Velocity Change Calculation for an Object Moving on a Rotating Spherical Surface

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Abstract

If an object is constrained to move on the surface of a sphere that is rotating at constant angular velocity, it is well known that within the rotating frame Coriolis and centrifugal accelerations appear. Assuming that there are no physical forces acting on the object that are directed tangentially, one might suppose that the change in velocity of the object over a finite time interval could thus be determined by integrating the sum of the Coriolis and centrifugal accelerations over that time interval. This procedure, however, does not yield the correct value for the velocity change. Owing to performing the calculation in spherical coordinates, additional acceleration contributions must also be included. The constraint forces cannot be the sources of these additional contributions, because the constraints are directed radially, while the additional accelerations are directed tangentially. The required additional acceleration terms are derived from first principles in a manner that would be suitable for presentation in either an undergraduate or graduate mechanics course.

1. Introduction

The problem of interest is the calculation of the components of the velocity change that occur over a finite time interval for an object moving on the surface of a rotating sphere, as determined within a reference frame that is rotating with the sphere. The object is constrained to remain on the sphere's surface, and it is assumed no physical forces act on the object in the tangential direction. For definiteness, the sphere is assumed in example calculations to be similar to the Earth. However, the sphere is assumed to always remain exactly spherical, and there is no atmosphere. The sphere is assumed to complete one full rotation is exactly 24 hours, and to have a radius of exactly 4000 miles. The only physical forces that are present are radially directed, and include gravity and the normal force of the spherical surface on the object. It is well known that under the conditions of interest, fictitious accelerations arise within the rotating frame. These are the Coriolis and centrifugal accelerations. One might suppose that since these two fictitious accelerations are the only ones that arise, it should be possible to compute the desired velocity change components simply by integrating the components of these two fictitious accelerations over the time interval of interest. This procedure, however, does not produce the correct result. This fact can be valuable to discuss with students when presenting the Coriolis concept, and to include in more general discussions of noninertial frames.

There has been much work done in the area of noninertial frames [1-10]. Reference 1 considers the problem of an object moving on the Earth's surface in the absence of tangential physical forces. However, the problem was solved numerically, not analytically. Here the exact analytical solution is presented, although for a less general case. The problem is also considered analytically in Ref. 2, including radial motion. The calculation of velocity changes was also considered in Ref. 2, although the results given there are expressed in approximate expansions, not the exact analytical results given here. It is worthwhile noting that the need to include accelerations other than Coriolis and centrifugal were also found to be required in Ref. 2. This result is termed a "curvilinear effect."

The behavior of an object in a rotating two-dimensional frame is considered in Ref. 3. It can be worthwhile for students to consider the methods of this reference before moving on to the more difficult problem of motion on a curved surface. The problem of motion in a spherical noninertial frame is considered in Ref. 4 through Ref. 9. These references consider the more difficult case in which the motion in the radial direction is unconstrained. They also consider the presence of additional forces such as air resistance. The work presented in Ref. 10 is worthwhile because it describes demonstrations and experiments that can be done in the classroom and also used in a physics laboratory course.

In section 2 the approach used in determining the velocity change calculation of interest is described, and the two coordinate systems used here are presented. The equations of motion in the rotating frame are given in section 3. The expressions for the velocity change components over a finite time interval are developed in section 4, and thus the principal results of this work are contained in this section. The results of section 4 are discussed in section 5 in terms of a simple example that illustrates why integrating over only the Coriolis and centrifugal accelerations produces the wrong result. A more complicated special-case example is described and the exact analytical solution of it are given in section 6. Numerical examples are given in section 7 and the conclusion is given in section 8.

2. Coordinate Systems

The approach used here involves analyzing the equations of motion in the rotating spherical reference frame to aid in the calculation of the velocity change over a finite time interval. Starting from the well-known forms of these equations that include Coriolis and centrifugal acceleration terms, the desired change in velocity is calculated. It is then shown that a direct integration of the sum of the Coriolis and centrifugal accelerations produces a result that differs from that obtained by direct integration of the equations of motion.

