Functional differential equations. 3: Radiative damping

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Abstract

What is the solution of the equation of motion of a single classical charged particle with radiative damping? Contrary to the physical expectation, the mathematical solution is anti-damped! Attempts to curb these runaway solutions lead to pre-acceleration. Worse, despite a century of effort, there is still no way to obtain a proper solution in a general context. This failure of classical electrodynamics is intrinsic, irrespective of the hydrogen atom, and hence needs to be remedied. We outline a general method to resolve the infinities of quantum electrodynamics (renormalization problem). The same method was recently applied to resolve the infinities of classical electrodynamics. This involves a modification of Maxwell's equations at the microphysical level. The resulting equations of motion of even a single charged particle with radiative damping are functional differential equations (FDEs). These FDEs can and have been solved. The implications for quantum mechanics are postponed to the next article.

1 Recap

In two earlier articles[1, 2] in this series, we saw that functional differential equations (FDEs) are fundamentally different from ordinary differential equations (ODEs). Hence, doing physics with FDEs leads to a paradigm shift in physics. To solve retarded FDEs, for example, we need to specify past history, not initial data alone, as is the case with the ODEs of Newtonian mechanics. Again, with FDEs, volume in phase space is *not*, in general, preserved (so fine-grained entropy does *not* stay constant), so we must reconsider sta-

tistical mechanics. These significant qualitative differences between FDEs and ODEs also mean that FDEs cannot, in general, be validly approximated by ODEs.

We also saw that FDEs arise naturally in physics, so that this new physics does *not* involve any new physical hypothesis, but is a matter of doing the math right. That is because FDEs are equivalent to a *coupled* system of ODEs and PDEs. Hence, FDEs arise naturally in the context of the electrodynamic 2-body problem, which involves a coupling between the Heaviside-Lorenz force law (ODEs) (according to which each particle moves) and Maxwell's equations which are partial differential equations (PDEs) (according to which each particle acts on the other).

This understanding of FDEs also explains the need for past data. To solve Maxwell's equations we need to specify the appropriate Cauchy data, which is the counterpart of initial data for PDE. That is, we need to specify the electric and magnetic fields on a hypersurface (i.e., at an "instant" of time). If we use retarded Green functions, actually specifying these fields on an entire hypersurface requires data for the entire past world lines of the particles which produce those fields. This requirement of past data, subtly hidden by the field picture, is only made manifest by using the particle pictures and FDEs.

How exactly does this affect electrodynamics? To this end we re-examined the question of the classical hydrogen atom. Physicists are taught in high-school that classical electrodynamics cannot describe the hydrogen atom. The argument for this proceeds as follows. It first supposes that in the *absence* of radiation damping, central orbits are stable for the electrodynamic two body problem. It then concludes, heuristically, that due to radiation damping those orbits are actually unstable.

With our new understanding of FDEs it is clear that this conclusion is based on faulty reasoning. The Coulomb force does not equal the full electrodynamic force. The full electrodynamic force leads to FDEs, so approximating it by the Coulomb force involves approximating FDEs by ODEs, a process known to be incorrect in general. Therefore, the claim that central orbits are stable in the absence of radiation damping was never properly established.

The first actual solution of the FDEs of the electrodynamic 2-body problem, with the full electrodynamic force, was carried out by this author only in 2004.[3] It showed that the solution with the Coulomb force is, in fact, incorrect. Heuristically, we observed that retardation leads to a delay torque, so that an electron tends to fall out of the atom, in the absence of radiation damping.

We concluded with the natural question: what happens in the presence of radiation damping? Are there motions (not necessarily circular orbits) for which the delay torque and the radiation damping cancel (either exactly or on an average)?

