.8^o	1.88	-0.05	1.88	-1.5^o
180^o	1.94	0	1.94	0^o	1.88	0	1.88	0^o

(92)

All this tediuos work will have been fruitful if we could show that the surface force density, when integrated over the entire surface Σ, will give us back the familiar Coulomb force between the two charges. The surface force density is the same as the stress vector on this surface. We shall work with the "reduced" surface force density, same as $\boldsymbol{\mathcal{T}}_r$.

The Coulomb force of attraction (if Q, q are of opposite signs) or repulsion (if they are of the same sign) will be along the line OA joining the two charges. Since this line coincides with the Z axis, we shall integrate the Z compomnent of $\boldsymbol{\mathcal{T}}_r$, which we shall denote as $\tilde{f}_z$. We go back to Eqs. (98) and (90) to compute this force, and get the following results after some simplification.

$$\begin{aligned}
&\tilde{f}_z = \mathbf{e}_z \cdot \boldsymbol{\mathcal{T}}_r \\
&= (\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta)\cdot(\mathbf{e}_r\mathcal{T}_{rr} + \mathbf{e}_\theta\mathcal{T}_{\theta r}) && (a)\\
&= \cos\theta\,\mathcal{T}_{rr} - \sin\theta\,\mathcal{T}_{\theta r} && (b)\\
&= \tilde{f}_z(Q^2) + \tilde{f}_z(Qq) + \tilde{f}_z(q^2),\ \text{where} && (c)\\
&\tilde{f}_z(Q^2) = \tfrac{1}{2}\tfrac{Q^2}{r^4}\cos\theta. && (d)\\
&\tilde{f}_z(Qq) = \tfrac{Qq[r\,\cos\theta - a]}{r^2(r^2+a^2-2ra\,\cos\theta)^{3/2}}. && (e)\\
&\tilde{f}_z(q^2) = \tfrac{1}{2}\tfrac{q^2[(r^2+a^2)\cos\theta - 2ra]}{(r^2+a^2-2ra\,\cos\theta)^3}. && (f)
\end{aligned} \quad (93)$$

The expressions in lines (d) and (f), involving Q^2 and q^2, are "self terms", whereas the expression in line (e) involving Qq is the "interaction term" The reader should complete the steps leading from line (b) to these lines. We shall soon show that the self terms will vanish upon integration, leaving the integrated stress force entirely a function of Qq.

The "reduced" force transmitted across the surface Σ, and hence acting on the charge Q, is the surface integral of $\tilde{f}_z$. Let us denote this integral as $\tilde{F}$. An area element on Σ is $da = r^2\,\sin\theta\,d\theta\,d\phi$. Therefore,

$$\begin{aligned}
&\tilde{F} = \iint_\Sigma \tilde{f}_z\, r^2 \sin\theta\, d\theta\, d\phi \\
&= 2\pi r^2 \textstyle\int_0^\pi \tilde{f}_z\,\sin\theta\,d\theta && (a)\\
&= 2\pi r^2[\mathcal{I}(Q^2) + \mathcal{I}(Qq) + \mathcal{I}(q^2)], && (b)\\
&\text{where}\ \mathcal{I}(Q^2) = \textstyle\int_0^\pi \tilde{f}_z(Q^2)\,\sin\theta\,d\theta = 0. && (c)\\
&\mathcal{I}(Qq) = \textstyle\int_0^\pi \tilde{f}_z(Qq)\,\sin\theta\,d\theta = -\frac{2Qq}{a^2r^2}. && (d)\\
&\mathcal{I}(q^2) = \textstyle\int_0^\pi \tilde{f}_z(q^2)\,\sin\theta\,d\theta = 0. && (e)\\
&\text{Hence,}\quad \tilde{F} = -\tfrac{4\pi Qq}{a^2}. && (f)
\end{aligned} \quad (94)$$

The integral given in line (c) is easy to evaluate. The other integrals have been worked out in the Appendix. They can be worked out more easily using Maxima with a computer.

To get the true force we go back to (88), multiply $\tilde{F}$ with the factor $\frac{1}{16\pi^2\epsilon_0}$, and get the force $\mathbf{F}_Q$ acting on the charge Q.

$$\mathbf{F}_Q = \frac{1}{16\pi^2\epsilon_0}\tilde{F}\mathbf{e}_z = -\frac{Qq}{4\pi\epsilon_0 a^2}\mathbf{e}_z. \qquad (95)$$

This force is the familiar Coulomb force on the charge Q located at the origin, exerted on it by another chareg q located at a distance a on the positive Z axis. It is repulsive, i.e., towards the negative Z axis, if Qq is positive, and attractive i.e., towards the positive Z axis, if Qq is negative.

5 Maxwell's Stress tensor for the Magnetostatic Field

This section is the magnetostatic analogue of the electrostatic stress tensor presented in Sec. 4.3. The steps are parallel, so that we shall avoid detailed explanation.

5.1 Volume force density in terms of the field

We shall construct Maxwell's stress tensor for the magnetostatic field, represent it by the symbol $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$. The volume force density in a magnetic field is $\mathbf{f}^{(\mathrm{M})} = \mathbf{J} \times \mathbf{B}$. Therefore we need to construct the tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$ under the specification

$$\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})} \equiv \mathbf{f}^{(\mathrm{M})} = \mathbf{J} \times \mathbf{B}. \qquad (96)$$

This is now an easy task, thanks to the identity (76) we had established in Sec. 4. We set $\mathbf{B}$ for $\mathbf{A}$ in that equation, and use the field equations: $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$; $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$, leading to:

$$\boxed{\begin{array}{rcll} \mathbf{f}^{(\mathrm{M})} & = & \mathbf{J} \times \mathbf{B} = \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}, & (a) \\ \text{where } \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})} & = & \frac{1}{\mu_0} \left[\mathbf{BB} - \frac{1}{2} B^2 \widehat{\mathbf{1}} \right]. & (b) \end{array}} \qquad (97)$$