Two coordinate systems are used in the analysis. These are termed the "unprimed" and "primed" coordinate systems. The systems are depicted in Fig. 1.The unprimed coordinate system is an inertial frame at rest with respect to the fixed stars. The Cartesian coordinates of this system are denoted (x, y, z). Not shown is a related unprimed spherical coordinate system (r, θ, ϕ). However, the angle ϕ of this system is depicted, and expresses the angle between the x and x' axes, and between the y and y' axes. The primed system also consists of Cartesian and spherical coordinates as shown. However, the primed system rotates about the common z, z' axes at constant angular speed $\dot{\phi} = \omega$.

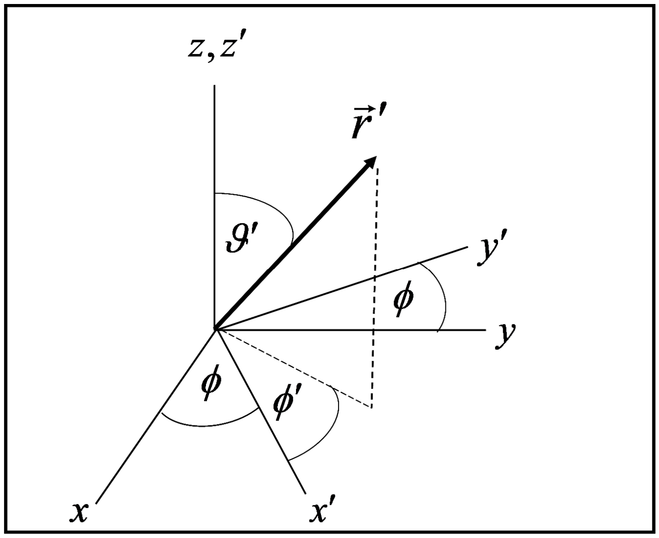

Fig 1 Primed and unprimed coordinate systems

3. Equations of Motion

In the primed coordinate system, the well-known equations of motion of an object moving on the surface with no physical tangential forces can be expressed as

$$R\ddot{\theta}' - R\dot{\phi}'^{2} \sin\theta' \cos\theta' = [a_{COR}]_{\theta'} + [a_{CENT}]_{\theta'} \qquad (1)$$

and,

$$R\ddot{\phi}' \sin\theta' + 2R\dot{\theta}'\dot{\phi}' \cos\theta' = [a_{COR}]_{\phi'}. \qquad (2)$$

Here, the notations $[a_{COR}]$ and $[a_{CENT}]$ denote the Coriolis and centrifugal accelerations, respectively. It is assumed that $r' = R$, the constant radius of the sphere. The Coriolis acceleration is $[\vec{a}_{COR}] = -2\vec{\omega}\times\vec{v}'$ and the centrifugal acceleration is $[\vec{a}_{CENT}] = -\vec{\omega}\times(\vec{\omega}\times\vec{r}')$, with $\vec{v}'$ denoting velocity of the object of interest in the primed frame. These are expressed in component forms in Eq. (3) through Eq. (5)

$$[a_{COR}]_{\theta'} = 2\omega R\dot{\phi}' \sin\theta' \cos\theta', \quad (3)$$

$$[a_{COR}]_{\phi'} = -2\omega R\dot{\theta}' \cos\theta', \quad (4)$$

and,

$$[a_{CENT}]_{\theta'} = R\omega^2 \sin\theta' \cos\theta'. \quad (5)$$

It should be noted that $[a_{CENT}]_{\phi'}$ is zero.

4. Velocity Change Calculation

Next, expressions are developed for determining the velocity changes of the object moving on the surface over a finite time interval attributable to each of the acceleration types that appear in the problem. Students may think that these velocity changes could be computed by simply integrating the Coriolis and centrifugal accelerations over the time interval of interest. However, this is not sufficient, so the following considerations would be worthwhile addressing in class.

First, it is worthwhile defining what is meant by "velocity change" here. The velocity vector in the spherical coordinates of the primed coordinate system, assuming no radial motion, takes the form

$$[\vec{v}']_{\theta',\phi'} = R\dot{\theta}'\hat{\theta}' + R\dot{\phi}'\sin\theta'\hat{\phi}', \quad (6)$$

with $\hat{\theta}', \hat{\phi}'$ being the usual spherical unit vectors. The velocity change of interest is taken here to be the change in the numerical value of the coefficients of the unit vectors in Eq. (6) that occur during a finite time interval. It does not include changes in the unit vectors themselves. Thus, the velocity change considered here represents the change in velocity in a given compass direction. For example, the numerical value of the coefficient of $\hat{\theta}'$ in Eq, (6) is the velocity in the northerly (or southerly) direction. As the object moves to a different latitude over the time interval of interest, the absolute meaning of "northerly" clearly changes. Nonetheless, if the object has a velocity component of 1000 miles/hour in the northerly direction at the start of the time interval, then reduces to a velocity of 800 miles/hour in the northerly direction at the end of the interval, the velocity change will be -200 miles/hour in the northerly direction, or 200 miles/hour in the southerly direction. It is not of concern here that the northerly direction points in a different direction in absolute space.