2 The problem of radiative damping

2.1 The formula for radiative damping

A peculiar difficulty arises in trying to give a rigorous answer to this question. First, we obviously need a quantitative account of radiation damping. Now, standard physics texts (e.g., [4], equation 11.80, p. 467) give a formula (Abraham-Lorentz formula) for the force due to radiative damping

$$\boldsymbol{F}_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\boldsymbol{a}}, \tag{1}$$

where μ_0 is the permeability of free space, q is the charge, c is the speed of light, $\boldsymbol{a}$ is the acceleration of the charge, and, as before, dots denote derivatives with respect to time. The physical understanding of this formula is that an accelerating charge radiates energy, and therefore its motion must be damped. This formula describes the self-force on the electron responsible for the damping.

2.2 The equation of motion of a charge

But this force has a peculiarity: it depends upon the derivative or rate of change of *acceleration*. Thus, the equation of motion of an accelerated charged particle under the influence of an external force $\boldsymbol{F}_{\rm ext}$ is

$$\begin{aligned} m\boldsymbol{a} &= \boldsymbol{F}\text{ext} + \boldsymbol{F}_{\rm rad} \\ &= \boldsymbol{F}\text{ext} + \frac{\mu_0 q^2}{6\pi c}\dot{\boldsymbol{a}}. \end{aligned} \tag{2}$$

Because of the appearance of $\dot{\boldsymbol{a}} = \dddot{\boldsymbol{x}}$ this is a *third* order ODE, unlike the ODEs of classical mechanics which are all of second order. Hence, to solve for the motion of a single charged particle, one must now prescribe also $\dot{\boldsymbol{a}}(0)$ or the initial acceleration of the charge.

2.3 The runaway solutions

It is not clear on what principles the specification of $\boldsymbol{a}(0)$ would be based, but let us see what happens in the simplest case. That is we consider a particle moving in one dimension, without any external forces, so that $\boldsymbol{F}_{\rm ext} = 0$. In this case the equation (2) can be rewritten as the simple equation

$$\tau\dot{a} = a, \tag{3}$$

where

$$\tau = \frac{\mu_0 q^2}{6\pi mc}. \tag{4}$$

The equation (3) has an equally simple solution

$$a(t) = a_0 e^{\frac{t}{\tau}}, \tag{5}$$

where a_0 is the initial acceleration.

But this is catastrophic! It is evident that so long as $a_0 \neq 0$, no matter what its value, the acceleration of the particle, a, *increases* exponentially. Thus, an arbitrary non-zero initial acceleration blows up, so these are known as runaway solutions. Under its own self-action, due to radiation damping, the particle continuously accelerates! Instead of damping (as expected on physical grounds), the mathematics tells us what we have here is unbounded anti-damping!

This difficulty has been known for so long that physicists have become a bit blasé about it, and easily satisfied with various partial solutions which have been offered. To understand these, let us first find a general solution of (2).

2.4 General solution of the equations of motion

To this end, let us rewrite (2) as

$$m(\boldsymbol{a} - \tau\dot{\boldsymbol{a}}) = \boldsymbol{F}_{\text{ext}}. \tag{6}$$

To help solve the equations, we introduce a new variable $\boldsymbol{a}_1(t)$ by

$$\boldsymbol{a}_1(t) = e^{-\frac{t}{\tau}}\boldsymbol{a}(t) \quad \text{or} \quad \boldsymbol{a} = e^{\frac{t}{\tau}}\boldsymbol{a}_1(t). \tag{7}$$

Then

$$\dot{\boldsymbol{a}} = \frac{1}{\tau}e^{\frac{t}{\tau}}\boldsymbol{a}_1 + e^{\frac{t}{\tau}}\dot{\boldsymbol{a}}_1. \tag{8}$$

Hence,

$$\boldsymbol{a} - \tau\dot{\boldsymbol{a}} = -\tau e^{\frac{t}{\tau}}\dot{\boldsymbol{a}}_1, \tag{9}$$

so that (6) can be rewritten

$$\dot{\boldsymbol{a}}_1 = -\frac{1}{m\tau}e^{-\frac{t}{\tau}}\boldsymbol{F}_{\text{ext}}. \tag{10}$$