Note the similarity between the stress tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$ written above and the stress tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}$ written in Eq. (79) on page 24. The former converts into the latter if we replace $\mathbf{E}$ with $\mathbf{B}$ and ϵ_0 with $\frac{1}{\mu_0}$. In the same way the matrix form given in Eq. (80) converts to the matrix form of $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$. Consequently, the stress vector changes from normal outward, to tangential, to normal inward, as the angle between the plane and the direction of the $\mathbf{B}$ field changes from 90^o to 45^o to 0^o, as shown in Eqs. (84) and illustrated in Fig 7. and the "Conclusion" written on page 25 carries over to the case of a magnetic field without any change. Each point in the conclusion is well illustrated in Fig. 9 (see next section) if the reader compares the direction of the field vector $\mathbf{B}$ in Fig(d) with the direction of stress vector $\boldsymbol{\mathcal{T}}_r$ in Fig(e).

5.2 Example: Force transmitted between two magnetic diploles across a spherical boundary

The smallest denomination of the source of a magnetic field is a magnetic dipole, consisting of a tiny current loop. We shall therefore think of the force between two magnetic dipoles. We have placed these dipoles along the Z axis, oriented them in the positive direction of this axis. Fig. 9(a) shows the geometry of this configuration. The dipoles are shown by tiny spherical blobs with an arrow pointing in the direction of this vector. As in the electrostatic example, we shall illustrate Maxwell's stress tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$ by finding the stress vector on the surface of an imaginary sphere Σ of radius r surrounding the point magnetic dipole $\mathbf{M}$ which is placed at a distance a from the other point magnetic dipole $\mathbf{m}$ such that $r < a$, and then integrate this stress vector over the spherical surface to obtain the force $\mathbf{F}_{\text{M}}$ on $\mathbf{M}$ exerted by $\mathbf{m}$.

We shall first obtain the $\mathbf{B}$ field at any arbitrary point P (r, θ, ϕ) located on the spherical surface Σ, at the displacement vector $\boldsymbol{\eta}$ from A and $\mathbf{r}$ fromO. In order to avoid repeated appearance of the constant $\frac{\mu_0}{4\pi}$, we shall set $\mathbf{B} = \frac{\mu_0}{4\pi}\boldsymbol{\mathcal{B}}$. Note that

$$\begin{array}{rrcll} & \boldsymbol{\eta} & = & \mathbf{r} - \boldsymbol{a} = \mathbf{r} - a\mathbf{e}_z, & (a) \\ \text{so that} & \eta^2 & = & r^2 + a^2 - 2ra\cos\theta, & (b) \\ \text{and} & \mathbf{e}_z & = & \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta. & (c) \end{array} \tag{98}$$

Let $\boldsymbol{\mathcal{B}}^{(M)}(r, \theta, \phi)$, $\boldsymbol{\mathcal{B}}^{(m)}(r, \theta, \phi)$ be the fields[15] produced by the dipoles $\mathbf{M}$ and $\mathbf{m}$ respectively, at any coordinate point (r, θ, ϕ). Adding them we get the total field $\boldsymbol{\mathcal{B}}(r, \theta, \phi)$.

$$\begin{array}{l} \boldsymbol{\mathcal{B}}(r,\theta,\phi) = \boldsymbol{\mathcal{B}}^{(M)}(r,\theta,\phi) + \boldsymbol{\mathcal{B}}^{(m)}(r,\theta,\phi). \\ \boldsymbol{\mathcal{B}}^{(M)}(r,\theta,\phi) = \frac{3\,(\mathbf{M}\cdot\mathbf{r})\,\mathbf{r} - \mathbf{M}r^2}{r^5} \\ = \mathcal{B}_r^{(M)}\mathbf{e}_r + \mathcal{B}_\theta^{(M)}\mathbf{e}_\theta, \text{ where,} \\ \mathcal{B}_r^{(M)} = \frac{2M\cos\theta}{r^3}; \quad \mathcal{B}_\theta^{(M)} = \frac{M\sin\theta}{r^3}. \\ \boldsymbol{\mathcal{B}}^{(m)}(r,\theta,\phi) = \frac{3\,(\mathbf{m}\cdot\boldsymbol{\eta})\,\boldsymbol{\eta} - \mathbf{m}\eta^2}{\eta^5} \\ = \mathcal{B}_r^{(m)}\mathbf{e}_r + \mathcal{B}_\theta^{(m)}\mathbf{e}_\theta, \text{ where,} \\ \mathcal{B}_r^{(m)} = \frac{m[2(r^2+a^2)\cos\theta - (3+\cos^2\theta)ar]}{\eta^5}, \\ \mathcal{B}_\theta^{(m)} = \frac{m(r^2-2a^2+ar\cos\theta)\sin\theta}{\eta^5}. \end{array} \tag{99}$$

For future convenience we write

$$\begin{array}{l} \boldsymbol{\mathcal{B}} = \mathcal{B}_r\mathbf{e}_r + \mathcal{B}_\theta\mathbf{e}_\theta, \quad \text{where,} \\ \mathcal{B}_r = \left(\frac{M}{r^3}\right)\alpha + \left(\frac{m}{\eta^5}\right)\beta. \quad \alpha = 2\cos\theta. \\ \mathcal{B}_\theta = \left(\frac{M}{r^3}\right)\gamma + \left(\frac{m}{\eta^5}\right)\delta. \quad \gamma = \sin\theta. \\ \beta = 2(r^2+a^2)\cos\theta - (3+\cos^2\theta)ar. \\ \delta = (r^2 - 2a^2 + ar\cos\theta)\sin\theta. \end{array} \tag{100}$$