Referring to Eq. (6), it can be seen that in terms of components, the velocity changes of interest can be expressed as

$$\Delta[\vec{v}']_{\theta'} = \Delta[R\dot{\theta}'] = [R\dot{\theta}']_0^t, \quad (7)$$

and,

$$\Delta[\vec{v}']_{\phi'} = \Delta[R\dot{\phi}'\sin\theta'] = [R\dot{\phi}'\sin\theta']_0^t. \quad (8)$$

Here, Δ has the usual meaning of "change," and the "*0*" and "*t*" notations on the right-hand sides of the final brackets of these equations indicate the time interval of interest. Of course the lower limit need not be zero, but that is the case of interest here, and there is no loss in generality in restricting the lower limit to be zero.

The velocity changes of Eq. (7) and Eq. (8) can be re-expressed using the equations of motion as given by Eq, (1) and Eq.(2). Determining the velocity change in the θ' direction initially involves simply rearranging Eq. (1) to isolate the $R\ddot{\theta}'$ term on the left-hand side and integrating over the time interval $(0,t)$, giving

$$\int_0^t R\ddot{\theta}'\,dt = \int_0^t R\dot{\phi}'^2 \sin\theta'\cos\theta'\,dt + \int_0^t [a_{COR}]_{\theta'}\,dt + \int_0^t [a_{CENT}]_{\theta'}\,dt \,. \qquad \textbf{(9)}$$

Carrying out the integral on the left-hand-side of Eq. (9) and comparing with Eq. (7) gives

$$\left[R\dot{\theta}'\right]_0^t = \Delta[\vec{\mathrm{v}}']_{\theta'} = \int_0^t R\dot{\phi}'^2 \sin\theta'\cos\theta'\,dt + \int_0^t [a_{COR}]_{\theta'}\,dt + \int_0^t [a_{CENT}]_{\theta'}\,dt. \qquad \textbf{(10)}$$

It is immediately apparent from Eq.(10) that the velocity change $\Delta[\vec{\mathrm{v}}']_{\theta'}$ cannot be determined simply by summing the time integrals of the Coriolis and centrifugal accelerations, since an additional term has appeared. This term arises from the second term on the left-hand-side of Eq. (1).

Determining the velocity change in the ϕ' direction involves using Eq.(2). The term $R\ddot{\phi}'\sin\theta'$ is first isolated on the left-hand-side, and again integrating both sides of the resulting equation over the interval $(0,t)$, gives

$$\int_0^t R\ddot{\phi}'\sin\theta'\,dt = -\int_0^t 2R\dot{\theta}'\dot{\phi}'\cos\theta'\,dt + \int_0^t [a_{COR}]_{\phi'}\,dt\,. \qquad \textbf{(11)}$$

It is next useful to apply integration-by-parts to the integral on the left-hand-side of Eq. (11), as shown in Eq. (12)

$$\int_0^t R\ddot{\phi}'\sin\theta' dt = \left[R\dot{\phi}'\sin\theta'\right]_0^t - \int_0^t R\dot{\phi}'\dot{\theta}'\cos\theta' dt \,. \qquad \textbf{(12)}$$

It can be seen that the first term on the right-hand-side of Eq. (12) is precisely the desired velocity change in the ϕ' direction as expressed by Eq.(8). Thus, substituting the right-hand-side of Eq. (12) for the left-hand-side of Eq. (11) and rearranging terms gives

$$\left[R\,\dot{\phi}'\sin\,\theta'\right]_0^t = -\int_0^t R\,\dot{\theta}'\dot{\phi}'\cos\,\theta'\,dt + \int_0^t \left[a_{COR}\right]_{\phi'}dt\,. \tag{13}$$