Equation (10) can be solved just by integrating it.

$$\boldsymbol{a}_1(t) = -\frac{1}{m\tau}\int_0^t e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt' + \boldsymbol{a}_1(0), \tag{11}$$

where $\boldsymbol{a}_1(0)$ is a constant of integration or the initial value. Note that $\boldsymbol{a}_1(0) = \boldsymbol{a}(0)$ by the definition (7). Hence, the general solution of (6) can be rewritten

$$\boldsymbol{a}(t) = \boldsymbol{a}(0)e^{\frac{t}{\tau}} - \frac{1}{m\tau}\int_0^t e^{\frac{(t-t')}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'. \tag{12}$$

2.5 Dirac's proposal

Now, nothing in earlier physics tells us what principles we should use to fix the value of $\boldsymbol{a}(0)$. Therefore, Dirac[5] in 1938 suggested we should fix it by the formula

$$\boldsymbol{a}(0) = \frac{1}{m\tau}\int_0^\infty e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'. \tag{13}$$

To see the point of this, let us plug in this value of $\boldsymbol{a}(0)$ into (12). We now obtain

$$\begin{aligned}
\boldsymbol{a}(t) &= e^{\frac{t}{\tau}}\left(\boldsymbol{a}(0) - \frac{1}{m\tau}\int_0^t e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'\right) \\
&= \frac{e^{\frac{t}{\tau}}}{m\tau}\left(\int_0^\infty e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt' - \right. \\
&\qquad \left.\int_0^t e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'\right) \\
&= \frac{e^{\frac{t}{\tau}}}{m\tau}\left(\int_t^\infty e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'\right) \\
&= \frac{1}{m\tau}\left(\int_t^\infty e^{\frac{(t-t')}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'\right).
\end{aligned} \tag{14}$$

Since $(t - t') < 0$ for $t' \in (t, \infty)$, the integral will converge for any reasonable external force described by an integrable (or even slowly increasing) function $\boldsymbol{F}_{\text{ext}}$. We can see this more clearly, by making the change of variables $t' = t + \tau s$, to rewrite (14) as

$$\boldsymbol{a}(t) = \frac{1}{m\tau}\int_0^\infty e^{-s}\boldsymbol{F}_{\text{ext}}(t + \tau s)ds. \tag{15}$$

Hence, the acceleration remains finite for all time.

2.6 Pre-acceleration

However, Dirac's proposal has a peculiar side-effect. It is clear from (15) that the accelera-

tion at time t is decided by a weighted average over all *future* forces. Hence, if, for example, an impulsive force is applied to a charged particle, it would start moving *before* the force is applied.

To see this, consider the case of 1-dimensional motion, and suppose a δ function force is applied at time $t = 0$. That is, $F_{\text{ext}}(t) = \delta(t)$. For the integral in (15) to be non-zero, we must have $t + \tau s = 0$, and this is possible only for $t < 0$. So, the solution is

$$a(t) = \begin{cases} \frac{1}{m\tau} e^{\frac{t}{\tau}} & \text{if } t < 0 \\ 0 & \text{if } t \geq 0. \end{cases} \qquad (16)$$

That is, the particle accelerates *before* the force is applied, and stops accelerating when it is applied. Hence, this is called pre-acceleration. This is considered "unphysical" since non-causal.

The usual defence is that the "violation" of causality takes place over a small time. For the case of an electron, the constant τ in (4) may be rewritten as

$$\tau = \frac{2}{3} \cdot \frac{q^2}{4\pi\epsilon_0 mc^3} = \frac{2}{3}\frac{r_e}{c}, \qquad (17)$$

where r_e is the classical radius of the electron, so that τ is of the order of the time it takes for light to traverse the classical radius of the electron; $\tau \approx 10^{-23}$s is also called the relaxation time of the electron. So, the argument is that the violation of causality takes place over such small times that it is of no consequence.