From Eq. (97) the stress tensor is

$$\begin{array}{l} \widehat{\boldsymbol{\mathcal{T}}}^{(\text{M})} = \frac{1}{\mu_0}\left(\mathbf{BB} - \frac{1}{2}E^2\widehat{\mathbf{1}}\right) \\ = \frac{\mu_0}{16\pi^2}\left(\boldsymbol{\mathcal{BB}} - \frac{1}{2}\mathcal{B}^2\widehat{\mathbf{1}}\right) = \frac{\mu_0}{16\pi^2}\tilde{\boldsymbol{\mathcal{T}}}^{(\text{M})} \\ \text{where} \quad \tilde{\boldsymbol{\mathcal{T}}}^{(\text{M})} = \boldsymbol{\mathcal{BB}} - \frac{1}{2}\mathcal{B}^2\widehat{\mathbf{1}}. \end{array} \tag{101}$$

which we may refer to as the "reduced stress tensor". The non-zero components of this tensor needed by us are

$$\begin{array}{l} \mathcal{T}_{rr} = \mathcal{B}_r^2 - \frac{1}{2}\mathcal{B}^2 = \frac{1}{2}(\mathcal{B}_r^2 - \mathcal{B}_\theta^2) \\ \mathcal{T}_{r\theta} = \mathcal{T}_{\theta r} = \mathcal{B}_r\mathcal{B}_\theta. \end{array} \tag{102}$$

In order to illustrate the above equations, and to see how the magnetic field vector $\mathbf{B}$ and the Maxwell's stress vector $\widehat{\boldsymbol{\mathcal{T}}}^{(\text{M})}$ look like on the surface of the imaginary sphere surrounding the charge Q, we shall make a numerical example, setting $M = 2, m = 1, a = 3, r = 1$ in Eqs. (100) and (102). For this purpose we have prepared the following table, after evaluating the corresponding quantities in the columns 1-9 using Maxima. The angles $\phi_{\text{B}}, \phi_{\text{T}}$ appearing in this table have been explained in Fig. 9(c). See also Eq. (90).

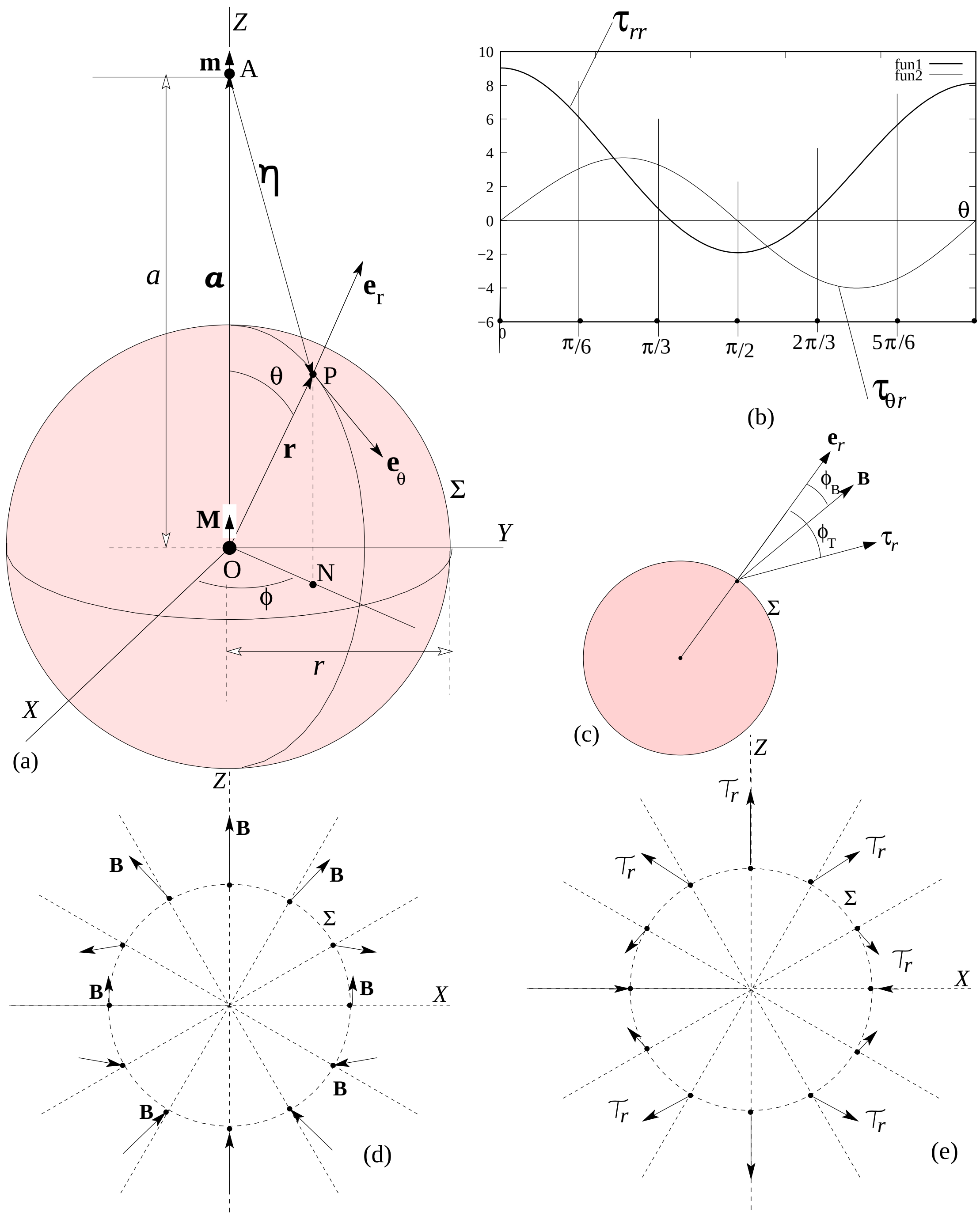

Figure 9: *Stress vector on a spherical surface.*

TABLE : $\boldsymbol{\mathcal{B}}$ AND $\boldsymbol{\mathcal{T}}_r$ VECTORS ON THE SURFACE OF THE SPHERE

