Transcending three dimensions in physics

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Abstract

We usually study or teach physics in the three dimensional or four dimensional space in classrooms. But now a days there is a growing interest among researchers [1-11] in multidimensional physics. In this article, we make a study of the Blackbody Radiation(BBR), Bose Einstein Condensation(BEC) and Pauli para magnetism in different dimensions. The reader can see that the study of physics is very much enjoyable with interesting surprises when we study some phenomena in higher or lower dimensions.

1 Introduction

The purpose of this article is to present three simple but interesting phenomena in statistical mechanics from the dimensional point of view. We organize the paper as follows. First we describe the Planck radiation law, Stefan Boltzmann law and Wien's law in d spatial dimensions and then discuss Bose Einstein condensation in arbitrary dimensions. Finally we discuss Pauli para magnetism in three dimensions. Although some of these results about black body radiations are known in literature[1-6] we approach the derivations in a method as given in Pathria[3] which may be familiar to most of the students. BEC studies in different dimensions has been done earlier[7,8] but mainly it was done with massive bosons. We here study BEC with different energies in different dimensions. Pauli para magnetism in arbitrary dimensions has also been reported[9] but we present here a simple method for its dimensional variation in 1, 2 and 3 dimensions. Nowadays it has become rather common to study the phenomena and underlying physics in arbitrary dimensions. The existence of extra dimensions has been a subject of intensive research study during the past few years [10-11]. The inclusion of extra dimensions plays a crucial role in many physical concepts, mostly in the construction of various models such as super string theory and general relativity [12-14]. The d dimensional dependence of physical laws would help us to understand their nature more profoundly and may give an answer to why our universe possesses three dimensions and not some other dimensions. Besides, from the point of view of physics education we can formulate various such simple problems in class rooms on higher dimensional physics which may stimulate the students' curiosity and imagination.

1.1 Planck's distribution law(PDL)

A black body cavity can be imagined to be filled with a gas of identical and indistinguishable quanta called photons with zero rest mass and with energy $E = \hbar\omega$. The energy of photons vary from 0 to infinity. Here we first analyze the blackbody radiation in a universe with 1, 2, 3 and d-spatial dimensions. Such a study was started by De Voss A in 1988[1] where no explicit expression for Stefan-Boltzmann constant in d dimension was obtained. Later an exact derivation of Planck distribution law[PDL], Wien's displacement law and Stefan Boltzmann law were given by Peter T Landsberg and Alexis De Vos [2] based on principles of electrodynamic waves in cavities. In 2005 there were 2 papers[5] and [6] which also gives the same ideas from different point of view. We approach the derivation in a pedagogical way based on the phase space principles in statistical mechanics as given by Pathria[3]. Such a study will help the students directly study any multidimensional problem other than BBR. The number of micro states in phase space is given by

$$\Omega = \frac{\pi^{\frac{d}{2}} R^d L^d}{h^d \left(\frac{d}{2}\right)!}$$

Substituting $R = p = \frac{h\nu}{c}$ the number of states between ν and $\nu + d\nu$ is

$$g(\nu)d\nu = \frac{d\pi^{\frac{d}{2}} \nu^{d-1} L^d}{c^d (\frac{d}{2})!}$$

Internal energy is given by

$$U = kT^2 \frac{\partial}{\partial T} \ln \mathcal{Z}$$

$$\ln \mathcal{Z} = -g_I \int_0^\infty g(\nu)\, d\nu \ln(1 - e^{-\beta h\nu})$$

where $\mathcal{Z}$ is the grand partition function, $\beta = \frac{1}{kT}$ and g_I is the internal degree of freedom. Taking 2 internal degrees of freedom for photons

$$\frac{U}{L^d} = \int_0^\infty \frac{2d\pi^{\frac{d}{2}} h\nu^d}{c^d (\frac{d}{2})!} \frac{1}{e^{\frac{h\nu}{kT}-1}} d\nu$$

$$u(\nu)d\nu = \frac{2d\pi^{\frac{d}{2}}h\nu^d}{c^d(\frac{d}{2})!}\frac{1}{e^{\frac{h\nu}{kT}-1}}d\nu$$

This is the PDL in d dimensions and we get Planck ′ distribution functions as

$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{\frac{h\nu}{kT}}-1}$$

$$u(\nu)d\nu = \frac{4\pi h\nu^2}{c^2}\frac{d\nu}{e^{\frac{h\nu}{kT}}-1}$$

$$u(\nu)d\nu = \frac{2\,h\nu}{c}\frac{d\nu}{e^{\frac{h\nu}{kT}}-1}$$

Among the three first is the conventional Planck's distribution law in 3 dimensions and others are in 2 and 1 dimension respectively.