2.7 Discussion

Now, I have been advocating the rejection of perfect "causality" for over 35 years,[6] and for the last 20 years I have been arguing that rejecting (mechanistic) "causality" in physics is a matter of elementary commonsense,[7] and the only way to explain mundane "causality" or the mundane experience of billions of people repeated thousands of time each day. So I cannot regard the failure of (mechanistic)[1] "causality" as some kind of a "violation" to be concerned about.

However, the real problem with the above solution is this: the formula does not do its basic job. It is small consolation to know that the acceleration is finite, because we cannot calculate its value! This matter has been subject to a long debate, and several variants on Dirac's technique and the Lorentz-Dirac equation have been proposed. But this objection also applies to other proposed "solutions" such as the one suggested by Plass:[9] the initial or boundary values are required to hold *exactly*. The slightest variation from it restores the runaways, so these methods of supposedly taming the infinities of classical electrodynamics are impractical for the problem at hand which is this: to calculate the motion of the electron in a hydrogen atom with radiative damping.

[1]The term "causality" is vague and can have diametrically opposite meanings, which are often confounded in philosophical discussions. Specifically, mechanistic "causality" as used by physicists as physicists means the exact opposite of mundane "causality", as used by physicists as human beings! Conflating these two causes great confusion see, e.g., [8].

A simple-minded way out is to say that Maxwell's equations anyway fail for the hydrogen atom where quantum mechanics applies. However, the failure of Maxwellian electrodynamics, as analysed above, is intrinsic, and makes no reference to the hydrogen atom. Therefore, we need to understand why this celebrated Maxwellian theory, which is otherwise useful, fails so miserably that it cannot describe the motion of even a single electron! More pragmatically, understanding the reasons for the intrinsic failure of Maxwell's equations may suggest an appropriate correction which opens the path to a fresh understanding of quantum mechanics, which is needed to resolve the problems facing quantum computing today.

Therefore, let us press on with our enquiry into the infinities which arise in the motion of a single charged particle in Maxwellian electrodynamics, their origin and their resolution.

2.8 The finite size electron

The most obvious suspicion is that these infinities have something to do with the assumption that charged particles must be like idealised geometric points. Radiation damping is attributed to the self-action of a charge. In the field picture, that self-action is described by a charge interacting with its own field. That field, however, blows up at the position of the particle if that is assumed to be a point.

The next obvious step is to suppose that the point-charge description is a simplification, and what we really have is a finite distribution of charge. That, in fact, was the first proposed solution to this problem, ironically proposed by Lorentz.

However, this notion of a finite-size electron encountered several serious problems. In the first place, suppose we simply smear out the electron charge over a sphere or shell. The Coulomb repulsion of one part of the charge distribution acting on another would blow apart the charge distribution. What holds it together?

One could get around this problem by postulating some new forces which hold the electron together. There is, however, a far more serious problem with this solution: it is not Lorentz invariant. We can hardly abandon Lorentz invariance because the requirement of Lorentz-invariance is tied to the current definition of time measurement, as I have explained in my book[10] and in an earlier article in this journal, and we cannot do any physics without a way to measure time. A finite distribution of charge cannot easily be described in a Lorentz invariant way. A sphere in one frame would not remain a sphere in another, for the Lorentz transform distorts a sphere into an ellipsoid. The problem of a Lorentz invariant or covariant extended electron has resisted attempts at a solution for the past century.

2.9 Is the limiting procedure valid?

We could get around this problem too, but there is another subtle problem which has not been noticed, but is rather serious. The usual

derivation of the third order radiation reaction force (1) does start by assuming a finite distribution of charge (for example, [4] starts with a dumb-bell charge distribution). Since, however, this finite charge distribution cannot be described in a Lorentz invariant way, the usual derivation proceeds to the limit of a point charge. The problem of Lorentz invariance disappears in the limit.

