Note on the magnetic energy of a rotating charged metal sphere

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Abstract

Contribution of the magnetic energy to the energy of field of a rotated charged conducting spheroid is calculated by a method which does not use either the integration of the magnetic energy density as the surface integration of the scalar product of the current density vector or the vector potential of field. The method may be interesting for a student, studying the classical electrodynamics course.

1. Introduction

The problem of magnetic field calculation, created by a rotating charged conducting sphere, is a traditional part of university textbooks on electrodynamics ([1]). If the sphere's radius is a, it's angular velocity is ω and the net charge on sphere is $Q = \sigma_0 \cdot 4\pi a^2$, then, in Gaussian units, the solution of the Poisson equation for the vector potential $\boldsymbol{A}(\boldsymbol{R})$

$$\nabla^2 \boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{j}, \tag{1}$$

with the current density

$$\boldsymbol{j}(\boldsymbol{R}) = \sigma_0 \cdot \omega a \sin\theta \cdot \delta(R-a)\boldsymbol{e}_\varphi \tag{2}$$

is

$$\boldsymbol{A}(r<a) = \frac{1}{2}B \cdot r\sin\theta \cdot \boldsymbol{e}_\varphi, \tag{3}$$

$$\boldsymbol{A}(r \geq a) = \frac{\Re}{r^2} \cdot \sin\theta \cdot \boldsymbol{e}_\varphi. \tag{4}$$

Here

$$\Re = \frac{Qa^2}{3c}\boldsymbol{\omega}, \tag{5}$$

is the sphere's dipolar moment,

$$\boldsymbol{B} = \frac{2}{a^3}\Re, \tag{6}$$

is the magnetic field induction inside it, R, θ, φ are the spherical coordinates, defined so that Oz axis is along the vector $\boldsymbol{\omega}$ and the center of sphere corresponds to $\boldsymbol{R} = \boldsymbol{0}$, $\boldsymbol{e}_\varphi$ is the correspondent unit vector.

To tell strictly, even in the zero electron mass approximation, which will be used further, the action of the Lorenz force on electrons will disturb the surface charge density σ_0 (we can consider the thin metal film sputtered on a dielectric ball instead of to consider the metal ball). However, it is easy to show (as in [2]), that the relatively rearrangement $\Delta\sigma/\sigma_0$ of the charge density at any point of sphere is proportional to β^2 where $\beta = \omega a / c$ (c is the speed of light in vacuum). Considering $\beta \ll 1$ we will not pay attention on this effect beneath.

To spin the ball with charged metal film, it's necessary to spend the work W against the eddy electric field among other. The quantity W is called the magnetic field energy. Contrary to (3) – (6), the W value is not presented in [1]. Meanwhile this quantity is called for the theoretical physics as it is seen from the original journal articles. In [3]

(Appendix), author uses for the calculation of W the formula

$$W = \frac{1}{8\pi}\int_{R^3} \boldsymbol{B}_{\mathrm{rot}}^2 dV\,, \qquad (7)$$

(R^3 is the symbol of integrating over all space, $\boldsymbol{B}_{\mathrm{rot}}(\boldsymbol{R})$ is the magnetic field induction in an arbitrary point) and derives

$$W = \frac{\Re^2}{a^3}. \qquad (8)$$

In [4] (Appendix) authors computed (8) with the help of formula

$$W = \frac{1}{2c}\int_{R^3} \boldsymbol{A}(\boldsymbol{R})\cdot \boldsymbol{j}(\boldsymbol{R}) dV\,, \qquad (9)$$

what is the more simple way owing to Dirac function presence in (2). The aim of this note is to show that an undergraduate student studying the classical electrodynamics may not spend the time for reproducing the routine algebraical calculations and to derive (8) more simply than in [4] if he orients freely in the theme of magnetostatics of ferromagnets in the volume, for example, of [5]. The method of calculation of the rotating charged body magnetic energy will be applied to the spheroid. For auditory purposes an educator may adapt this method turning the spheroid to sphere primarily and expelling a part of mathematics beneath.

