Relativistic equations of motion and the Newtonian limit

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Abstract

Working in an arbitrary inertial frame S, relativistic vector equations of motion of a particle for a constant acting force on it (as seen from S) are derived in cartesian coordintes following Newton's second law, and the equations are reduced to the well known kinematic Newtonian equations of motion under the non-relativistic limit.

1 Introduction

This article is dedicated to newcomers in special relativity and the aim is to derive the three vector equations of motion for a relativistic particle moving with instantaneous 3-velocity $\mathbf{v}$ as observed by some inertial frame S, and to rederive the famous kinematics equations by taking the non-relativistic limit which is $\lim c \to \infty$. The beginning point is Newton's second law: $\mathbf{F_S} = \dfrac{d\mathbf{p}}{dt}$, where $\mathbf{p}$ and $\mathbf{F_S}$ are the 3-momentum and the 3-force respectively as observed in frame S, and t is the coordinate time of S. The basic vector definitions of velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are taken as $\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$ and $\frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}$ respectively, where $\mathbf{r}$ and t are coordinate spatial seperation vector and coordinate time respectively for the frame S. Some common terminology associated to relativistic mechanics is used and is assumed that the reader is aware of it.

The vector expression of relativistic 3-momentum for a massive particle, which can be obtained from the general lorentz transformations in minkowski space-time background[1-4], is given by

$$\mathbf{p} = \frac{m_o \mathbf{v}}{\sqrt{1 - \frac{\mathbf{v.v}}{c^2}}} \tag{1}$$

where m_o is the rest mass of the particle and $\mathbf{v}$ is its 3-velocity, w.r.t S.

Assuming that a 3-force $\mathbf{F_S}$, as observed from S acts on the particle, Newton's second law gives,

$$\mathbf{F_S} = \frac{d}{dt}\frac{m_o \mathbf{v}}{\sqrt{1 - \frac{\mathbf{v.v}}{c^2}}} \tag{2}$$

Differentiating the above expression, one gets

$$\mathbf{F_S} = \frac{c^2\, m_o \mathbf{a} + m_o\, \mathbf{v} \times \mathbf{a} \times \mathbf{v}}{c^2 \left(1 - \frac{\mathbf{v.v}}{c^2}\right)^{3/2}} \tag{3}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the 3-acceleration. As one can notice, the vector expression for acceleration of the particle acted upon by this force $\mathbf{F_S}$ as seen from frame S is not a trivial one. However it is apparent that although acceleration is absolute for different inertial frames (i.e. if it is zero in one inertial frame, it is zero in every other inertial frame), its value changes from one to another[2-4]. We now derive the relativistic equations of motion for the frame S, assuming that the 3-force $\mathbf{F_S}$ is a constant.

2 Relativistic kinematic equations of motion

2.1 First

From (2) we have

$$\int d\frac{\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v.v}}{c^2}}} = \frac{\mathbf{F_S}}{m_o}\int dt \tag{4}$$

which gives

$$\frac{\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v.v}}{c^2}}} - \frac{\mathbf{u}}{\sqrt{1 - \frac{\mathbf{u.u}}{c^2}}} = \frac{\mathbf{F_S}}{m_o}(t - t_1) \tag{5}$$

on integration with the initial condition that at the instant $t = t_1$, the 3-velocity of the particle was $\mathbf{u}$. We set t_1 to zero. The equation can be elegantly written as

$$\mathbf{v}\gamma_v - \mathbf{u}\gamma_u = \frac{\mathbf{F_S}}{m_o}t \tag{6}$$

In the non-relativistic limit i.e. when $|\mathbf{u}|, |\mathbf{v}| \ll c$, it can be easily seen from (3) that acceleration is approximately a constant and is given by the expression $\mathbf{a} \approx \frac{\mathbf{F_S}}{m_o}$. From this approximation, we obtain the first Newtonian equation of motion

$$\mathbf{v} - \mathbf{u} \approx \mathbf{a}t \tag{7}$$

2.2 Second

From equation (6), $\mathbf{v}$ can easily be found to be equal to

$$\mathbf{v} = \frac{\mathbf{k}}{\sqrt{1 + \frac{\mathbf{k}.\mathbf{k}}{c^2}}} \tag{8}$$

where $\mathbf{k} = \frac{\mathbf{F_S}}{m_o}t + \mathbf{u}\gamma_u$.