But is this limiting procedure valid? The question was first raised by me long ago, in this very journal.[11] The doubt about the validity of the limiting procedure may be explained in simple terms as follows. In a finite charge distribution, when one part of the charge distribution acts on another, there is a retardation or delay involved. Therefore, the equations involved are FDEs; we have seen that. However, when we proceed to the limit of a point charge, the final equation of motion with radiative damping is just an ODE, as above.

So, mathematically, the limiting procedure amounts to "Taylor" expanding in powers of the delay, and then proceeding to the limit as the delay goes to zero. This limiting process converts an FDE into a higher-order ODE. We have seen[1] that this is an incorrect procedure, therefore the limiting process is not valid, even though it looks plausible, and is followed by all texts in electrodynamics today! Thus, there is a fundamental problem concerning the derivation of very formula for radiation damping (1).

3 Infinities of classical and quantum electrodynamics

3.1 A connection?

How to correct the derivation of radiation reaction? Long ago, Wheeler and Feynman thought that the infinities of quantum electrodynamics might be fixed by correcting the corresponding infinities in classical electrodynamics. Today, physicists believe that the infinities in quantum electrodynamics have been fixed through what is called renormalization. But the infinities of classical electrodynamics still stay unfixed! Nevertheless, the hunch of a connection between the two infinites was right.

Thus, long ago, I suggested a novel method of renormalization in quantum field theory. The method was presented at my guide's festschrift, and published in the proceedings,[12] but never advertised, or further developed. Hence, it is hardly known, and the following is a brief explanation.

3.2 The renormalization problem

The propagators of quantum electrodynamics are what mathematicians call fundamental solutions of PDEs. The fundamental solution of the Dirac equation is the spinor propagator, while that of the relativistic wave equation or Klein-Gordon equation is the photon

propagator. These are also known as Green functions.

These propagators involve entities like the Dirac δ function, which are regarded as generalised functions or Schwartz distributions. On the orthodox formalist exposition of the Indian calculus, as found in university calculus texts today, the derivative is defined as a limit. This definition forces a differentiable function to be continuous. However, in a situation like that of shock waves, the need arises in physics to differentiate a discontinuous function.

The Schwartz theory of distributions modifies the conventional calculus of limits, by allowing discontinuous functions (like the Heaviside jump function) to be infinitely differentiated. However, the limitation of the new theory is that generalised functions or distributions cannot be multiplied: the Schwartz theory assigns a meaning to δ, but not to the product $\delta \cdot \delta$.

Some writers on shock waves, such as Taub,[13] have wrongly maintained (on "physical grounds") that this is a trivial problem . Taub wrote "Fortunately the product of such distributions (as arise) is quite tractable." He argues as follows. Let θ denote the Heaviside function,

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0, \end{cases} \qquad (18)$$

(the value at 0 does not matter[2]). Then, $\theta^2 = \theta$, so that differentiation gives $2\theta \cdot \dot{\theta} = \delta$. But $\dot{\theta} = \delta$ hence $\theta \cdot \delta = \frac{1}{2}\delta$. The problem with this is that we also have $\theta^3 = \theta$, so that $3\theta^2\delta = \delta$, so that, since $\theta^2 = \theta$, $\theta \cdot \delta = \frac{1}{3}\delta$. Another example is that of $(x^{-1} \cdot x) \cdot \delta = \delta \neq 0 = x^{-1} \cdot (x \cdot \delta)$. Thus, neither the product rule for differentiation nor the associative law may be safely assumed in dealing with products of Schwartz distributions.

The infinities of quantum field theory have long been believed to arise because (Fourier transforms of) products of propagators (generalised functions) enter into the S-matrix expansion. Thus, for example, if $\hat{}$ denotes the Fourier transform, $(\delta \cdot \delta)\hat{} = \hat{\delta} \otimes \hat{\delta} = 1 \otimes 1 = \int_{-\infty}^{\infty} 1 dx$, where $\otimes$ denotes convolution, and it is blindly assumed (as in quantum field theory) that a Fourier transform carries products to convolutions (even when the former is undefined!).