2.Magnetic energy of a rotating charged metal spheroid

Let the thin metal film is sputtered onto the dielectric so that the equation of the external metal surface is:

$$\left(\frac{\rho}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1\,, \qquad (10)$$

where $\rho = \sqrt{x^2 + y^2}$. The distribution of the surface charge density $\sigma(z)$ is ([6]):

$$\sigma(z) = \frac{Q}{4\pi b}\frac{1}{\sqrt{\rho^2 + \left(\frac{a}{b}\right)^4 z^2}}\,. \qquad (11)$$

After the body began to rotate the linear current density of the surface charge is

$$i(z) = \sigma(z)\cdot\omega\rho(z) = \frac{Q\omega}{4\pi b}\cdot\cos\alpha(z)\,, \qquad (12)$$

where the angle α is defined in the Figure 1 and the equation $\mathrm{tg}\alpha = d\rho\,/\,d|z| = z\cdot a^2/\rho\cdot b^2$ for the spheroid surface was used.

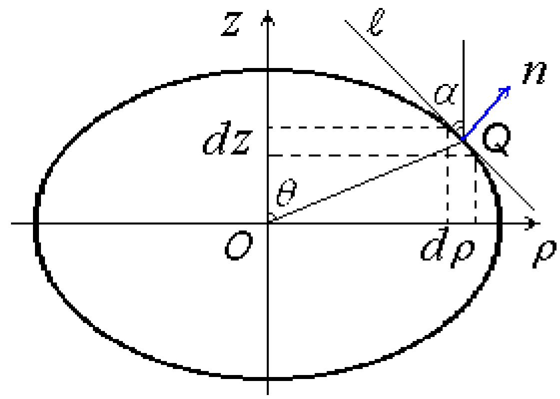

FIG. 1: Vertical section of metal spheroid (the dielectrical core is not shown). Q – arbitrary point on it's surface. n – normal to surface in Q, ℓ is the line, tangent to spheroid in Q.

The magnetic field induction $\boldsymbol{B}_{\mathrm{rot}}(\boldsymbol{R})$ created in all space by such current distribution coincide with the just one $\boldsymbol{B}_{\mathrm{magn}}(\boldsymbol{R})$ created by the spheroidal permanent magnet of the same shape (10) with the magnetization

$$\boldsymbol{M} = \frac{1}{c}\cdot\frac{Q\omega}{4\pi b}\boldsymbol{e}_z\,, \qquad (13)$$

in accordance with the well-known statement ([6]) that the uniform magnetization $\boldsymbol{M}$ is equivalent to molecular current with the linear surface density

$$\boldsymbol{i} = c\cdot[\boldsymbol{M};\boldsymbol{n}], \qquad (14)$$

where $\boldsymbol{n}$ is the unit vector of outer normal at the given point of magnet surface. Let us call such a magnet as the equivalent one for our rotating charged metal spheroid.

Let the two ideal conductors, screaning completely any external variable magnetic field inside it at zero temperature, move from infinity to the equivalent magnet as it is shown in Figure 2. Everyone of the two has one flat surface with the deepening as the half of our spheroid. When the conductors will taught the magnet, the field of screaning currents $\boldsymbol{B}_{\mathrm{j}}(\boldsymbol{R})$ will compensate the field of magnet in all space so that $\boldsymbol{B}_{\mathrm{j}}(\boldsymbol{R}) + \boldsymbol{B}_{\mathrm{magn}}(\boldsymbol{R}) = 0$. So the magnetic energy (7) of our rotating spheroid is equal to the magnetic energy of currents in conductors which is, by definition, the work of external forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ being spended for the conductors transition from