1	2	3	4	5	6	7	8	9
θ	$\mathcal{B}_r$	$\mathcal{B}_\theta$	$\mathcal{B}$	ϕ_B	$\mathcal{T}_{rr}$	$\mathcal{T}_{\theta r}$	$\mathcal{T}$	ϕ_T
0^o	4.25	0	4.25	0^o	9.03	0	9.03	0^o
30^o	3.58	0.86	3.69	13.5^o	6.05	3.09	6.80	26.9^o
60^o	2.00	1.64	2.59	39.3^o	0.66	3.28	3.34	78.5^o
90^o	-0.03	1.96	1.96	-89.4^o	-1.91	-0.06	1.91	1.7^o
120^o	-2.03	1.71	2.66	-40.1^o	0.60	-3.48	3.53	-76.8^o
150^o	-3.5	0.99	3.63	-16.0^o	5.62	-3.47	6.60	-31.5^o
180^o	-4.03	0	4.03	0^o	8.13	0	8.13	0^o

(103)

We have plotted $\mathcal{T}_{rr}, \mathcal{T}_{\theta r}$ as functions of the polar angle θ in Fig. 9(b), using Maxima, and have drawn the vectors $\mathbf{B}$ and $\boldsymbol{\mathcal{T}}_r$ on the sphere Σ in Fig. 9(d) and (e) (using two different scales for the two sets of vectors.)

All this tediuos work will have been fruitful if we could show that the surface force density, when integrated over the entire surface Σ, will yield the same force between the two dipoles that we can calculate using the standard formulas of megnetostatics. Let us then first apply the "standard formula"

The force $\mathbf{F}_M$ on $\mathbf{m}$ is given by the formula $\mathbf{F} = (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}$, in which $\mathbf{B}$ is the field created by $\mathbf{M}$. The $\mathbf{m}$ vector is in the Z direction. Therefore, $\mathbf{m}\cdot\boldsymbol{\nabla} = m\frac{\partial}{\partial z}$, which means that we can treat the (x, y) coordinates as constant and equal to zero. Therefore,

$$\begin{aligned}&\mathbf{F}_m = m\tfrac{\partial \mathbf{B}}{\partial z}\big|_{x=y=0,z=a},\\ &\text{where, } \mathbf{B}(0,0,z) = \tfrac{\mu_0 M}{4\pi}\left[\tfrac{3z^2-z^2}{z^5}\right]\mathbf{e}_z.\\ &\tfrac{\partial \mathbf{B}}{\partial z}\big|_{x=y=0,z=a} = -\tfrac{3\mu_0 M}{2\pi}\tfrac{1}{a^4}\mathbf{k}.\\ &\text{Hence, } \mathbf{F}_m = -\tfrac{3\mu_0 mM}{2\pi a^4}\mathbf{e}_z.\end{aligned} \tag{104}$$

By Newton's third law of motion,

$$\mathbf{F}_M = -\mathbf{F}_m = \frac{3\mu_0 Mm}{2\pi a^4}\mathbf{e}_z. \tag{105}$$

Now we shall calculate the same force using the stress tensor. The surface force density is the same as the stress vector on this surface. We shall work with the "reduced" surface force density, same as $\boldsymbol{\mathcal{T}}_r$.

The force of attraction between the dipoles will be along the line OA joining them, which lies on the Z axis. Therefore we need the Z component of the surface force density $\tilde{f}_z$:

$$\begin{aligned}\tilde{f}_z &= \mathbf{e}_z \cdot \boldsymbol{\mathcal{T}}_r\\ &= (\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta)\cdot(\mathbf{e}_r\mathcal{T}_{rr} + \mathbf{e}_\theta\mathcal{T}_{\theta r})\\ &= \cos\theta\,\mathcal{T}_{rr} - \sin\theta\,\mathcal{T}_{\theta r}.\end{aligned} \tag{106}$$

We shall break up this force density into three components: (1) $\tilde{f}_z(M^2)$ representing self term for $\mathbf{M}$, (2) $\tilde{f}_z(Mm)$ representing interaction term between $\mathbf{M}$ and $\mathbf{m}$, (3) $\tilde{f}_z(m^2)$ representing self term for $\mathbf{m}$. From Eqs. (100), (102) and (106):

$$\tilde{f}_z(M^2) = \left[\left(\alpha^2 - \gamma^2\right)\cos\theta - 2\alpha\gamma\sin\theta\right]\frac{M^2}{2r^6}.$$
$$\tilde{f}_z(Mm) = [(\alpha\beta - \gamma\delta)\cos\theta \\ -(\alpha\delta + \beta\gamma)\sin\theta]\tfrac{Mm}{r^3\eta^5}.$$
$$\tilde{f}_z(m^2) = \left[\left(\beta^2 - \delta^2\right)\cos\theta - 2\beta\delta\sin\theta\right]\frac{m^2}{2\eta^{10}}. \tag{107}$$

The "reduced" force $\tilde{F}$ transmitted across the surface Σ, and hence acting on the dipole $\mathbf{M}$, is the surface integral of $\tilde{f}_z$, which is the sum of the integrals of $\tilde{f}_z(M^2)$, $\tilde{f}_z(Mm)$, and $\tilde{f}_z(m^2)$. Each integral is difficult to evaluate, because $\alpha, \beta, \gamma, \delta$ are complicated functions of r, a, θ. We have evaluated these integrals using Maxima. The result is as follows.