What I showed long ago was that this belief is wrong: the problem does *not* lie with products of distributions alone. I defined a natural product of distributions[14], still the only such definition which works for both classical physics and quantum field theory.[15] This definition earlier used non-standard analysis, but the definition actually works perfectly well with a so-called non-Archimedean ordered field, such as the number system of "unexpressed fractions" (rational functions) routinely used from the 5th c. by traditional Indian mathematicians while developing the calculus. Anyway, with my definition, all propagator products arising in quantum field theory are finite, in one dimension.[16] My analysis[12] identified the problem as really that of defining *compositions* not products.

[2]since the Schwartz theory is based on the Lebesgue integral where the value of a function at one point is irrelevant, since a point has measure zero.

What difference does that make? The difficulties which arise with compositions are different from those that arise with products. Thus, I did define compositions along with products,[14] but the issue is as follows. The propagators need to be defined on the null cone $\lambda = 0$. How should we define the composition $\delta(\lambda)$? For any hypersurface $\Sigma = 0$, we can naturally define $\delta(\Sigma)$ just as $\delta(n)$, locally, wherever there is a unique normal to Σ and n denotes the coordinate normal to Σ in Gaussian normal coordinates, so that the equation of Σ locally is $n = 0$. That is we can define $\delta(\lambda)$ everywhere on the null cone except at its vertex. For the particular case of the δ function, we can extend the definition of $\delta(\lambda)$ even to the vertex of the null cone. But in the case of a general distribution f, there is a geometrical difficulty in defining $f(\lambda)$ at the vertex of the null cone because there is no unique normal vector there.

This understanding immediately suggests a very simple and elegant solution to the problem of the infinities of quantum field theory.[12] Namely, eliminate that vertex and replace the support of the propagators by a Lorentz-invariant hyperboloid. Unlike a cutoff, this preserves the Lorentz invariance of the theory, which is *essential* for all current physics as already noted. Unlike a regularisation left on, the support of the propagators is not fuzzy, so interactions do not creep outside the null cone, and positivity of energy is preserved.

Changing the propagators is equivalent to changing the underlying PDEs (Dirac equation, Klein-Gordon equation), of which these propagators are fundamental solutions. That does not really matter, since all calculations are actually done only with the propagators. For example, look at the way we use the Green function to get solutions of Maxwell's equations. Once we have the propagator, or the Green function, we also have the solution, and we don't really need to refer back to the equation.

3.3 Back to classical electrodynamics

The point of this long digression into quantum electrodynamics is this. Can this solution to the problem of infinities in quantum electrodynamics be applied to get rid of the infinities of classical electrodynamics? Indeed it can! This was done some time ago.[17] This is described below using the covariant formulation of electrodynamics to emphasize that everything is done in a Lorentz covariant way.

As regards the Lorentz-invariant hyperboloid, which replaces the null cone, there are two possibilities. A hyperboloid of one sheet would give a Lorentz-covariant model of a spatially extended particle, but this does not give radiation reaction. To get radiation reaction, we need a hyperboloid of two sheets, or what one might call "particles extended in time". We denote the separation by d, assumed to be a constant for the moment.

4 Modified Maxwell equations

4.1 Obtaining the new equations of motion

Thus, the new retarded Green function for classical electrodynamics is given by

$$G_r(\mathbf{x},\mathbf{y}) = \delta((\mathbf{x}-\mathbf{y})^2 + d^2)\theta(x^0 - y^0), \quad (19)$$

where θ is, as before, the Heaviside step function, and δ is its derivative the Dirac delta. Exactly how this changes Maxwell's equations has been worked out, but is irrelevant as explained above.

We follow the original article,[17] and use the metric diag$(-c^2, 1, 1, 1)$, i.e.,

$$||\mathbf{x}||^2 = x^\mu x_\mu = -c^2(x^0)^2 + \sum_i (x^i)^2. \quad (20)$$

For vectors satisfying $(\mathbf{x}-\mathbf{y})^2 = -d^2$, a Lorentz transformation cannot change the sign of $x^0 - y^0$. Hence, the Green function in (19) is Lorentz invariant.

