Doppler Shift of the de Broglie Waves- Some New Results from Very Old Concepts

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Abstract

The Doppler shift of de Broglie wave is obtained for fermions and massive bosons using the conventional form of Lorentz transformations for momentum and energy of the particles. A formalism is developed to obtain the variation of wave length for de Broglie waves with temperature for individual particles using the classic idea of Wien in a many body Fermi gas or massive Bose gas.

1 Introduction

It is well known that the pitch of an audible acoustic wave appears to change if there is a relative motion between the source and the observer. Such apparent change in frequency of sound waves when there is a non-zero relative velocity is known as the Doppler effect in the case of acoustics. Same kind of physical effect is also been observed in the case of optics with an apparent change in wavelength or color of the light. With simple theoretical calculation one can show that the effect is common to all kinds of travelling waves. However, to make appreciable change in frequency or wavelength for light waves the relative velocity must be a considerable fraction of the velocity of light. Since the speed of light is extremely high compared to the speed of any terrestrial object, it is difficult to detect any measurable or observable change in wavelength or color of the emitted light from any source having relative motion with respect to the observer. Whereas many of the stellar objects are moving with very high ve-

locity, therefore wavelength of light emitted from these heavenly bodies, moving either towards or away from the earth show measurable amount of blue or redshift respectively in the spectral lines. The possibility of shift in the position of a spectral line due to relative motion of the source and the observer was first pointed out by Doppler in the year 1842. From the observed red shift and blue shift of spectral lines, it has been reported that the stars like Sirius, Caster, Regulus are moving away from the earth, whereas, the stars like Arcturus, Vega and α Cygni are moving towards the earth. An extremely interesting application of the Doppler effect is to the discovery of spectroscopic binaries. The Doppler principle has also been applied to determine the nature of Saturn's ring.

Since the effect of relative motion of source and observer on apparent change in frequency or wavelength is common to all kinds of waves. We expect that the de Broglie waves will also be Doppler shifted if there is a relative motion between the emitter and the detector (see [1] for a very nice discussion). In this article, based on three old classic pieces of discoveries- the Doppler effect in the year 1842, the Wien's displacement law in the year 1893 and the matter wave or the de Broglie wave in the year 1923 [2, 3, 4], we shall study the Doppler shift of de Broglie waves associated with fermions or bosons inside many body Fermi system or Bose system respectively. In our investigation, instead of conventional photon or phonon gas, we have considered bosons of non-zero mass. Then using the classic idea of Wien we shall develop a formalism to express the variation of de

Broglie wavelength for the individual particles with the temperature of the system for both fermions and bosons. The relations may be called as the modified form of the Wien's displacement law for black body Fermi gas or Bose gas. To the best of our knowledge this problem has not been addressed before.

2 Doppler Shift of de Broglie Waves

We consider a many body quantum system consisting of either fermions or bosons. Then without the loss of generality, we may assume that just like a black body system of electromagnetic waves or photon gas, the system is essentially a gas of a large number of de Broglie waves of fermions or bosons. Hence we may call the system as a black body Fermi gas or Bose gas. We have further assumed that the collisions among the particles is elastic in nature. To obtain the change in wavelength for de Broglie waves due to Doppler effect, we start with the well known form of Lorentz transformations of particle momentum and energy. We assume two frame of references, S, the rest frame and the frame S' is moving with respect to S with a uniform velocity V along x-direction. We further assume that the motion of the particle is on $x - y$ plane. Then we have from the standard text book results [5], the Lorentz transformation for the x-component of parti-

cle momentum

$$p'_x = \gamma \left(p_x - \frac{VE}{c^2}\right) \ \text{ with } \ \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2} \tag{1}$$

where E is the energy of the particle in S-frame and c is the velocity of light. If θ and θ' are the angles subtended by the particle momenta $\vec{p}$ and $\vec{p'}$ with the x-direction in S and S' frames respectively, then we have from eqn.(1)

$$p' \cos\theta' = \gamma \left(p \cos\theta - \frac{VE}{c^2}\right) \tag{2}$$

and since $p'_y = p_y$, hence we can write $p' \sin\theta' = p \sin\theta$.

