

# Functional differential equations.

## 4: Retarded gravitation

C. K. Raju

ckr@ckraju.net

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### Abstract

Are functional differential equations (FDEs) only about electrodynamics? No. They apply also to gravitation. We explain a recent reformulation of gravitation, called retarded gravitation theory (RGT), which is Lorentz covariant, and uses functional differential equations. RGT modifies the Newtonian “inverse square law” gravitational force: the RGT force depends upon (a) retarded distance, and (b) includes a velocity-dependent term. RGT, since Lorentz covariant, theoretically improves on Newtonian gravitation. At the same time, RGT has the practical advantage over general relativity theory (GRT) that a solution of the many-body problem is feasible in RGT. Hence, RGT can and ought to be applied to the galaxy where Newtonian physics apparently fails but GRT cannot be applied. The tiny velocity dependence of the RGT force is amplified across a hundred billion co-rotating stars in the galaxy, so that non-Newtonian velocities of stars in spiral galaxies are to be expected on RGT, even without dark matter. Possible experimental tests of RGT include the flyby anomaly observed for NASA spacecraft which depends systematically on velocity-effects due to the rotation of the earth.

We further clarify that Laplace’s objection to pre-relativistic naive theories of retarded gravitation (NRG) does not apply to RGT. We solve the 2-body FDEs of RGT for the sun-Jupiter case: the system is stable despite tiny differences from Newtonian gravitation. Thus, FDEs are a general feature of post-relativity physics.

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## 1 Recap

In three earlier articles[1, 2, 3] in this series, we saw that functional differential equations (FDEs) are fundamentally different from ordinary differential equations (ODEs). Hence, doing physics with FDEs leads to a paradigm shift in physics. Further, FDEs arise naturally in classical electrodynamics: *without* any new physical hypotheses but just by doing the math right. The right way to solve for the classical hydro-

gen atom, even without radiation damping, is to use FDEs and that changes the qualitative features of the solution.

What happens if we have radiation damping? The problem of the motion of even a single charged particle, in classical electrodynamics, with radiation damping has remained mathematically unsolved for a century because of runaways. These runaways can be controlled by modifying Maxwell’s equations at the microphysical level, so that the equations of motion

of even a single charged particle become FDEs.

Before proceeding further to quantum mechanics, there is one doubt which needs to be settled. Are FDEs only about classical electrodynamics? No. They are about resolving a fundamental conceptual flaw in Newtonian physics. I have dealt with this issue of Newtonian physics in detail in previous articles in this very journal,[4, chp. 2, chp. 3a, chp. 3b] and will only summarise the key points here.

## 2 The problem of time in Newtonian physics

Consider Newton's first law of motion. It states that, in the absence of external forces, a body continues in its state of rest or uniform motion. Is this meaningful? It is easy to understand "rest", but what is "uniform motion"? A body is said to be in uniform motion if it covers equal distances in equal times. But what are equal times?

When we say that one hour in the past is equal to one hour in the future, there is no way to verify it empirically. Obviously, we cannot bring back one hour in the past and compare it in the present with one hour in the future. We must use a clock. But, clocks differ, so *which* clock should one use? Uniform motion according to my heart beats would not be uniform motion according to a simple pendulum, and vice versa. Without a definition of equal intervals of time, or the definition of an "ideal clock", there is no basis on which to say that a mechanical clock is "better" than heart beats.

So, what exactly is an ideal clock? In his *Principia*, Newton admitted that days and nights are unequal, as are the swings of a pendulum, and that no natural phenomenon would provide an ideal clock. But he reached the strange conclusion that it was unnecessary to define equal intervals of time. He said that he was concerned only with "absolute, true, and mathematical time, which flows on without regard to anything external". Each of these adjectives: "**absolute**", "**true**", "**mathematical**", and "**without regard to anything external**" makes clear that Newton took time as an aspect of meta-

physics. In short, he thought it was all right if God knew what equal intervals of time were, even if humans did not.

This was a mistake because to do physics, humans too need to know what equal intervals of time are. Indeed, Newton's predecessor and mentor Barrow had emphasized the need for a clear physical definition of equal intervals of time, saying those who did physics without it were "quacks".[5] *Why* did Newton make time metaphysical? Newton thought that making time metaphysical was the way to make "perfect" the notion of  $\frac{d}{dt}$  needed for his second law. This related to the European misunderstanding of the Indian calculus imported into Europe in the 16th c. This is an interesting but long story, which I have told elsewhere.[6, 7]

For common applications of Newtonian mechanics, to planetary motion and ballistics, many common clocks "work". However, Newton's failure to provide a physical definition of equal intervals of time, became prominent during attempts to reconcile electrodynamics with Newtonian physics at the turn of the 19th c. The solution provided by relativity was to define equal times in a way which preserved electrodynamics but required a modification of Newtonian physics.