Scalar and vector potentials are obtained as usual.

$$A_\mu(\mathbf{x}) = \frac{1}{2\pi\epsilon_0 c}\int j_\mu(\mathbf{y})G(\mathbf{x},\mathbf{y})d^4\mathbf{y} + \partial_\mu\chi, \quad (21)$$

where χ is an arbitrary scalar function which vanishes in the Lorenz gauge.

For a point charge q, with worldline $\alpha^\mu(s)$ and proper time, s,

$$F_{\mu\nu}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0 c\,(\zeta\cdot\dot\alpha)^2}\left(\ddot\alpha_{[\mu}\zeta_{\nu]} - \frac{\dot\alpha_{[\mu}\zeta_{\nu]}\,(c^2 + \zeta\cdot\ddot\alpha)}{\zeta\cdot\dot\alpha}\right). \quad (22)$$

Dots now denote derivatives with respect to proper time, evaluated at retarded/advanced time, τ_r, τ_a, obtained as the solution of

$$||\mathbf{x}^\mu - \alpha^\mu(\tau)||^2 + d^2 = 0. \quad (23)$$

The retarded time τ_r is the solution for which $\mathbf{x}^0 > \alpha^0(\tau_r)$, while the advanced time τ_a is the solution satisfying the opposite inequality. Further, the vector ζ is defined as the retardation vector pointing from the retarded position to the current position: $\zeta^\mu = \mathbf{x}^\mu - \alpha^\mu(\tau_r)$, and similarly in the advanced case, using the advanced time τ_a instead. For a slow moving particle, the delay $\tau_d \equiv \tau - \tau_r \sim \frac{d}{c}$.

The equation of motion of a charged particle obeying the modified Maxwell equations is

$$\ddot\alpha^\mu = \frac{q}{m}\dot\alpha_\nu F^{\mu\nu}, \quad (24)$$

where $F^{\mu\nu}$ is the net field strength and includes the field from the particle.

This equation looks the same as in the Maxwellian theory, but because of the separation constant d there is retardation involved even in the case of self-action, where $F^{\mu\nu}$ is solely the self-field. Hence, the resulting equations of motion for even a *single* accelerating charged particle is now an FDE, *not* a third-order ODE.

4.2 Consequences of changing Maxwell's equations

What difference does that make? The difference is this. Unlike the runaway solutions of the 3rd-order ODE, arising from Maxwell's equations, this FDE arising from

the modified Maxwell's equations has globally bounded solutions. However, locally the value of the radiation reaction remains roughly the same as that described by the third order term. This is as expected, since the modification of Maxwell's equations is "small", since the delay involved, $\tau_d \sim \frac{d}{c}$, is very small, and roughly the same as the relaxation time of the electron defined in (17), if we suppose that $d \sim r_e$.

Most importantly, we reiterate that this gives us a way to actually solve the problem of the motion of a classical charged particle with radiation reaction. Thus, unlike all the previous attempts in the past century, this modification of Maxwell's equations resolves the problem of how to actually calculate the motion of a charged particle with radiative damping.

There are no doubt technical difficulties in obtaining a numerical solution. For example, for the case of the hydrogen atom, the problem is numerically stiff: there are two widely different time-scales in the problem: the time scale of the radiation reaction and the time scale of orbital motion. Nevertheless, there does exist a code called RADAR to solve numerically stiff FDEs,[18] and we have actually used it. The details are in the original paper.[17]

However, it should be clear by now that the rigorous solution of the classical electrodynamic 2-body problem, and even the 1-body problem, is a complex matter. But, why should we bother to find a solution to such FDEs? Don't we already know that quantum mechanics is the right theory? Is it worth the effort? Is any of this going to lead to quantum mechanics?

We will see the answers to these questions in the next part of this series of articles.

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