$$\tilde{F} = \iint_\Sigma \tilde{f}_z\, r^2 \sin\theta\, d\theta\, d\phi = 2\pi r^2 \int_0^\pi \tilde{f}_z\, \sin\theta\, d\theta$$
$$= 2\pi r^2[\mathcal{I}(M^2) + \mathcal{I}(Mm) + \mathcal{I}(m^2)], \text{ where}$$
$$\mathcal{I}(M^2) = \int_0^\pi \tilde{f}_z(M^2)\, \sin\theta\, d\theta = 0.$$
$$\mathcal{I}(Mm) = \int_0^\pi \tilde{f}_z(Mm)\, \sin\theta\, d\theta = \tfrac{12Mm}{a^4 r^2}.$$
$$\mathcal{I}(m^2) = \int_0^\pi \tilde{f}_z(m^2)\, \sin\theta\, d\theta = 0.$$
$$\text{Hence,} \quad \tilde{F} = \tfrac{24\pi Mm}{a^4}. \tag{108}$$

Because of the relation (101) the true force $\mathbf{F}_{\rm M}$ acting on the dipole $\mathbf{M}$ is $\frac{\mu_0}{16\pi^2}$ times the force $\tilde{F}$. Hence

$$\mathbf{F}_{\rm M} = \frac{3\mu_0 Mm}{2\pi a^4}\mathbf{e}_z. \tag{109}$$

We have thus verified that the stress tensor has given us the same force that we obtained in Eq. (105) using standard formulas of magnetostatics.

We have worked out three examples to bring out the meaning of Maxwell's stress tensor for electric and magnetic fields. The reader may wonder why we should go through such a tortuous road to get answers that can be easily obtained using simpler formulas of electrostatics and magnetostatics? Isn't it like demolishing a mud wall with a cannon?

Every cannon needs a mud wall to ensure its trust-worthiness before deployment in a true situation. Maxwell's stress tensor is destined to play a bigger role, in constructing the conservation equation for field momentum, and later under the watchful eye of Special Relativity, in building up the covariant expression for conservation of energy and momentum. The three examples we have worked out were intended to be an intellectual exercise to instil confidence in the mathematical expressions of $\widehat{\boldsymbol{\mathcal{T}}}^{(\rm E)}$ and $\widehat{\boldsymbol{\mathcal{T}}}^{(\rm M)}$ before crowning them for their majestic role.

6 Maxwell's Stress Tensor and Momentum Conservation

We had introduced Maxwell's stress tensor for static electric and static magnetic fields, with suitable applications, in Sections 4 and 5. These applications demonstrated that the force acting on static distributions of electric charges and currents lying within a bounded volume $\mathcal{V}$ is equal to the stress vector integrated over the surface $\mathcal{S}$ bounding this volume. The attribute "static" implied that the objects considered in our discussion. e.g., isolated charges and isolated current carrying loops, were fixed with a kind of "glue" making them immobile inspite of the electric and magnetic forces acting on them. We shall now remove that glue and see what role can now be played by the same stress tensors.

At this point we shall make a subtle distinction between force and stress. Force acts on material objects which may be discrete charged particles or a localized continuous material media, e.g., a plasma. The stress considered here acts on the field, which is a kind of etherial medium, as conceived by Maxwell and his contemporary physicists. In the absence of any glue holding them, the charges (e.g., electrons, nuclei) and currents (e.g., current loops) will be free to move and gain momentum. However, the momentum need not be confined to material objects. It can be shared by the field as well. Therefore we shall make the following conjecture.

Conjecture 1 *There exists a Maxwell's Stress Tensor* $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}$ *for the Electromagnetic field, and it is given as*

$$\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})} \equiv \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} + \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})} = \epsilon_0 \left[\mathbf{EE} - \frac{1}{2}E^2\widehat{\mathbf{1}}\right] + \frac{1}{\mu_0}\left[\mathbf{BB} - \frac{1}{2}B^2\widehat{\mathbf{1}}\right], \tag{110}$$

such that

$$\frac{d}{dt}\left(\iiint_{\mathcal{V}} \boldsymbol{\Pi}\, d^3r\right) + \frac{d}{dt}\left(\iiint_{\mathcal{V}} \mathbf{P}\, d^3r\right) = \iint_{S} \widehat{\mathbf{T}}^{(\mathrm{EM})}\cdot \mathbf{n}(\mathbf{r})\, da. \tag{111}$$

where $\boldsymbol{\Pi}$ *and* $\mathbf{P}$ *are, respectively, the field momentum density and the material momentum density, the latter being governed by Newton-Minkowski-Lorentz-force equation*

$$\frac{\partial \mathbf{P}}{\partial t} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}. \tag{112}$$

The right side of Eq. (111) gives the stress transmitted across the boundary $\mathcal{S}$. The right side of Eq. (112) gives the density of Lorentz force acting on all charged matter lying within the volume $\mathcal{V}$. We shall convert the surface integral on the right side of (111) into a volume intregral, using Gauss's theorem (see Sec. 3.3) so that each term in this equation is a volume integral, and then remove the integral sign reducing the same equation to an equality among three density functions.

$$\begin{aligned}\frac{\partial \mathbf{\Pi}}{\partial t}+\frac{\partial \mathbf{P}}{\partial t} &= \nabla\cdot\widehat{\mathbf{T}}^{(\mathrm{EM})}. \quad \text{Or,}\\ \frac{\partial \mathbf{\Pi}}{\partial t}+\rho\mathbf{E}+\mathbf{J}\times\mathbf{B} &= \nabla\cdot\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}+\nabla\cdot\widehat{\mathbf{T}}^{(\mathrm{M})}.\end{aligned} \tag{113}$$

We shall now show that the above conjecture is right, that starting from Maxwell's equations we are able to find an expression for the field momentum density such that the momentum conservation of matter and field together falls into the scheme suggested in Eq. (113). Our task is made simple by the identity the identity (76) we had established in Sec. 4. We shall do the work in two stages: (1) set $\mathbf{E}$ for $\mathbf{A}$ in (76), and use Maxwell's equations: $\nabla\cdot\mathbf{E}=\rho/\epsilon_0$; $\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$, (2) set $\mathbf{B}$ for $\mathbf{A}$ and use Maxwell's equations: $\nabla\cdot\mathbf{B}=0$; $\nabla\times\mathbf{B}=\mu_0(\mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$.