Again, comparing the left-hand-side of Eq. (13) with Eq. (8) gives,

$$\Delta\left[\vec{\mathrm{v}}'\right]_{\phi'} = -\int_0^t R\,\dot{\theta}'\dot{\phi}'\cos\,\theta'\,dt + \int_0^t \left[a_{COR}\right]_{\phi'}dt\,. \tag{14}$$

Again, keeping in mind that the centrifugal contribution in the ϕ' direction is zero, it is immediately apparent from Eq.(14) that the time integral of Coriolis and centrifugal terms alone is insufficient for computing the velocity change $\Delta\left[\vec{\mathrm{v}}'\right]_{\phi'}$, since once again another term has appeared. The new term arises from both the first and second terms on the left-hand-side of Eq. (2).

Introducing some new notation into Eq. (10) and Eq, (14) gives for the desired velocity changes

$$\left[\Delta\vec{\mathrm{v}}'\right]_{\theta'} = \int_0^t \left[a_{COR}\right]_{\theta'}dt + \int_0^t \left[a_{CENT}\right]_{\theta'}dt + \int_0^t \left[a_{KIN}\right]_{\theta'}dt\,, \tag{15}$$

and,

$$\left[\Delta\vec{\mathrm{v}}'\right]_{\phi'} = \int_0^t \left[a_{COR}\right]_{\phi'}dt + \int_0^t \left[a_{KIN}\right]_{\phi'}dt\,. \tag{16}$$

Here the notation $\left[a_{KIN}\right]$ is introduced as a way to allow convenient reference to the terms that appear in the velocity change calculations that are neither Coriolis nor centrifugal accelerations, with the subscript intended to mean "kinematic." The term kinematic is used because these terms appear purely due to describing the motion in a spherical reference frame. The components of the kinematic acceleration are given by

$$\left[a_{KIN}\right]_{\theta'} = R\,\dot{\varphi}'^2\sin\theta'\cos\theta', \tag{17}$$

and

$$\left[a_{KIN}\right]_{\phi'} = -R\,\dot{\theta}'\,\dot{\varphi}'\cos\theta'. \tag{18}$$

One aspect of the kinematical components that distinguishes them from the Coriolis and centrifugal components is that they do not depend explicitly on the angular velocity of rotation, but only implicitly through their dependence on the coordinates and their time derivatives.

It should be noted that similar non-Coriolis and non-centrifugal acceleration terms were described previously [2]. The previous work introduced approximate expansions for the additional terms, and the terminology "curvilinear effects" was used to describe these terms.

5. Discussion

To understand why integrating the sum of the Coriolis and centrifugal accelerations does not produce the true velocity change as seen in the rotating frame, it is helpful to consider first an object at rest in the unprimed, or absolute rest, frame. Suppose further that such an object is positioned at some latitude between the equator and the northern pole, and that the object in question is in contact with the surface of the sphere. Bearing in mind that the contact point is assumed to be frictionless, the object will remain perpetually motionless in the inertial frame, with the gravitational force and the normal force of the surface on the object exactly canceling. Since the object is at rest and is unaccelerated in the unprimed frame, it follows that in the primed frame $\dot{\vartheta}' = 0$ and $\dot{\phi}' = -\omega$. That is, in the rotating frame the object is seen to be moving in the direction opposite to that of the surface and with a rotational speed equal in magnitude to that of the surface, but oppositely directed.

Referring now to Eq. (4), and also recalling that $\left[a_{CENT}\right]_{\phi'}$ is zero as noted following Eq. (5), it is seen that the vanishing of $\dot{\vartheta}'$ guarantees that the contributions of the Coriolis and centrifugal accelerations to the acceleration component $\left[a\right]_{\phi'}$ in this example is zero. The kinematic contribution as given by Eq. (18) is also seen to be zero. Thus, there is zero net acceleration in the ϕ' direction. This is not surprising, since the object in question is stationary in the absolute inertial frame, so it makes sense that its acceleration in the rotating frame would vanish. However, in the θ' direction the net contribution to the acceleration due to the Coriolis and centrifugal accelerations alone is not zero. This can be seen by setting $\dot{\phi}'$ equal to $-\omega$ in Eq. (3) and adding the resulting Coriolis acceleration to the centrifugal acceleration determined by Eq. (5), giving

$$\left[a_{COR} + a_{CENT}\right]_{\theta'} = -R\omega^2 \sin\theta' \cos\theta' . \qquad \textbf{(19)}$$

