Vector Addition in Physics

Leonid Minkin[1], Alexander S. Shapovalov[2]

[1]*Department of Physics, Portland Community College, Portland OR, USA,*
www.pcc.edu/staff/lminkin
[2]*Department of Physics, Saratov State University, Saratov, Russia*

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Abstract

Introductory physics textbooks usually give some information about vectors and rules for their addition (triangle rule) and multiplication. In mathematics, addition of vectors is a matter of definition while in physics, in some cases, it must be done based on the meaning of addition and experimental confirmation. In physics, the triangle rule might be worthless if the meaning of addition is not strictly defined.

Keywords: vectors' addition, triangle rule, polar and axial vectors, sliding and bound vectors, center of boyuoncy, superposition

Introduction

Many fundamental principles of physics are presented in vector form. In mathematics, a vector is invariant with respect to rotation of coordinate axes and displacement from origin. Due to this property of vectors, the equations which express physics laws in vector form do not depend on the choice of inertial system of coordinates. This is the reason why it is simple and convenient to present physics laws in vector form. However, using vectors raises questions of transformation of different vector quantities under transition from one inertial (and non-inertial) system to another. The triangle rule is a mathematical rule for addition of two vectors. Nevertheless, this rule should be used with caution because in some cases, the physical meaning of vector addition must be first clarified.

In mechanics textbooks, the triangle rule of vector addition, including addition of velocities and accelerations, is usually described. This formal rule is applicable in Newtonian mechanics – for non-relativistic motion in inertial frames. On the contrary, in relativistic kinematics, where physical principles of measurements of length and time must be specified, the triangle rule of vector addition is not valid and the rule of vector addition should be based on physical principles of the special theory of relativity. In non-relativistic mechanics, the triangle rule for addition of accelerations in the frame rotating with respect to inertial one, in general, is not valid either.

The triangle rule is not always operational for addition of parallel vectors. For example, for addition of two collinear angular velocities, the location of instantaneous axis of rotation and magnitude of sum of angular velocities cannot be found based only on principles of vector algebra. Physical principles of kinematics must be used. Similar situation arises for addition of pressure forces applied to the object submerged in fluid. In this case, rules of vector algebra are not enough to find the point of application of the resultant of these forces – physical principles must be used.

Some examples, which illustrate the statements given in the abstract and introduction sections, are presented below.

Specifics of Vector Addition in Physics

1. Let us consider stationary frame K_1 and frame K, moving with the constant velocity $\boldsymbol{u}$ with respect to frame K_1. Imagine that the point object is moving with velocity $\boldsymbol{v}$ with respect to frame K. What is the velocity of the point object, $\boldsymbol{v_1}$, with respect to frame K_1? In this case, the motion of the object is considered in two different systems and clocks and rulers in different frames measure velocities. In classical mechanics, addition of velocities follows the triangle rule (mathematical vector addition) and $\boldsymbol{v_1} = \boldsymbol{v} + \boldsymbol{u}$, while in relativistic kinematics this rule cannot be applied ($\boldsymbol{v_1} \neq \boldsymbol{v} + \boldsymbol{u}$). Addition is based on the experimentally confirmed principles of special relativity theory. In one-dimensional case

$$v_1 = \frac{v+u}{1+\frac{vu}{c^2}} \qquad (1)$$

Here, c is the speed of light in vacuum.

2. In non-relativistic classical mechanics, addition of vectors of acceleration, in general, does not follow the triangle rule. Let the frame K be moving with acceleration $\boldsymbol{a}$ with respect to frame K_1. Imagine that a point like object is moving with acceleration a' with respect to frame K. In this case, the acceleration, $\boldsymbol{a_1}$, of the moving point in the system K_1 can be found by using the triangle rule only if both motions (K system and the point with respect to K frame) are translational. In this case $\boldsymbol{a_1} = \boldsymbol{a} + \boldsymbol{a}'$. In general,