$$\begin{aligned}&\nabla\cdot\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}=\nabla\cdot\epsilon_0\left[\mathbf{EE}-\tfrac{1}{2}E^2\widehat{\mathbf{1}}\right]\\&=\epsilon_0\left[(\nabla\cdot\mathbf{E})\mathbf{E}-\mathbf{E}\times(\nabla\times\mathbf{E})\right]\\&=\rho\mathbf{E}+\epsilon_0\mathbf{E}\times\tfrac{\partial\mathbf{B}}{\partial t}.\\&\nabla\cdot\widehat{\mathbf{T}}^{(\mathrm{M})}=\nabla\cdot\tfrac{1}{\mu_0}\left[\mathbf{BB}-\tfrac{1}{2}B^2\widehat{\mathbf{1}}\right]\\&=\tfrac{1}{\mu_0}\left[(\nabla\cdot\mathbf{B})\mathbf{B}-\mathbf{B}\times(\nabla\times\mathbf{B})\right]\\&=-\mathbf{B}\times(\mathbf{J}+\epsilon_0\tfrac{\partial\mathbf{E}}{\partial t})=\mathbf{J}\times\mathbf{B}+\epsilon_0\tfrac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}.\\&\nabla\cdot\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}=\tfrac{\partial}{\partial t}\left(\epsilon_0\mathbf{E}\times\mathbf{B}\right)+(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}).\end{aligned} \tag{114}$$

The last equation is obtained by adding the first two, and using definition of $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}$ as given in (110). It confirms validity of our conjecture and identifies the field momentum density as

$$\boxed{\mathbf{\Pi}=\epsilon_0(\mathbf{E}\times\mathbf{B}).} \tag{115}$$

We shall like to recast Eq. (113a) into the general format of conservation equation

$$\frac{\partial}{\partial t}(\text{volume density})+\nabla\cdot(\textbf{flux density})=0. \tag{116}$$

In this case the momentum flux density $\widehat{\mathbf{\Phi}}$ is to be identified as

$$\widehat{\mathbf{\Phi}}=-\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}. \tag{117}$$

Eq. (113a) now reads like a true momentum conservation equation:

$$\frac{\partial}{\partial t}(\mathbf{\Pi}+\mathbf{P})+\nabla\cdot\widehat{\mathbf{\Phi}}=\mathbf{0}. \tag{118}$$

It may be easier to comprehend the meaning of the above conservation equation by writing its three cartesian components. For example, the x-component of the above equation will be

$$\begin{aligned}&\tfrac{\partial P_x}{\partial t}+\tfrac{\partial \Pi_x}{\partial t}+\nabla\cdot\mathbf{\Phi}_x=0,\\&\text{where } \mathbf{\Phi}_x=\widehat{\mathbf{\Phi}}\cdot\mathbf{e}_x=-\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}\cdot\mathbf{e}_x\\&=-\epsilon_0[\mathbf{e}_x\tfrac{1}{2}(E_x^2-E_y^2-E_z^2)\\&+\mathbf{e}_yE_yE_x+\mathbf{e}_zE_zE_x]\\&-\tfrac{1}{\mu_0}[\mathbf{e}_x\tfrac{1}{2}(B_x^2-B_y^2-B_z^2)\\&+\mathbf{e}_yB_yB_x+\mathbf{e}_zB_zB_x].\end{aligned} \tag{119}$$

The first two terms in the first line give the rate of increase of the x-component of total momentum (consisting of field momentum and material momentum) per unit volume, the third term gives the rate of outflux of the x- component of the field momentum per unit volume. Conservation of momentum implies that the sum of the two must be zero.

Before leaving this topic let us take a look at the field energy density $\mathcal{U}$ and the field energy flux density $\mathbf{S}$ (i.e. the Poyning's vector) written in Eq. (**??**). It is immediately noticed that

$$\mathbf{S} = c^2\mathbf{\Pi}. \tag{120}$$

When the electromagnetic field is a radiation field, $E = cB$ and $\mathbf{E} \times c\mathbf{B} = E^2\mathbf{n}$ where $\mathbf{n}$ is the direction of the Poynting.s vector, giving the direction of the flow of radiation energy. For such radiation fields,

$$\mathcal{U} = \epsilon_0 E^2;\ \mathbf{S} = c\mathcal{U}\,\mathbf{n};\ \mathbf{\Pi} = \frac{\mathcal{U}}{c}\mathbf{n};\ \mathcal{U} = c\,\Pi. \tag{121}$$

The last equality is a reninder of the relation $E = cp$ between the energy E and the momentum p of a photon.

We are still not too clear about the true meaning of the momentum flux density $\widehat{\mathbf{\Phi}}$. To get familiarity with it let us consider a plane electromagnetic wave propagating in the x-direction, polarized in the y-direction. For such a field $\mathbf{E} = E\mathbf{e}_y$, $c\mathbf{B} = E\mathbf{e}_z$. It is a simple exercise to evaluate $\widehat{\mathbf{\Phi}}$ by setting $E_x = 0, E_y = E, E_z = 0;\ cB_x = 0, cB_y = 0, cB_z = E$ in the expression for $\mathbf{\Phi}_x$ in Eq. (119c) and similar expressions for $\mathbf{\Phi}_y, \mathbf{\Phi}_z$ and obtain

$$\begin{aligned}\widehat{\mathbf{\Phi}} &= \mathbf{\Phi}_x\mathbf{e}_x + \mathbf{\Phi}_y\mathbf{e}_y + \mathbf{\Phi}_z\mathbf{e}_z = (\epsilon_0 E^2\mathbf{e}_x)\,\mathbf{e}_x \\ &= c\,\Pi\,\mathbf{e}_x\mathbf{e}_x = c\,\mathbf{\Pi}\,\mathbf{e}_x = \mathbf{\Pi}\,\mathbf{c}.\end{aligned} \tag{122}$$

Here $\mathbf{c} = c\mathbf{e}_x$ represents the "velocity" of light, being the speed c multiplied with a unit vector in the direction of prpagation. If we now consider a plane perpendicular to the X-axis, so that $\mathbf{n} = \mathbf{e}_x$, then the outflux of field momentum per unit area across the plane will be $\widehat{\mathbf{\Phi}} \cdot \mathbf{n} = \widehat{\mathbf{\Phi}} \cdot \mathbf{e}_x = c\,\mathbf{\Pi}$.