From $\int d\mathbf{r} = \int \mathbf{v}dt$ and the above expression for velocity, we get

$$\int d\mathbf{r} = \int \frac{\frac{\mathbf{F_S}t}{m_o} + \mathbf{u}\gamma_u}{\sqrt{1 + \left(\frac{\mathbf{F_S}t}{m_o c} + \frac{\mathbf{u}\gamma_u}{c}\right)^2}} dt \tag{9}$$

Although the integration is trivial, the answer is a mess:

$$\mathbf{r} = \frac{c}{\sqrt{\mathbf{F_S}.\mathbf{F_S}}} \left[\mathbf{F_S}\sqrt{(t + m_o A)^2 + B} \right.$$
$$+ m_o (\gamma_u \mathbf{u} - A\,\mathbf{F_S}) \ln \left(\frac{t + m_o A}{\sqrt{B}} \right.$$
$$\left. \left. + \frac{\sqrt{(t + m_o A)^2 + B}}{\sqrt{B}} \right) \right] + \mathbf{k}'$$

where

$$A = \frac{\mathbf{F_S}.\mathbf{u}\gamma_u}{\mathbf{F_S}.\mathbf{F_S}}; \qquad B = m_o^2 \left(\frac{\gamma_u^2 c^2}{\mathbf{F_S}.\mathbf{F_S}} - A^2 \right) \tag{10}$$

and $\mathbf{k}'$ is a constant of integration found by plugging the initial seperation vector as $\mathbf{r} = \mathbf{r_1}$ at $t = t_1$ which we have set to zero. This is the required second relativistic equation of motion and describes a hyperbola in the r, t space[5]. The non-relativistic limit gives us back the regular second kinematic equation of motion:

$$\mathbf{r} \approx \mathbf{r_1} + \mathbf{u}t + \frac{\mathbf{a}}{2}t^2 \tag{11}$$

This describes a parabolic trajectory in the r, t space. The true hyperbolic trajectory has been approximated by a parabolic one in the non-relativistic limit.

2.3 Third

We begin with equation (2) and dot it with $d\mathbf{r}$ to give

$$d\mathbf{r}.\frac{d}{dt}(\gamma_v\,\mathbf{v}) = \frac{1}{m_o}(\mathbf{F_S}.d\mathbf{r}) \tag{12}$$

This is equivalent to

$$\mathbf{v}.d(\gamma_v\,\mathbf{v}) = \frac{1}{m_o}(\mathbf{F_S}.d\mathbf{r}) \tag{13}$$

For integration (note that it is a line integral), one can do integration by parts to get

$$|\gamma_v \mathbf{v}.\mathbf{v}|_{boundary} - \int \gamma_v \mathbf{v}.d\mathbf{v} = \frac{1}{m_o}\int \mathbf{F_S}.d\mathbf{r} \tag{14}$$

Now the boundaries are offcourse the final and initial velocities ($\mathbf{v}$ and $\mathbf{u}$) respectively for the left hand side and the final and initial positions ($\mathbf{r}$ and $\mathbf{r_o}$) respectively for the right hand side. The second integral is tivial (note that $\mathbf{v}.d\mathbf{v} = d(\mathbf{v}.\mathbf{v})/2$) and the whole expression turns out to be

$$m_o\, c^2 \left(\gamma_v^2 - \gamma_u^2\right) = \mathbf{F_S}.\Delta\mathbf{r} \tag{15}$$

where $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r_o}$. Here too, one can easily get the Newtonian limit by taking $c \to \infty$:

$$2\mathbf{a}.\mathbf{r} \approx \mathbf{v} - \mathbf{u} \tag{16}$$

which is the third non relativistic kinematic equation of motion.

3 Conclusion

These three relativistic equations of motions must be used when dealing with particles moving with velocities comparable to that of light.They can be efficiently reduced to Newtonian equations of motion when working with low velocities.

References

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