Now defining the de Broglie wavelengths in these two frames as $\lambda = h/p$ and $\lambda' = h/p'$, and using $E = (p^2 c^2 + E_0^2)^{1/2}$, the particle energy in S frame and a similar expression for E' in S' frame, with $E_0 = m_0 c^2$, the rest mass energy, we have

$$\frac{\lambda}{\lambda'} \cos\theta' = \gamma \left[\cos\theta - \frac{V}{c}\left\{1 + \left(\frac{\lambda}{\lambda_c}\right)^2\right\}^{1/2}\right] \tag{3}$$

where $\lambda_c = h/m_0 c$, the Compton wavelength. Then it can very easily be shown that the aberration is given by

$$\tan\theta' = \frac{\sin\theta}{\gamma \left[\cos\theta - \frac{V}{c}\left\{1 + \left(\frac{\lambda}{\lambda_c}\right)^2\right\}^{1/2}\right]} \tag{4}$$

Now it is a matter of simple algebra to verify that for the mass-less case when $\lambda_c = \infty$, the usual results for electromagnetic waves or photons can be obtained from the eqns.(3) and (4).

To obtain the Doppler shift of the de Broglie waves for the particles it is more convenient to start from the Lorentz transformation for the particle energy, given by

$$E' = \gamma(E - V p_x) \tag{5}$$

Squaring both the sides and using the definition of de Broglie waves in both S and S' frames, we have

$$\frac{\lambda}{\lambda'} = \gamma \left[1 - \frac{2V}{c}\cos\theta\left\{1 + \left(\frac{\lambda}{\lambda_c}\right)^2\right\}^{1/2} + \left(\frac{V}{c}\right)^2\left\{\cos^2\theta + \left(\frac{\lambda}{\lambda_c}\right)^2\right\}\right]^{1/2} \tag{6}$$

Again it is very easy to verify that the results for electromagnetic waves follow from here for $\lambda_c = \infty$.

It is also obvious that the transverse form

of Doppler shift with $\theta = \pi/2$ is non-vanishing in the case of matter waves and is given by

$$\frac{\lambda}{\lambda'} = \gamma \left[1 + \left(\frac{V}{c} \right)^2 \left(\frac{\lambda}{\lambda_c} \right)^2 \right]^{1/2} \qquad (7)$$

3 Variation of de Broglie Wavelength with Temperature

To obtain the variation of de Broglie wavelength with temperature, we consider either a Fermi gas or a Bose gas of non-zero mass in an enclosure, just like the black body chamber of a photon gas. For the sake of simplicity the enclosure is assumed to be spherical in nature. Further the wall of the enclosure is assumed to be a perfect reflector and moving outward adiabatically with a velocity V, where V is small enough compared to the velocity of light. The moving wall of the enclosure may be treated as S' frame, whereas the S frame is at rest inside the enclosure with a fictitious observer sitting there. The tangential plane at some arbitrary point on the outer surface of the wall is assumed to be in the $y - z$ plane. Then the normal drawn from the centre to this point of intersection is the x-direction. The particle which is hitting the wall at this point of intersection is as before assumed to be moving in $x - y$ plane. Then for a de Broglie wave of wavelength λ, the point of intersection on the moving wall at which it is hitting is equivalent to an observer moving away from the source, which is

radially outward along x-direction. As a consequence the received de Broglie wave at the moving wall will be red shifted and is given by

$$\frac{\lambda}{\lambda'} = \left[1 - \frac{2V}{c} \frac{\lambda}{\lambda_c} \cos\theta \right]^{1/2} \qquad (8)$$

where we have neglected the term $(V/c)^2$ and assumed that $\lambda \gg \lambda_c$ in the non-relativistic approximation. When the particle is reflected back from the point of incidence, since the wall is moving outward, it is equivalent to the emission from a source moving away from the observer. Therefore in this case also the de Broglie wave of the particle will be red shifted as observed from S frame. Combining these two effects, the relation between the final red shifted wavelength to that of the original one is given by

$$\frac{\lambda''}{\lambda} = \frac{\left[1 + \frac{2V}{c} \frac{\lambda''}{\lambda_c} \cos\theta \right]^{1/2}}{\left[1 - \frac{2V}{c} \frac{\lambda}{\lambda_c} \cos\theta \right]^{1/2}} \qquad (9)$$

Since $V/c \ll 1$, we have approximately

$$\frac{\lambda''}{\lambda} \approx \frac{\left[1 + \frac{V}{c} \frac{\lambda}{\lambda_c} \cos\theta \right]}{\left[1 - \frac{V}{c} \frac{\lambda}{\lambda_c} \cos\theta \right]} \qquad (10)$$