Generalization of Eq. (122) is obvious. If there is a source of radiation at the origin (say, an antenna, or an accelereting charged particle), then far away from the origin, the momentum flux density tensor $\widehat{\mathbf{\Phi}}$ has the form

$$\widehat{\mathbf{\Phi}} = c\,\Pi\,\mathbf{e}_r\mathbf{e}_r = \mathbf{\Pi}\,c\,\mathbf{e}_r = \mathbf{\Pi}\,\mathbf{c}, \tag{123}$$

where $\mathbf{e}_r$ is the unit vector in the radial direction, also midentified with the direction of propagation of the electromagnetic wave. The tensor $\widehat{\mathbf{\Phi}}$ gives the measure of how much momentum is crossing a spherical surface per unit area per unit time. The momentum density is $\mathbf{\Pi} = \Pi\,\mathbf{e}_r$, and it is propagating in the radial direction with velocity $\mathbf{c} = c\,\mathbf{e}_r$.

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Appendix: Useful Integrals

We shall write derive the values of some integrals required in this book. The integrands of all the integrals will have in their denominators integer/half-integer powers of the expression $(r^2 + a^2 - 2ra\,\cos\theta)$, the integration variable will be θ, and the range of integration $[0, \pi]$. We shall do some preliminary work by changing the variable of integration from θ to η, accompanied by the change of the limits of integration, and conversion of the numerators for the first two cases.

$$\begin{aligned}
\eta^2 &= r^2 + a^2 - 2ra\,\cos\theta, & (a)\\
\eta\,d\eta &= ar\,\sin\theta\,d\theta. & (b)\\
a - r\cos\theta &= \tfrac{a^2 - r^2 + \eta^2}{2a} & (c)\\
(r^2 + a^2)\cos\theta - 2ra &= \tfrac{(a^2 - r^2) - (a^2 + r^2)\eta^2}{2ra} & (d)\\
\text{Lower limit: } \theta = 0 \Rightarrow \eta &= \{(a - r),\ \text{if } a > r\}; \quad \{(r - a),\ \text{if } r > a\}. & (e)\\
\text{Upper limit: } \theta = \pi \Rightarrow \eta &= a + r. & (f)
\end{aligned} \qquad (124)$$

6.1 Direct Evaluation

Using the above conversions hints it should not be difficult for the reader to establish the following integrals.

Integral # 1

$$\Psi_1(a,r) \equiv \int_0^\pi \left[\frac{(a - r\cos\theta)}{(r^2+a^2-2ra\,\cos\theta)^{\frac{3}{2}}}\right]\sin\theta\,d\theta = \begin{cases} \dfrac{2}{a^2} & \text{if} \quad a > r; \\ 0 & \text{if} \quad a < r. \end{cases} \tag{125}$$

Integral # 2

$$\Psi_2(r,a) \equiv \int_0^\pi \left[\frac{(r^2+a^2)\cos\theta - 2ra}{(r^2+a^2-2ra\,\cos\theta)^3}\right]\sin\theta\,d\theta = 0. \tag{126}$$

6.2 Evaluation using Maxima

We have evaluated the following three integrals, using Maxima (version 5.13.0). We shall first write down the values of the integrals, and then show the commands used in Maxima to obtain these results.

Let us write

$$\begin{array}{ll} \alpha = 2\cos\theta; & \beta = 2(r^2+a^2)\cos\theta - (3+\cos^2\theta)ar. \\ \gamma = \sin\theta; & \delta = (r^2 - 2a^2 + ar\cos\theta)\sin\theta. \end{array} \tag{127}$$

Integral # 3

$$\Psi_3(r,a) = \int_0^\pi \left[\left(\alpha^2 - \gamma^2\right)\cos\theta - 2\alpha\gamma\sin\theta\right]\,\sin\theta\,d\theta = 0. \tag{128}$$

Integral # 4

$$\Psi_4(r,a) = \int_0^\pi \left[\frac{(\alpha\beta - \gamma\delta)\cos\theta - (\alpha\delta+\beta\gamma)\sin\theta}{(r^2+a^2-2ra\,\cos\theta)^{5/2}}\right]\sin\theta\,d\theta = \begin{cases} \dfrac{12r}{a^4}; & (a > r) \\ 0; & (a < r). \end{cases} \tag{129}$$

Integral # 5

$$\Psi_5(r,a) = \int_0^{\pi} \left[\frac{(\beta^2 - \delta^2)\cos\theta - 2\beta\delta\sin\theta}{(r^2 + a^2 - 2ra\ \cos\theta)^5} \right] \sin\theta\, d\theta = 0. \tag{130}$$

6.3 Maxima Commands, Inputs and Outputs

We shall write the interactive commands and prompts between the user and the Maxima so that the reader can verify the values of the integrals #4 and # 5. Note the following:
1. Some output lines (e.g., %o5, %06 in Ex.#4) are spread over two lines in which the first line contains the "indices", e.g., "to the power 2". These indices get displaced and detached from the base when the output is copied into any text file. To avoid this anomaly we have brought them to one line using mathematical mode.
2. If the output is an expression of a definite integral, it is spread over seven lines (e.g., as in %o9 in Ex.#4), and the integral sign becomes unintelligible when copied. We have replaced these outputs and other outputs that appear too long and complicated with "...". All outputs except the final one are non-essential.