1.1.1 Thermodynamics of photon gas in d- dimensions

It is always informative to find the thermodynamics of photons and we do this here in different dimensions.

Pressure

We have

$$\ln \mathcal{Z} = -g_I \int_0^\infty g(\nu)d\nu \sum_{l=1}^{\infty} \frac{(-1)e^{-\beta h\nu l}}{l}$$

From[3] we know

$$\frac{PL^d}{kT} = \ln \mathcal{Z}$$

On integrating we get

$$P = \frac{2\,d!\pi^{\frac{d}{2}}}{\left(\frac{d}{2}\right)!}\frac{k^{(d+1)}T^{(d+1)}}{c^d\,h^d}\sum_{l=1}^{\infty}\frac{1}{l^{d+1}}$$

$$\boxed{P \propto T^{d+1}}$$

which is Stefan-Boltzmann law in d dimensions.

Energy density

Using the equation for energy we get

$$\frac{U}{L^d} = \frac{2\,d\,d!\pi^{\frac{d}{2}}}{\left(\frac{d}{2}\right)!}\frac{k^{d+1}T^{d+1}}{c^d h^d}\sum_{l=1}^{\infty}\frac{1}{l^{d+1}} \qquad (1)$$

$$\boxed{U \propto T^{d+1}}$$

Then the relationship between the pressure and energy density for a photon gas is

$$P = \frac{1}{d}\frac{U}{L^d}$$

For 3 dimensions we get $P = \frac{1}{3}\frac{U}{V}$

Entropy

Using the relation $A = U - TS$, where A is the Helmholtz free energy, S is the entropy and with $A = -kT\ln\mathcal{Z}$ (since chemical potential of photon gas is zero) we get

$$S = \left(1 + \frac{1}{d}\right)\frac{2dd!\pi^{\frac{d}{2}}}{\left(\frac{d}{2}\right)!}\frac{k^{d+1}T^d}{c^d h^d}L^d\sum_{l=1}^{\infty}\frac{1}{l^{d+1}}$$

$$\boxed{S \propto L^d\,T^d}$$

1.1.2 Wien's displacement law

PDL curve rises and become a maximum and then decreases and there is a $\lambda = \lambda_{max}$ for which intensity is a maximum. From the distribution law for frequency, using $c = \nu\lambda$ we get

$$u(\lambda)d\lambda = \frac{-2d\pi^{\frac{d}{2}}hc}{\lambda^{d+2}(\frac{d}{2})!}\frac{1}{e^{\frac{hc}{\lambda kT}-1}}d\lambda$$

When $\lambda = \lambda_{max}$,

$$\frac{du(\lambda)}{d\lambda} = 0$$

we get

$$\frac{xe^x}{e^x - 1} = d + 2$$

where $x = \frac{hc}{\lambda_{max}kT}$. This is a transcendental equation for whom some solutions are

$$x_1 = 2.8214$$

$$x_2 = 3.9207$$

$$x_3 = 4.9651$$

$$x_4 = 5.9849$$

$$x_5 = 6.9936$$

In 3 D

$$\frac{xe^x}{e^x - 1} = 5$$

Using these equations we can show that the color of the sun with surface temperature 6000 K will be yellow in 3 dimensions, near red in 2 dimensions and infra red in 1 dimension.

1.1.3 Stefan Boltzmann law

This law was deduced from experimental observation by Stefan in 1879; five years later Boltzmann derived it from thermodynamic consideration. The energy radiated per unit area per unit time is related to energy density of the body as[2],

$$R = \frac{U}{L^d}R^d$$

where

$$R^d = c\frac{\Gamma(\frac{d}{2})}{2\sqrt{\pi}\Gamma(\frac{(d+1)}{2})}$$

Substituting the value of U we get

$$R = \sigma_d T^{d+1} \qquad (2)$$

where d dimensional Stefan- Boltzmann constant is

$$\sigma_d = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{(d+1)}{2})}\frac{\Gamma(d+1)\zeta(d+1)}{h^d c^{d-1}}R^{d+1}$$

Equation[2] is the d dimensional Stefan Boltzmann law. In 3 D

$$\sigma_3 = 5.67 \times 10^{-8} Wm^{-2}K^{-4}$$

In 2 D

$$\sigma_2 = 1.92 \times 10^{-10} Wm^{-1}K^{-3}$$

In 1 D

$$\sigma_1 = 9.46 \times 10^{-13} WK^{-2}$$

These equations show that σ is a dimensional dependent constant. The 1 D equation expresses the thermal noise power transfer in one-dimensional optical systems and we get a similar equation for Johnson noise or Nyquist noise[2]. As already indicated in the abstract we can see some interesting and surprising results like the dimensional dependence of the color of the sun, dimensional dependence of Stefan Boltzmann constant etc.