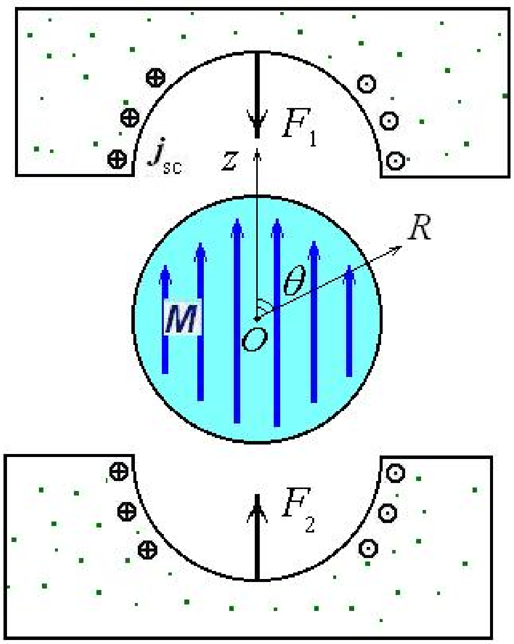

FIG. 2: Permanent magnet and the ideal conductors. $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are the external forces transferring the last two. ***J***sc are the screening currents appeared by this.

infinity. This work will increase the free energy $\Im$ of the system “magnet + ideal conductors”:

$$\Im = W + \Im_{\mathrm{ext}} , \qquad (15)$$

where $\Im_{\mathrm{ext}}$ is the free magnetostatic energy of the retired equivalent magnet. As it was discussed in [7],

$$\Im_{\mathrm{ext}} = -\frac{1}{2}\boldsymbol{M}\boldsymbol{B}'\cdot V = -\frac{1}{2}\boldsymbol{M}\boldsymbol{H}\cdot V + k\boldsymbol{M}^2 , \qquad (16)$$

where $\boldsymbol{B}'$ is the micro field, acting on the magnetic moments inside the spheroid, $\boldsymbol{H}$ is the magnetic field strength there, k depends only of a crystal lattice type of material of magnet and V is it’s volume. (k is ignored often in literature as it is made in [5]) Formula for $\Im$ is presented in [8]:

$$\Im = -\frac{1}{2}\boldsymbol{M}\left(\boldsymbol{B}' + \boldsymbol{B}_{\mathrm{j}}\right)\cdot V , \qquad (17)$$

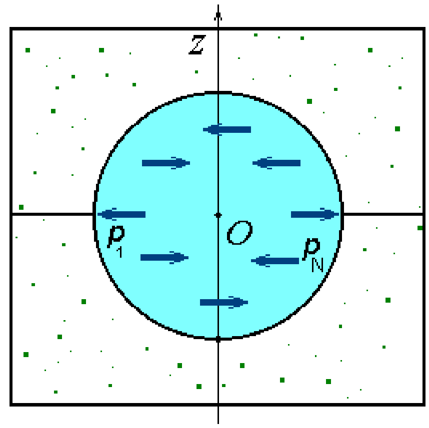

FIG. 3: The demagnetized magnet inside the system of two ideal conductors. ***P***1 – ***P***N are the magnetic moments of atoms.

without any comments and references what reflects it’s evidence for professional physics-theoreticians. For students the next comment may be done. Let us imagine that the equivalent magnet is magnetized in the presence of the ideal conductors enveloping it from the initial state where the magnetic moments $\boldsymbol{p}_i$ of atoms (i = 1 … N) lie chaotically in the Oxy plane (Figure 3). During the magnetization process $\boldsymbol{p}_i$ vectors rotate up to Oz axis so that the value of angle θ between $\boldsymbol{p}_i$ and Oz axis is just the same for all atoms at any moment of time ($\theta = \pi / 2$ in Figure 3). Then the vectors $\boldsymbol{B}'$ and $\boldsymbol{B}_{\mathrm{j}}$ became functions of θ and ($\boldsymbol{d\theta}_i$ is the vector of infenitizemal rotation of $\boldsymbol{p}_i$):

$$\Im = N\cdot\int_{\theta=\pi/2}^{\theta=0}\left[\boldsymbol{p}_i ; \boldsymbol{B}'(\theta) + \boldsymbol{B}_{\mathrm{j}}(\theta)\right]\cdot \boldsymbol{d\theta}_i =$$

$$= N\cdot\int_{\theta=\pi/2}^{\theta=0}\left(\boldsymbol{B}'(\theta) + \boldsymbol{B}_{\mathrm{j}}(\theta)\right)\cdot\left[\boldsymbol{d\theta}_i ; \boldsymbol{p}_i\right] . \qquad (18)$$