This acceleration in the θ' direction (directed away from the equator), which is nonzero for all test locations except the equator and the poles, conflicts with the known motionlessness as seen from the inertial, or unprimed, frame. This difficulty is resolved if the kinematic contribution, as given by Eq. (17), is added to the result shown in Eq. (19). Again setting $\dot{\phi}' = -\omega$ in Eq. (17) gives for the kinematic contribution

$$\left[a_{KIN}\right]_{\theta'} = R\omega^2 \sin\theta' \cos\theta' . \qquad \textbf{(20)}$$

Including the kinematic contribution, that is, adding the results given by Eq. (19) and Eq. (20) together, gives a net zero acceleration in the θ' direction, in agreement with the known object behavior in the inertial frame. The only way to reconcile the motionlessness in the inertial frame with the behavior in the θ' direction of the noninertial frame is through the inclusion of the kinematic contribution.

6. Example Problem and Analytical Solution

A specific example will next be considered, and its exact analytical solution presented. The problem is illustrated in Fig. 2.

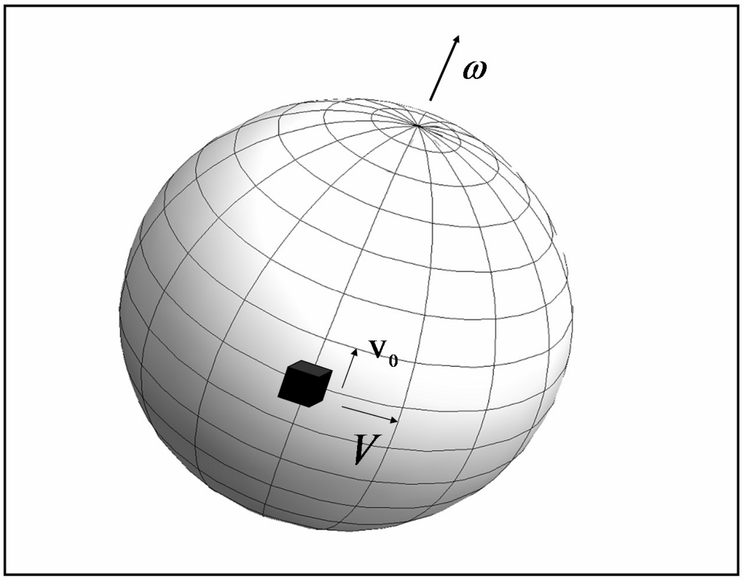

Fig 2 A sphere of radius *R* rotates uniformly at angular speed ω.

The block shown is in frictionless contact with the surface of the sphere of radius *R*. The sphere is rotating at constant angular speed ω, and the block is assumed to be constrained to remain on the surface so that the radial speed of the object is always zero. The sphere is assumed to have properties approximating those of the Earth and, as such, the pole appearing at the top of the illustration is taken to define "north," while the direction of rotation is taken to be from the "west" and toward the "east." The velocities V and v_0 are the initial velocity components as seen in the unprimed, or inertial, frame of Fig.1. At $t = 0$ the block is located at the equator, and appears to be motionless in the east-west direction from the perspective of the rotating frame. Thus, initially, $V = R\omega$. The block is seen to have the same initial northward speed v_0 in both the primed and unprimed coordinate systems.

The initial conditions to be satisfied by the solutions of Eq.(1) and Eq. (2) are

$$(\theta', \phi')_{t=0} = \left(\pi/2, 0\right),\ \dot{\theta}'_{t=0} = -v_0/R, \text{ and } \dot{\phi}' = 0.$$

The methods of Great Circle analysis [2] are effective and straightforward for obtaining the solutions to this problem in the unprimed frame. And the solutions in the primed frame can be obtained straightforwardly from the solutions in the primed frame using appropriate rotation matrices. In summary, when an object is given an initial velocity at any point on the surface of the sphere, from the external inertial frame the resulting motion can be described as that of an object executing uniform circular motion about a

Great Circle. In the present case, the object moves along the Great Circle at the constant speed $\sqrt{v_0^2+V^2}$. The Great Circle is located in a plane that is tilted with respect to the plane of the equator at by the tilt angle $\arctan\left(\frac{V_0}{V}\right)$.

The tilt angle is taken about the x axis of Fig. 1, and is rotated from the +y axis toward the –y axis.