2 Some Low dimensional problems

2.1 Bose Einstein Condensation

Now we consider another topic BEC which is now an active research problem. All the particles in nature may be classified as either bosons or fermions according to the value of their spin angular momentum. Particles with integer spin are bosons and particles with half integer spin are called fermions. Most of the fundamental building blocks of matter (e.g. electrons, neutrons, and protons) are fermions. A composite particle comprising an even number of fermionic building blocks (such as an atom) are also bosons and with odd number are fermions. The wave function describing the state of a system of particles will be symmetric for bosons and anti symmetric for fermions. The properties of ultracold atomic gases are dramatically different for bosons and fermions. Below a critical temperature, bosons undergo a phase transition and a macroscopic number of the atoms are forced into the lowest energy state of the system. This phenomenon is called Bose Einstein Condensation. Simply speaking Bose Einstein Condensation is the piling up of particles in the lowest energy level, below a particular temperature called critical temperature. We can see that equations for BEC is different for different energies.

2.1.1 Massive non relativistic bosons

Consider a gas of bosons with energy $\frac{p^2}{2m}$ where p is the momentum and m is the mass of the particle.

Three Dimension

In grand canonical formulation

$$\ln \mathcal{Z} = -g_I \sum_p \ln\left(1 - ze^{-\beta\varepsilon_p}\right)$$

where ε_p is a function of p. Here $\mathcal{Z}$ is the grand partition function z is the fugacity which is related to the chemical potential μ as $z = e^{\beta\mu}$ and g_I is the internal degree of freedom which is 1 for a classical particle. Taking all these

$$\ln \mathcal{Z} = -\sum_p \ln\left(1 - ze^{-\beta\frac{p^2}{2m}}\right)$$

On simplifying using the number of states between p and $p + dp$ as $g(p)dp = \frac{4\pi p^2 dpV}{h^3}$ we get

$$\ln \mathcal{Z} = g_{\frac{5}{2}}(z)\frac{V}{\lambda^3} \tag{3}$$

where $g_{\frac{5}{2}}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{\frac{5}{2}}}$ and thermal De Broglie wavelength $\lambda = \frac{h}{(2\pi mkT)^{\frac{1}{2}}}$. Here k is Boltzmann constant.

The total number of bosons in a given state can be obtained by using the expression

$$N = z\frac{\partial}{\partial z}\ln \mathcal{Z} \qquad (4)$$

$$N = \frac{V}{\lambda^3} g_{\frac{3}{2}}(z)$$

For the Bose particles there is no restriction on the number of particles to occupy any level in the system. Let N_0 be the number of particles in the ground state. For temperature very much greater than critical temperature, the number of particles in the ground state will be very very small. Hence we can write

$$N = \frac{V}{\lambda^3} g_{\frac{3}{2}}(z) \ + \ N_0$$

at $T = T_c$, $z = 1$[3] $V = \frac{N\lambda_c^3}{g_{\frac{3}{2}}(1)}$

Substituting this in the equation for N we get,

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3$$

This is the equation of BEC. The right hand side of the equation is the fraction of total number of particles in the ground state. We can see that at $T = T_c$, $N_0 = 0$ which means no particle in the ground state. When $T < T_c$, $N \approx N_0$, which means the significant fraction of total number of particles are in the lowest possible energy state. When $T = 0$, $N = N_0$ all the particles are in the ground state which is BEC. Now this curious phenomenon can be done in 2 and 1 dimensions.

Two and one Dimension

For 2 dimensions we will get $\ln \mathcal{Z} = \frac{A}{\lambda^2}\ g_2(z)$ and $N = \frac{A}{\lambda^2}\ g_1(z) = \frac{A}{\lambda^2}\zeta(1)$. For one dimension we will get $\ln \mathcal{Z} = \frac{L}{\lambda}\, g_{\frac{3}{2}}(z)$, $N = \frac{L}{\lambda} g_{\frac{3}{2}}(1) = \frac{L}{\lambda}\zeta(\frac{1}{2})$ for $\mu = 0$. The expressions for N are non physical or the condensation for massive bosons in 2D and 1-D does not occur.