Input/Ouptut for Integral #4

```
(%i1)   aa: 2*cos(x);
(%o1)   2 cos(x)
(%i2)   bb: 2 *( r^2 + a^2) *cos(x) - ( (cos(x))^2 +3)* a* r ;
```
(%o2) $2(r^2 + a^2)\cos(x) - ar(\cos^2(x) + 3)$

```
(%i3)   cc: sin(x);
(%o3)   sin(x)
(%i4)   dd: (r^2 -2* a^2 + a* r *cos(x) )*sin(x) ;
```
(%o4) $(ar\cos(x) + r^2 - 2a^2)\sin(x)$

```
(%i5)   f: (aa*bb-cc*dd)* cos(x)-( aa*dd+bb*cc)* sin(x) ;
```
(%o5) cos(x) (2 cos(x) $(2(r^2 + a^2)\cos(x) - ar(\cos^2(x) + 3))$
- (a r cos(x) + $r^2 - 2a^2)\sin^2(x))$ - sin(x)
$((2(r^2 + a^2)\cos(x) - ar(\cos^2(x) + 3))\sin(x)$
+ 2 cos(x) (a r cos(x) + $r^2 - 2a^2$) sin(x))

```
(%i6)   et: abs(sqrt(r^2+a^2 - 2*r*a*cos(x)));
```
(%o6) sqrt(- 2 a r cos(x) + $r^2 + a^2$)

```
(%i7)   h: (f/(et^5))*sin(x) ;
(%o7)   ....
(%i8)   assume (a-r > 0) ;
(%o8)   [a > r]
(%i9)   'integrate (h, x) ;
(%o9)   ...
(%i10)  changevar (%, et - y, y, x) ;
        Is y positive, negative, or zero?
        pos;
        solve: using arc-trig functions to get a solution.
        Some solutions will be lost.
(%o10)  ...
(%i11)  %,nouns;
        Is sqrt(r + 2 a r + a ) - sqrt(r - 2 a r + a ) positive, negative, or zero?
        pos;
        Is r + a zero or nonzero?
        nonzero;
```

(%o11)

$$- \Big(\frac{\mathrm{sqrt}(r^2 - 2ar + a^2)(48a^2r^7 + 36a^3r^6 + 8a^4r^5 + 4a^5r^4)}{r - a} - \frac{\mathrm{sqrt}(r^2 + 2ar + a^2)(48a^2r^7 - 36a^3r^6 + 8a^4r^5 - 4a^5r^4)}{r + a}$$

$$+\mathrm{sqrt}(r^2 + 2ar + a^2)(-48a^2r^6 + 12a^3r^5 - 4a^4r^4)$$

$$-\ \mathrm{sqrt}(r^2 - 2ar + a^2)(-48a^2r^6 - 12a^3r^5 - 4a^4r^4)\Big)/(16a^6r^6)$$

To simplify the last output (%o11), set

$$\mathrm{sqrt}(\mathrm{r}^2 - 2\mathrm{ar} + \mathrm{a}^2) = \begin{cases} (\mathrm{a} - \mathrm{r}) & \text{if, } (\mathrm{a} > \mathrm{r}) \\ (\mathrm{r} - \mathrm{a}) & \text{if, } (\mathrm{a} < \mathrm{r}) \end{cases} \tag{131}$$

and get $\frac{4 \times 48\mathrm{a}^2\mathrm{r}^7}{16\mathrm{a}^6\mathrm{r}^6} = \frac{12\mathrm{r}}{\mathrm{a}^4}$ for the first case and 0 for the second.

Input/Ouptut for Integral #5

```
(%i1)  bb: 2*(r^2 + a^2)*cos(x) - ( (cos(x))^2 +3)*a*r;
(%o1)  2 (r^2+ a^2) cos(x) - a r (cos^2(x) + 3)
(%i2)  dd: (r^2 -2*a^2 + a*r*cos(x) )*sin(x) ;
(%o2)  (a r cos(x) + r^2- 2 a^2) sin(x)
(%i3)  f: (bb^2-dd^2)*cos(x) - 2*bb*dd*sin(x);
(%o3)  cos(x) ((2 (r^2+ a^2) cos(x) - a r (cos^2(x) + 3))^2
       - (a r cos(x) + r^2- 2 a^2)^2 sin^2(x)) - 2 (a r cos(x) + r^2- 2 a^2)
       (2 (r^2+ a^2) cos(x) - a r (cos^2(x) + 3)) sin^2(x)
(%i4)  ets: r^2+a^2 - 2*r*a*cos(x) ;
(%o4)  - 2 a r cos(x) + r^2+ a^2
(%i5)  h: (f/(ets^5))*sin(x) ;
(%o5)  (sin(x) (cos(x) ((2 (r^2+ a^2) cos(x) - a r (cos^2(x) + 3))^2
       - (a r cos(x) + r^2- 2 a^2)^2 sin^2(x)) - 2 (a r cos(x) + r^2- 2 a^2)
       (2 (r^2+ a^2) cos(x) - a r (cos^2(x) + 3)) sin^2(x)))/(- 2 a r cos(x) + r^2+ a^2)^5
(%i6)  assume (a-r > 0) ;
(%o6)  [a > r]
(%i7)  'integrate (h, x, 0, %pi ) ;
(%o7)  ...
(%i8)  changevar (%, abs(sqrt(ets)) - y, y, x) ;
       Is y positive, negative, or zero?
       pos;
       solve: using arc-trig functions to get a solution.
       Some solutions will be lost.
(%o8)  ...
(%i9)  %,nouns;
       Is sqrt(r^2+ 2 a r + a^2) - sqrt(r^2- 2 a r + a^2) positive, negative, or zero?
       pos;
       Is r + a zero or nonzero?
       nonzero;
(%o9)  0
```