Transforming the solutions of Eq.(1) and Eq.(2) obtained from the Great Circle methodology subject to the given initial conditions into the primed frame produces the results that are presented in Eq. (21) and Eq, (22)

$$\theta'(t) = \arccos\left[\frac{v_0}{\sqrt{v_0^2+V^2}}\sin\left(\frac{t\sqrt{v_0^2+V^2}}{R}\right)\right], \qquad \textbf{(21)}$$

and

$$\phi'(t) = \arctan\left[\frac{V}{\sqrt{v_0^2+V^2}}\tan\left(\frac{t\sqrt{v_0^2+V^2}}{R}\right)\right] - \omega t. \qquad \textbf{(22)}$$

Here again, $V = R\omega$. It is straightforward to verify these solutions satisfy Eq. (1) and Eq. (2) by direct substitution. Verifying that all four initial conditions are satisfied is also straightforward. Students can derive these solutions as an assignment or they can be derived in class. Carrying the calculations through is not difficult using a symbol manipulator such as Maple™ or Mathematica™. Solution verification is also straightforward using this kind of software.

Once these solutions have been presented, it would be worthwhile to point out to the students the advantage of such analytic representations relative to a purely numerical solution [1]. Numerical solutions are subject to error due to improperly chosen step size or insufficient numbers of digits used to represent the numbers being manipulated. There is even a possibility of bugs in the numerical software. These concerns are allayed with an available analytical solution. Calculations based on applying the formulas are also possible that could only be performed by trial and error using a purely numerical approach.

7. NUMERICAL CALCULATIONS

Two numerical examples are considered. In the first, the block has the initial northerly speed $V_0 = 50$ miles/hour. This is perhaps a realistic speed that may appear in the movement of the atmosphere. However, much of the interesting physics is only seen when the initial speed V_0 has an extreme value. Thus, in the second example the block has an initial speed of $V_0 = 5000$ miles/hour.

Considering the case where $V_0 = 50$ miles/hour, it should first be mentioned that the object will reach a maximum northerly distance from the equator, reverse direction, then eventually return to the equator and will then proceed to move into the southern hemisphere. One calculation of interest is the maximum northerly distance from the equator reached by the object. At this position $\dot{\theta}' = 0$. Differentiating Eq. (21) with respect to time determines $\dot{\theta}'$, and the first root of this expression after *t = 0* determines the first time when the northerly motion turns toward the south. The maximum northerly displacement is then determined from the expression $R\left(\pi/2 - \theta'\right)$, where θ' is computed using Eq. (21) evaluated at the just-determined time where $\dot{\theta}' = 0$. The maximum northerly displacement produced by this procedure for the current example is approximately 190.84 miles.

Since the object then reverses its direction of motion in the θ' direction, it is evident that the initial northerly speed of 50 miles/hour has been reduced to zero by the three accelerations operating in the θ' direction. Also of interest is the amount of velocity reduction contributed by each of these accelerations. This is determined by separately evaluating each of the three integrals appearing in Eq. (15), again using for the upper limit the just-determined time where $\dot{\theta}' = 0$. The results are $\left[\Delta v_{COR}\right]_{\theta'}$ = -0.151429 miles/hour, $\left[\Delta v_{CENT}\right]_{\theta'}$ = -49.8484 miles/hour, and $\left[\Delta v_{KIN}\right]_{\theta'}$ = -0.000138051 miles/hour, where the subscripts on the velocity-change symbols correspond to the subscript conventions for each acceleration appearing in Eq. (15). It is evident that the kinematic contribution is negligible in this case, which is one reason an extreme example is also considered.

One final calculation of interest in the 50 miles/ hour case is the maximum easterly speed reached by the object due to each of the two accelerations acting in that direction, and the contributions to that speed change by each acceleration component. The maximum easterly speed is obtained by evaluating the integrals appearing in Eq. (16). The upper time limit of these integrals is determined by finding the location of the first maximum of the function $R\dot{\phi}'\sin\theta'$ after *t = 0*. It is found that this first maximum occurs at the same time at which the maximum northerly displacement is reached, that is, when $\dot{\theta}' = 0$. To understand why this is so, it is helpful first to realize that when the function $R\dot{\phi}'\sin\theta'$ is differentiated with respect to time, one of the two resulting terms contains a factor of $\dot{\theta}'$ and the other term contains a factor of $\ddot{\phi}'$. By rearranging Eq. (2), $\ddot{\phi}'$ can be isolated on the left-hand side. Doing this, it is seen that all terms on the right hand side of the resulting equation contain a factor of $\dot{\theta}'$. It follows that the derivative of $R\dot{\phi}'\sin\theta'$ with respect to time vanishes when $\dot{\theta}'$ is zero.