Taking in (18) $\boldsymbol{B}_{\rm j} = 0$ and $\boldsymbol{B'}(\theta) \sim \cos\theta \cdot \boldsymbol{e}_z$, we derive (16). But for the screaning currents it must be $B_{\rm j}(\theta) \sim B'(\theta)$ as the consequence of the Maxwell equation rot $\boldsymbol{H} = 4\pi\boldsymbol{j}/c$ linearity. So at any θ, it will be $\boldsymbol{B}_{\rm j}(\theta) \sim \cos\theta \cdot \boldsymbol{e}_z$ and we came to (17).

Using (16) – (17) in (15) we obtain

$$W = \frac{1}{2}\boldsymbol{B}_{\rm magn}\cdot\mathfrak{R} = \frac{1}{2}\left(4\pi\boldsymbol{M} + \boldsymbol{H}\right)\cdot\mathfrak{R} =$$

$$= \mathfrak{R}\cdot\frac{1}{2}\left(4\pi\boldsymbol{M} - N_z\boldsymbol{M}\right) = N_x\cdot\frac{\mathfrak{R}^2}{V}, \qquad (19)$$

where N_x and N_z are the demagnetization factors of spheroid along Ox and Oz axes and, as it resulted from (13),

$$\mathfrak{R} = \boldsymbol{M}\cdot V = \boldsymbol{M}\cdot\frac{4}{3}\pi a^2 b = \frac{Qa^2}{3c}\boldsymbol{\omega}. \qquad (20)$$

For the sphere $N_x = 4\pi/3$ and we return from (19) to (8). In the general case of the arbitrary $m = b/a$ it is conveniently to present the result as the $W/W_{\rm el}$ dependence of m where $W_{\rm el} = Q^2/2C$ is the electrostatic energy of the charged metal spheroid and C is it's capacity. Taking the expressions for C from [6] and for N_x from [9] we have:

$$\frac{W}{W_{\rm el}} = \frac{\beta^2}{3}\cdot\frac{1}{1-m^2}\left\{1 - \frac{m\cdot\sqrt{1-m^2}}{\arccos m}\right\} + O(\beta^4), \quad (21)$$

for the oblate spheroid and

$$\frac{W}{W_{\rm el}} = \frac{\beta^2}{3}\cdot\frac{1}{m^2-1}\left\{\frac{m\sqrt{m^2-1} + \frac{1}{2}\cdot\ln\left(m - \sqrt{m^2-1}\right)}{\ln\left(m + \sqrt{m^2-1}\right)} - \frac{1}{2}\right\} + O(\beta^4). \qquad (22)$$

for the prolate one. Formulae (21) – (22) are illustrated graphically in the Figure 4.

Formula (19) with $\mathfrak{R}$ derived from (20) must became the strict one for infinite cylinder when the tangential component of the Lorenz force acting on the free electrons disappears. Inserting $\boldsymbol{B} = 4\pi\boldsymbol{M}$ in (20) and using (20) in (19) with $N_x = 2\pi$, we return to (7).

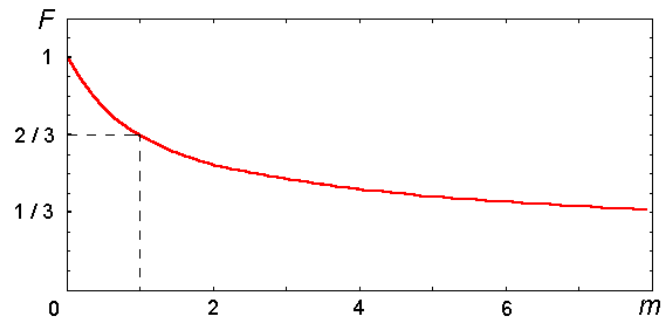

FIG. 4: The dependence of the value $F = (3/\beta^2)\cdot W/W_{\rm el}$ as function of the spheroid parameter $m = b/a$

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