$$\boldsymbol{a_1} \neq \boldsymbol{a} + \boldsymbol{a}'$$

and $\boldsymbol{a_1}$ is not defined by $\boldsymbol{a}$ and $\boldsymbol{a}'$ For example, if the coordinate system K is rotating with constant angular velocity $\boldsymbol{\omega}$ with respect to a fixed coordinate system K_1, the acceleration of the particle, $\boldsymbol{a_1}$, observed in the system K_1 is [1]

$$\boldsymbol{a_1} = \boldsymbol{a} + \boldsymbol{a_{cor}} + \boldsymbol{a}'$$

Here $\boldsymbol{a_{cor}} = \boldsymbol{2\omega}$ x $\boldsymbol{v_{rel}}$ is Carioles' acceleration, $\boldsymbol{v_{rel}}$ is the velocity of the point with respect to system K, $\boldsymbol{a}' = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ is the centripetal acceleration directed to the axis of rotation, and $\boldsymbol{r}$ is a vector position of the point in the system K.

3. In mathematics, the transformation of vector components is defined when one system of coordinate is turned with respect to the other. But the problem of transformation of vectors of electric and magnetic fields in moving systems of coordinates cannot be resolved using only mathematical principles. The transformation of electric and magnetic fields under

transition from one reference frame to another is not the same as transformation of vectors – electric and magnetic fields are coupled. Actually, electric and magnetic fields are second rank tensors [2] (although a vector can be considered as a first rank tensor). The special theory of relativity allows us to find the laws of conversion of electric and magnetic fields.

4. Superposition principle for electric and magnetic fields [2] is not obvious but is an experimental fact and it should be emphasized that this principle and the triangle rule of vector addition can be applied only to linear systems. Maxwell's equations, which are mathematical expressions of experimentally confirmed laws of nature, are linear and, therefore, the mathematical rule of vector addition is valid.

a. In mathematics, a directed line segment is called a vector [3]. Algebraically, a vector is a set of three numbers, which we call vector components [3]. These definitions are acceptable in physics although the given geometric and algebraic definitions suggest the concept of a sliding (free) vector. That is a vector whose initial point can be any point on a straight line that is parallel to the vector. In physics, statics, structures, and strength of materials, the point of vector application can be critical (bound vectors) [4] and the point of application of the resultant vector must be defined not only by mathematical rules but also by applying physics meanings. For example:

a. Any axial vector which is a cross product of two polar vectors and includes a vector position (torque, angular momentum) depends on the axis chosen and the point of force applied.

b. Deformation of solids depends on the point of force applied. It means that two parallel forces of equal magnitude (which are identical in mathematical meaning) cause different changes in the shape of an object.

c. The point of application of the resultant pressure force (center of boyuoncy, ***CB***) acting on the object submerged in fluid (buoyant force, $\boldsymbol{F_b}$) is important for studying rotational motion, equilibrium, and stability of the object in fluid. Typically, US introductory physics textbooks do not consider this topic. However, it may be difficult to predict ***CB*** intuitively. For example, let us consider a cylinder submerged in liquid (Fig. 1). The magnitude of the buoyant force acting upward on a partially or fully submerged object is equal to the weight of the liquid displaced by that object and ***CB*** point is the center of mass of the fluid displaced by the floating or submerged body [5]. The ***CB*** point for this particular case, which is the result of addition of parallel pressure forces, is located higher than all points of the appliedvertical pressure forces. The position of ***CB*** point is defined not by the mathematical rules of vector addition but by Newton's Principles of mechanics.

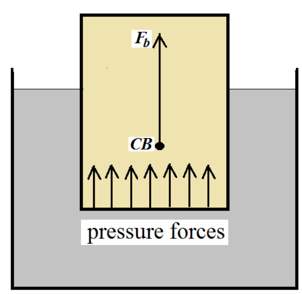

Fig 1: Vertical pressure forces, their resultant $\boldsymbol{F_b}$ (boyuant force), and center of boyuancy ***CB*** of the object submerged in liquid.