The results of carrying out the integrals in Eq. (16) are that the maximum easterly speed obtained by the object is 2.38461 miles/hour. The separate contributions to the result are $\left[\Delta v_{COR}\right]_{\phi'}$= 2.38325 miles/hour and $\left[\Delta v_{KIN}\right]_{\phi'}$= 0.00135752 miles/hour.

Although the example involving $v_0 = 50$ miles/hour is interesting since it considers a speed that is perhaps comparable to atmospheric winds, that example does not reveal all the interesting physics of the problem. We thus consider now a second example where $v_0 = 5000$ miles/hour. This example differs substantially from the 50 mile/hour case. Since the initial velocity is so great, rather than computing the maximum northerly displacement it is more interesting to compute the distance of closest approach to the northern pole. Considering the large starting speed, it might be supposed that the object would closely approach the pole. It may be surprising then to learn that the distance of closest approach in this case is 825.82 miles, still a considerable distance away.

As with the first example, it is also interesting to compute how much velocity change is contributed by each of the three accelerations in reducing the initial northerly velocity to zero. The results are $\left[\Delta v_{COR}\right]_{\theta'} = -742.85$ miles/hour, $\left[\Delta v_{CENT}\right]_{\theta'} = -115.28$ miles/hour, and $\left[\Delta v_{KIN}\right]_{\theta'} = -4141.87$ miles/hour. Thus unlike the previous example where the kinematic contribution was negligibly small, in the current case the kinematic contribution is rather dominant.

The maximum easterly speed and the contributions of each of the two accelerations that involved in attaining it are also of interest. The maximum easterly speed is 4893.82 miles/hour. The contributions of the individual accelerations are $\left[\Delta v_{COR}\right]_{\phi'} = 1665.06$ miles/hour and $\left[\Delta v_{KIN}\right]_{\phi'} = 3228.76$ miles/hour. So again unlike the previous example, the kinematic contribution dominates, though not as significantly as with the northerly velocity change.

8. CONCLUSION

The problem of computing the components of the velocity change in the rotating frame of an object moving on the surface of a rotating sphere under conditions of no physical tangential forces was solved exactly. In the rotating frame the well-known Coriolis and centrifugal fictitious accelerations appear. However, it was shown that integrating only these two acceleration terms over a finite time interval does not yield the correct result for the determination of the velocity change. The additional terms that must be included in the calculation were herein termed the "kinematic" acceleration for the purpose of easy reference. In the two numerical examples that were considered, it was found that the kinematic contribution is negligibly small when an initial velocity comparable to atmospheric winds was considered, but was found to be dominant when an extremely high initial velocity was considered. Such an extremely high atmospheric velocity may be realistic for some exoplanet that has a substantially thinner atmosphere than that of the Earth. Even within the solar system, very high wind speeds have been observed. For example, a wind speed of approximately 1300 miles/hour has been seen on Neptune [11].

It should be emphasized that the concept of kinematic acceleration introduced here is not some "new" acceleration term that has somehow been missed by previous workers in the area of spherical rotating reference frames. The physics of the problem is completely contained in the equations of motion as given in Eq.(1) and Eq. (2). These equations have been known at least since the time of Coriolis. However, if one is interested in the calculation of the change of velocity over a finite time interval, and one then asks the question, "How much of the velocity change is due to each acceleration component (?)," one may very well be surprised to find that the Coriolis and centrifugal contributions sum up to less than 100% of the change. One is then likely to ask, "Where does the rest of the velocity change come from?" It is in answering this last question where the concept of the kinematic acceleration is useful.

It is hoped that the results presented here will be of use to those who teach subjects related to rotating frames. The velocity change calculation of interest and the example problem considered were capable of exact solution using relatively elementary methods. Such exactly solvable problems are rare.

ACKNOWLEDGMENT

The drawing of the sphere shown in Fig. 2 was produced using *Mathematica*, from Wolfram Research.

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