If we assume that the amount of final red shift is infinitesimal, the above relation may further be approximated to

$$\frac{\lambda''}{\lambda} \approx 1 + \frac{2V}{c} \frac{\lambda}{\lambda_c} \cos\theta \qquad (11)$$

Writing the final red shifted wave length $\lambda'' = \lambda + d\lambda$, we have the resultant infinitesimal change in wave length

$$d\lambda = \frac{2V}{c} \frac{\lambda^2}{\lambda_c} \cos\theta \qquad (12)$$

To eliminate the arbitrary angle of incidence θ, we consider multiple reflection of de Broglie waves from the inner wall of the enclosure. For spherical geometrical structure, with radius r, a de Broglie wave travels a distance $2r\cos\theta$ between two successive collisions. Therefore the number of reflections per unit time is $v/2r\cos\theta$. Hence the change in wavelength per unit time is

$$d\lambda = \frac{V}{c}\frac{\lambda^2}{\lambda_c}\frac{dr}{r} \qquad (13)$$

where we have assumed that the particle travels δr distance in unit time and in the limiting case it is dr.

Now from the first law of thermodynamics we have for an adiabatic change, $dQ = dU + PdV_0 = 0$. Hence we know for a non-relativistic Fermi or Bose gas $PV_0^{5/3}$ =constant.

Now with the standard results from the textbook on statistical mechanics [6], the energy density for free Fermi gas or Bose gas is given by

$$\epsilon = \frac{3}{2}\frac{kT}{\lambda_0^3}f_{5/2}(z) \quad \text{and} \quad \epsilon = \frac{3}{2}\frac{kT}{\lambda_0^3}g_{5/2}(z) \quad \text{respectively,} \qquad (14)$$

$$\text{where} \quad f_{5/2}(z) = \sum_{l=1}^{\infty}(-1)^{l-1}\frac{z^l}{l^{5/2}}, \quad g_{5/2}(z) = \sum_{l=1}^{\infty}\frac{z^l}{l^{5/2}}, \quad \lambda_0 = \frac{\hbar}{(2\pi m kT)^{1/2}}$$

$$\text{and} \quad z = \exp\left(\frac{\mu}{kT}\right) \qquad (15)$$

the fugacity, with μ, the chemical potential of the constituents. The functions $f_\nu(z)$ and $g_\nu(z)$ are the Fermi function and Bose function respectively.

Since for a many body Fermi system, whether it is electron gas in a piece of metal or inside white dwarfs or neutron matter inside neutron stars, the chemical potential μ is always non-zero. Therefore it is quite obvious that from eqns.(14) and (15) an analytical expression for energy density for a Fermi gas can not be obtained. We therefore approximate the Fermi gas by Boltzmann statistics.

Then it is very easy to show that

$$P = \frac{2}{3}\epsilon \quad \text{and} \quad P \propto \exp\left(\frac{\mu}{kT}\right)T^{5/2}$$

Hence from the adiabatic relation between pressure and volume, we have

$$\exp\left(\frac{2\mu}{5kT}\right)Tr^2 = \quad \text{constant} \qquad (16)$$

where we have used $V_0 = 4\pi r^3/3$, the volume of the enclosure at some instant. Taking log

of both the sides and then differentiating all the terms and finally using eqn.(13), we have

$$\left(-\frac{2\mu}{5k}\frac{1}{T^2}+\frac{1}{T}\right)dT = -2\frac{dr}{r} = -2\frac{c}{V}\frac{\lambda_c}{\lambda}\frac{d\lambda}{\lambda}$$
$$(17)$$

Now integrating and rearranging the terms and finally redefining λ and T as λ^* and T^*, we have

$$\frac{1}{\lambda^*} - \frac{1}{T^*} = \ln\left(\frac{T}{T_0}\right) \tag{18}$$

where

$$\frac{1}{\lambda^*} = \frac{2\lambda_c c}{V}\frac{1}{\lambda}, \quad \frac{1}{T^*} = \frac{2\mu}{5k}\frac{1}{T} \quad \text{and} \quad T_0 \text{ is a positive constant}$$