Similar comments can be made about the position of the center of gravity – the addition of gravity forces is defined not only by the mathematical rules of vector addition but by the definitions of center of gravity and torque.

d. Introductory physics textbooks typically consider smooth rolling objects (without slipping) along a flat surface. This motion can be studied as rotational motion with respect to the center of mass with angular velocity $\boldsymbol{\omega}$ and translational motion of the center of mass, or as pure rotational motion with the same angular velocity $\boldsymbol{\omega}$ with respect tomoving axis that always extends through contact lineof two surfaces [6]. However, a slightly different case of rolling thefirst cylinder of radius r_1 on the external surface of the second stationary cylinder of radius r_2 (Fig. 2) may be considered as pure rotation of the first cylinder with respect to its center (instantaneous axis *A*) with angular velocity $\boldsymbol{\omega}_1$ and pure rotational motion of the *A*-axis with respect to axis *B* with a *different* angular velocity $\boldsymbol{\omega}_2$. $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are parallel vectors with respect to different frames (similar problem of rolling one cone along the surface of the other fixed one with both cones having common vertex is considered in [7] and [8]) and, according to the mathematical rule, their sum is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \quad (2)$$

However, in this case, the application of the mathematical rule for vector additionis not enough to find the location of axis of rotation for which resultant motion of any point of the first cylinder is purely rotational with angular velocity $\boldsymbol{\omega}$. Eq. (2) does not define $\boldsymbol{\omega}_2$ or the magnitude of the resultant instantaneous angular velocity $\boldsymbol{\omega}$. To resolve above problems, additional kinematics relationships must be used.

Taking into account thatforrotation of the point on the rim of the first cylinder, the length of arc $OO^{/}$ is equal to the distance traveled in the same time by the point due to rotation with respect to axis *A* (Fig. 2b), one can see that for rolling without sliding

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 \frac{r_1}{r_2} \quad (3)$$

Since the triangle rule foraddition of linear velocities can be applied in a non-relativistic approach, velocity $\boldsymbol{v}$ of any point ofthe first cylinder with respect to the second stationary cylinder is

$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$

where $\boldsymbol{v}_1$ is velocity with respect tothe *A* axis and $\boldsymbol{v}_2$ is the velocity caused by rotation of the *A* axis with respect to axis *B*.For a point on the axis of rotation $\boldsymbol{v} = 0$, $\boldsymbol{v}_1 = -\boldsymbol{v}_2$ (Fig. 2c) can be accomplished only if the axis of rotation is a line of contact of the two cylinders (*O*-axis) for which Eq. (3) holds. It means that the first cylinder motion in frame *B*can be considered as pure rotation with angular velocity $\boldsymbol{\omega}$ with respect to the *O*-axis.

For the case of rolling of the first cylinder inside the hollow second cylinder (Fig. 2d), Eq. (11) holds, while angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are antiparallel and Eq. (3) must be replaced by

$$\boldsymbol{\omega}_2 = -\boldsymbol{\omega}_1 \frac{r_1}{r_2}$$

For both cases (rolling along internal or external surfaces), angular velocity with respect to instantaneous O-axis can be written in the form

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1\left(1 \pm \frac{r_1}{r_2}\right)$$

As r_2 approaches infinity (flat surface), the last equation transforms to a well known equality $\boldsymbol{\omega} = \boldsymbol{\omega}_1$.