BEC for bosons with relativistic massless and harmonic oscillator energy

For massless relativistic, identical, non-interacting bosons the energy is given by $\varepsilon = c\,|p|$. Using the number of states as for massive bosons we get $\ln \mathcal{Z} = \left(\frac{V}{\lambda^3}\right) g_4(z)$ where $g_4(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^4}$ and $\lambda = \frac{hc}{2\pi^{\frac{1}{3}} mkT}$. Then $N = \left(\frac{V}{\lambda^3}\right) g_3(z) = \left(\frac{V}{\lambda^3}\right)\zeta(3)$ which has definite value and hence condensation is possible. For 2 dimensions $\ln \mathcal{Z} = \left(\frac{A}{\lambda^2}\right) g_3(z)$. With this $N = \left(\frac{A}{\lambda^2}\right) g_2(z) = \left(\frac{A}{\lambda^2}\right)\zeta(2)$ which has once again definite value .This result shows that massless bosons in 2D do indeed form a condensate. But for one dimension $N = \left(\frac{L}{\lambda}\right)\zeta(1) \to \ \infty$ which forbids condensation. For harmonic potential energy Hamiltonian is of the form $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$ Using this Hamiltonian as above we can show that BEC is possible in 3 and 2 dimensions= and not possible in in 1 dimension.

2.2 Pauli Para magnetism

Pauli para magnetism arises due to the alignment of the spin magnetic moments of free electrons. Here we consider low temperature(absolute zero), low field para magnetism of metals or free electron gas. We assume that the electrons with dipole moment μ will be either parallel to the field B or anti parallel. We thus have two groups of particles in the gas:

1. Electrons having μ parallel to B, with energy $\frac{p^2}{2m} - \mu B$
2. Electrons having μ anti-parallel to B, with energy $\frac{p^2}{2m} + \mu B$

At absolute zero, all energy levels up to the Fermi level ϵ_F will be filled, while all levels beyond ϵ_F will be empty. Accordingly, the kinetic energy of the particles in the first group will range between 0 and (ϵ_F + μ B), while the kinetic energy of the particles in the second group will range between 0 and (ϵ_F-μ B). The respective numbers of particles in the two groups will, therefore, be equal to the number of levels and then will be equal to

$$N^+ = \frac{4\pi V}{3h^3}(p_F^+)^3;\ \ N^- = \frac{4\pi V}{3h^3}(p_F^-)^3$$

for 3 dimensions

$$N^+ = \frac{\pi A}{h^2}(p_F^+)^2\ ; N^- = \frac{\pi A}{h^2}(p_F^-)^2$$

for 2 dimensions and

$$N^+ = \frac{L}{h}p_F^+\ ; N^- = \frac{L}{h}p_F^-$$

for 1 dimension where $p_F^+ = [2m(\epsilon_F + \mu B)]^{\frac{1}{2}}$, $p_F^- = [2m(\epsilon_F - \mu B)]^{\frac{1}{2}}$, V is the volume, A is the area and L is the length of the material. The intensity of magnetization $M = \mu(N^+ - N^-)$ and using the expression for susceptibility $\chi = \lim_{B\to 0} = \frac{M}{VB}$ we get

$$\chi_{3D} = C_1(\epsilon_F)^{\frac{1}{2}}$$

$$\chi_{2D} = C_2$$

$$\chi_{1D} = C_3(\epsilon_F)^{-\frac{1}{2}}$$

We can see that at low magnetic field and at absolute zero Kelvin, Pauli para magnetism in 2 dimension is a constant independent of Fermi temperature which indicates that it is independent of the material which is indeed a curious result demanding more investigations on para magnetism.

2.3 Conclusions

In Coulombs law the factor $4\pi r^2$ comes because of the 3 dimensional nature. For all spherical or 3 D variation this term will be there. If we express Coulomb's law in other dimensions what will be its nature is not always discussed in regular class rooms or the dimensionality dependences in the fundamental laws of physics are not described in most of the textbooks. Maxwell equations, Lorentz force, Coulomb law, the Schroedinger equation and Newton law of universal gravitation in d spatial dimensions were obtained[4] by Masaki Hayashi and Kazuo Katsuura. One can recognize how the dimensionality of the world is reflected in these equations and laws. One problem that exists is the visualization of the extra dimensions. If extra dimensions exist, either they must be hidden from us by some physical mechanism or we do not have proper techniques to identify them. Studies point out a possibility that the extra dimensions may be "curled up" and hence invisible.

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