The above equation (eqn.(18)) looks like the equation for a thin lens, where the right hand side is the inverse of focal length. Now it is well known that at high temperature a Fermi gas behaves like a Boltzmann gas and then the de Broglie wavelengths for the particles become infinitely large. Therefore the right hand side must be negative, i.e., $T < T_0$. Hence we can say that eqn.(18) may be compared with the equation for the convex lens. The temperature at which a fermion behaves like a classical particle can be obtained from the numerical solution of the equation $T^* \ln(T/T_0) + 1 = 0$. At this temperature, the de Broglie wavelength becomes infinitely large and above this temperature eqn.(18) does not hold. The system behaves classically. As $T \longrightarrow 0$, the magnitude of the second term on the left hand side of eqn.(18) becomes infinitely large much before the right hand side goes to $-\infty$. Therefore we may conclude that as temperature decreases, the de Broglie wavelength also decreases. Which means that the quantum mechanical effect becomes more and more important. In the extreme case, at $T \longrightarrow 0$, the Wave length tends to zero. If one compares eqn.(18) with the equation for a convex lens, then it is quite obvious that $\lambda = \infty$ corresponds to the object on the first focal plane of the convex lens in the object space, for which the real image is formed at infinity. In the case of a Fermi gas the temperature at which this happens, the de Broglie wavelength becomes infinitely large. This is also the limiting temperature separating the temperature space into a quantum zone and a classical zone. Now it is well known that beyond the first focal plane away from the convex lens, the images are always real, which in the present scenario corresponds to classical picture. In the case of a convex lens the object space between the focal plane and the lens always produce virtual images. Same kind of picture is true here also. The temperature zone between the upper critical value and $T \longrightarrow 0$ is the quantum mechanical region. So the quantum mechanical region of

temperature in the present scenario corresponds to the object space producing virtual images in the case of a convex lens. There is perhaps nothing wrong in such comparison of quantum mechanical temperature zone with the object space produce virtual images. In the quantum mechanical zone, because of uncertainty principle the exact location of the particle can not be predicted, only the probability of existence at a point can be obtained from the wave function of the particle. Therefore grossly speaking, a cloudy picture will be observed instead of a real location of the particle.

We next consider a massive Bose system. It may be a $\pi^+ - \pi^-$ matter or a neutral pion matter or even a system of extremely rarefied cold atoms. Since for these bosonic systems there is as such no conserved quantum number, the chemical potential for the constituents are exactly zero. Then we have from eqn.(18) (Now in the case of bosons, the chemical potential $\mu = 0$, therefore energy density can be obtained from the second expression as given by eqn.(14). The series $g_{5/2}(1)$ can be expressed in terms of known Zeta-function)

$$\frac{1}{\lambda} = \frac{V}{2\lambda_c c} \ln\left(\frac{T}{T_0}\right) \ \ \text{or} \ \ \lambda \ln\left(\frac{T}{T_0}\right) = \ \text{constant}$$

(19)

This is again the form of Wien's displacement law for the massive Bos gas. From the nature of the above equations and the properties of massive bosons, one can infer that T_0 is the minimum value of temperature for a Bose system at which $\lambda = \infty$. In this crude model calculation we may say that this is the tempera-

ture for Bose condensation of the gas. Therefore, T_0 in eqn.(18) and eqn.(19) are carrying quite different physical meaning. Since in the condensed phase all the bosons occupy the same quantum state, the spatial coherence length will be large enough in the atomic scale. In this simple model it is reflected by the extremely large value of de Broglie wavelength, which is large enough in the quantum scale. Then as the temperature increases, the system becomes more and more incoherent because of the randomness, and at very high temperature the system becomes a classical gas. Of course with this crude model calculation this can not be shown. For mass-less bosons, i.e. with $m = 0$, since $\lambda_c = \infty$, the above equation can not predict the condensation temperature. Which is already known for photon gas and phonon gas.

4   Conclusions

It is therefore quite surprising that based on three very old classic pieces of discoveries- the Doppler effect in the year 1842, the Wien's displacement law in the year 1893 and the matter wave or the de Broglie wave in the year 1923, it is possible to predict the variation of de Broglie wavelength with temperature for individual fermions and bosons in a many particle system. It is also possible to obtain the temperature beyond which the fermionic system behaves classically, the critical temperature for Bose condensation for massive Bose gas and also one can conclude that for mass-less bosons the critical temperature of Bose condensation can not be pre-

dicted.

References

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