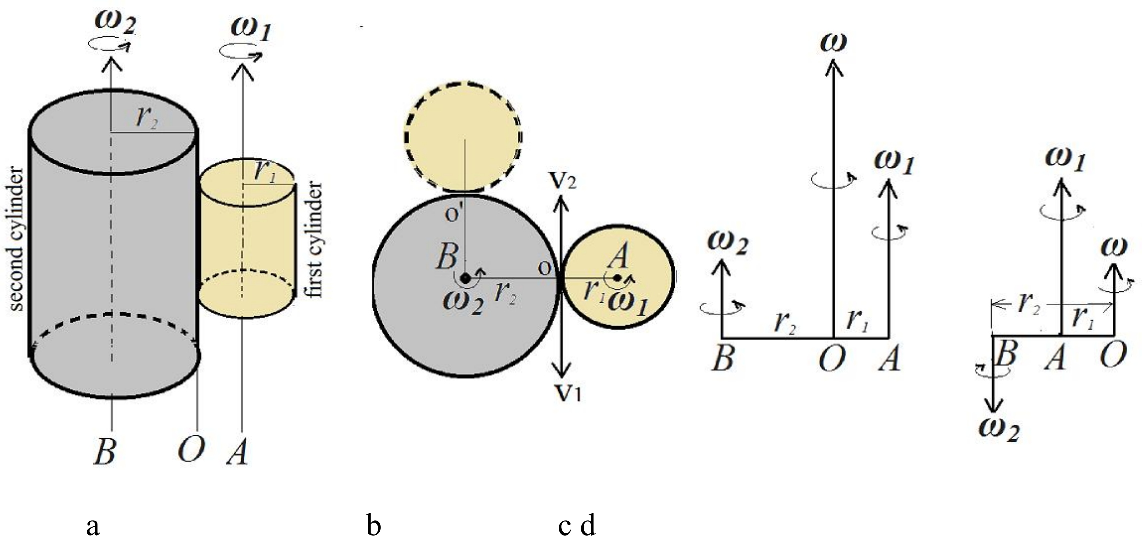

Fig 2: Smooth rolling of the first cylinder (radius r_1) on the external surface of the second stationary cylinder (a), its vertical projection (b) and vector diagram for addition of angular velocities (c); vector diagram of angular velocities for rolling of the first cylinder inside of the second hollow cylinder (d).

Conclusion

In mathematics, the rule of addition (triangle rule) of vectors is defined axiomatically, and conversion of vector components (three numbers) for transition from one Cartesian system of coordinates to another is defined by algebraic and trigonometric formulas. In physics, there are some occasions when operation of vector addition is based on its meaning and experimental justification.

It is obvious that physical restrictions on speeds cannot be explained by mathematics but special theory of relativity does set a limit for the speed of an object. Actually, all special relativity theory expressions are introduced with the use of a light signal exchange (the method of Einstein's synchronization). The light-signal method is used for time synchronization and for measuring length. This means that kinematics of the special theory of relativity is based on experiments (real or mental) which lead to rules of addition for velocities that are different from the triangle rule. Moreover, as a peculiarity of the special theory of relativity, the general law of velocity composition is not commutative [9] while the Galilean velocity composition law is commutative. Einstein's velocity addition is commutative only when **u** and **v** are parallel (Eq. (1)).

In some instances, the point of application of resultant force is important. However, vector algebra alone does not allow for locating of this point. For example, centers of gravity and buoyancy, which are the points of application of the forces of gravity and buoyancy, represent unique points of an object or system (the centers of gravity and buoyancy are not necessarily inside the object) which can be used to describe the system's response to external forces and torques. For example, the center of gravity of an extended body or system of masses is distinguished by the fact that it will remain at rest or moving at constant velocity unless the body is acted on by a net external force. The center of gravity may also be defined by implying that the torque about the origin would be the same if the entire weight acted through the center of gravity instead of acting through the individual masses. Similar rules should be used to find the center of bouncy. It means that principles of physics (Newton's Laws) must be exploited to locate the point of application of the resultant force. Another example, given in this paper, illustrates that to find instantaneous angular velocity (magnitude and location of axis of rotation) of a rolling cylinder on the surface of the second one, the vector addition rule of vector algebra is not enough – definitions of instantaneous axis of rotation and kinematics relationships are needed to be used. The transformation of electric and magnetic fields in moving systems of coordinates cannot be found using solely mathematical principles of vector operations. These transformations are not the same as transformation of vectors. When it comes to electric and magnetic field transformations, electric and magnetic fields are coupled. Electric and magnetic field vectors, as well as the vector of the electromagnetic force, are not invariant to different moving inertial frames, and the special theory of relativity allows us to find the laws of conversion of electric and magnetic fields and of Lorenz's force [2].

Typically, the considered specifics of vector addition are not the topic of introductory physics textbooks. It seems that some particularities of vector addition should be mentioned in the introductory physics class. Also, a brief review of polar and axial vectors, sliding and bond vectors would be helpful when students encounter problems similar to the ones presented here.

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