Physics texts teach relativity differently: they teach that relativity began with the Michelson-Morley experiment which proved the absence of ether and the constancy of the speed of light. That, however, is not correct: one cannot measure the speed of light or anything else without a clock, and a positive result in the experiment (as later found by Miller) is no evidence either for ether or for a varying speed of light.[4, p. 56-57] In fact, as explained in an earlier article in this journal, the Michelson-Morley experiment was NOT designed to test the existence of ether: it was designed to test between the two ether theories of Fresnel and Stokes. Amusingly, it came out in favour of the wrong theory: the Stokes theory, which involved a mathematical absurdity. Hence, Lorentz thought it was preferable to believe that the arm of the Michelson interferometer contracted in the direction of motion.

Now, a clock is required even to measure *lengths*: for a moving rod, one must note the positions of

both ends of the rod *simultaneously*, and simultaneity is decided by a clock. If one postulates that the speed of light is constant, then a photon bouncing between two parallel mirrors marks equal times between bounces, and this provides an ideal clock. The Lorentz-Fitzgerald length contraction is a natural consequence of using such a clock.

Note clearly that the constancy of the speed of light is a postulate, *not* an experimental fact. This postulate of constant speed of light automatically leads to the Lorentz transform which Poincaré derived and so named. That is, the special theory of relativity came about as the solution to the problem of equal intervals of time in Newtonian physics.

As for “ether”, the word is confusingly used in multiple senses. One sense is as an absolute reference frame. But the original sense of ether (= sky = *ākāśa*, as in the Vaiśeṣika sūtra) relates to action by contact (*saṃyoga*). Eliminating ether also eliminates action by contact, and admits, for example, delayed action at a distance. This is mathematically equivalent to replacing ODEs by FDEs (which Poincaré called “equations of finite differences” [4, p. 116]). Einstein, to whom special relativity is usually attributed, never understood this point, for he mistakenly kept approximating FDEs by ODEs, until late in his life.[4, p. 122]

This *process* of development of relativity, by identifying and resolving a conceptual flaw in Newtonian physics, as well as the connection of relativity with FDEs, are both obscured by usual accounts of the theory of relativity which focus on glorifying an individual, Einstein. (It is on record that Einstein knew of Poincaré’s work until 1902. In his 1905 special relativity paper, he casually used the strange term “longitudinal mass” first circumspectly used by Lorentz in 1904. Einstein also used the novel term “relativity” first used by Poincaré in his 1904 paper (instead of his earlier “principle of relative motion”). Einstein later denied reading both the 1904 papers, of Lorentz and Poincaré, and his 1905 paper on (special) relativity cites absolutely no references.)

### 3 Modifying gravitation

Special relativity modified Newton’s laws of motion; but that is not enough, Newtonian gravitation too must be modified for the two come as a package deal. Newtonian gravitation involves instantaneous action at a distance which is incompatible with special relativity, where the speed of light is a limiting speed. The general theory of relativity (GRT) did modify Newtonian gravitation. However, GRT is enormously complicated: in a century since GRT was formulated, even the two body problem could not be solved in it. This creates a peculiar problem as follows.

#### 3.1 Galactic rotation curves

Newtonian gravitation worked well for the solar system, but it fails for the galaxy. In the solar system the rotational speed of a planet of mass  $M_p$  is determined by

$$\frac{GM_S M_p}{r^2} = \frac{M_p v^2}{r} \quad (1)$$

where  $M_S$  is the mass of the sun,  $v$  is the rotational velocity of the planet round the sun, and  $r$  is the distance of the planet from the sun. This means that the rotational velocities of planets  $v \propto \frac{1}{\sqrt{r}}$  decline with distance  $r$  from the centre. Or, in terms of the time period  $T = \frac{2\pi r}{v}$ , we must have  $T \propto r\sqrt{r}$ . This accurately fits observations: Pluto some 39.5 times more distant from the sun than earth takes  $39.5 \times \sqrt{39.5} \approx 248$  earth years to complete an orbit round the sun.

However, what happens in a spiral galaxy is starkly different. In spiral galaxies, the rotational velocities of stars, instead of declining, are observed to *increase* as one moves out from the centre. (Fig. 1) This is contrary to what one expects from Newtonian gravitation.

#### 3.2 Dark matter

Of course, the Newtonian theory can be easily “saved” by supposing that there is invisible dark matter (DM) in the galaxy. Perhaps that is so: but at least we expect a decline in rotational velocities of

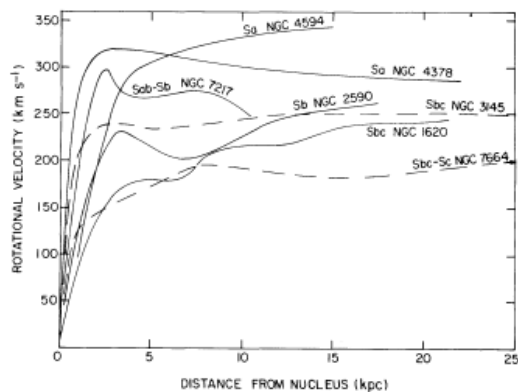


Figure 1: Rotation curves of various galaxies. The rotational velocities increase and then become roughly constant.[8] This is contrary to the expected behaviour on Newtonian gravitation that, sufficiently far from the nucleus, rotational velocities must decline as  $\frac{1}{\sqrt{r}}$  with distance  $r$  from the nucleus.

stars as we move to the edge of the visible galaxy. Unfortunately, even that expectation is belied. It is clear from Fig. 1 that the rotational velocities of stars, instead of declining, become approximately *constant* at the edge of the galaxy. Therefore, to “save” the theory we must make one more hypothesis: that the hypothetical invisible dark matter is distributed in a peculiar way in the form of a halo round the galaxy, with its density reaching a peak where the luminous matter thins out to zero.

Now why should that be so? The hypothetical, invisible dark matter, whatever its composition, has exactly the same gravitational properties as the luminous matter in galaxies. On the scale of the galaxy, gravitation is the dominant force which decides structure. So why should luminous matter and dark matter be distributed in such strikingly different ways? No clear explanation has emerged so far.

### 3.3 MOND

Dissatisfaction with the DM hypothesis led to the formulation of another theory: modified Newtonian dynamics (MOND).[9, 10] In its original form, the the-

ory simply supposed that the gravitational force law itself changed at the scale of the galaxy on the phenomenological grounds of observations. Moreover, it did not initially correct what we now know to be a critical conceptual defect in Newtonian physics.

Could GRT explain any part of the departure from Newtonian gravitation? To answer this we need to be able to apply GRT to the galaxy. Unfortunately, that is not feasible: GRT is too complex to be used to solve the many body problem of a galaxy typically involving hundreds of billions of stars. Even modelling a collection of discrete objects is very difficult in GRT. Therefore, the only thing available is to fall back on Newtonian gravitation just believing it to be a good approximation to GRT at those scales.

## 4 RGT

This situation motivates retarded gravitation theory.[7] We know that special relativity is an essential conceptual correction to Newtonian physics. Can we have a theory of gravitation compatible with special relativity? Poincaré did attempt to find such a theory (with a different motivation) but did not fix on a definite expression for the force, or try to solve the problem of galactic rotation curves, which was not known in his time.

Some people might ask: why look for such a theory when we already have the “ultimate” theory, namely GRT? One simple answer is this: it is no use having an ultimate theory which is not usable! It is like saying ultimately “God knows everything”, but we have no way to read the mind of God! In the context of the galaxy, the theory which is actually used is Newtonian gravitation. RGT, being a Lorentz covariant theory of gravitation, improves on that. As we will see, RGT does help us to bypass the additional hypotheses introduced by both DM and MOND.

### 4.1 Derivation of the expression for the force

How does one make gravitation Lorentz covariant? The derivation of the Lorentz-covariant gravitational

force is so simple that it can be reproduced in its entirety here.

We start with a reference frame in which the test particle (“attracted body”) is a mass point (at rest) at the origin. The “attracting body” is located at the retarded position described by the 4-vector  $X = (ct, \vec{x})$  and moving with a 4-velocity  $V = \gamma_v(c, \vec{v})$ , both at *retarded* time  $t = -\frac{r}{c}$ . Here,  $\vec{x} = (x, y, z)$ ,  $r = \sqrt{x^2 + y^2 + z^2}$ , and  $\gamma_v = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$  is the Lorentz factor.

Let  $F = (T, \vec{f})$  be the 4-force experienced by the attracted body. This 4-vector transforms in the same way as the 4-vectors  $X$  and  $V$ , so we take it to be given by a linear combination

$$F = aX + bV, \tag{2}$$

where  $a$ , and  $b$  are Lorentz invariants to be determined. Since  $a$  and  $b$  are Lorentz *invariant*, the expression (2) for the 4-force  $F$  would be Lorentz covariant, as required.

For the case where the attracting body is also at rest ( $\vec{v} = 0$ ), we require that the 3-force must approximately agree with the Newtonian gravitational force  $\vec{f} = k(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3})$ , where  $k = Gm_0m_1$ , the two (rest) masses are  $m_0$  and  $m_1$ , and  $G$  is the Newtonian gravitational constant. (Note that the sign conventions we are using are the opposite of the usual ones, since the “attracting body” is at  $X$ , and the force is in the direction of its retarded position.) Therefore,  $a \approx \frac{k}{r^3}$ . This suggests that  $a = -\frac{kc^3}{a_1^3}$  where  $a_1$  is the Lorentz invariant quantity  $a_1 = X.V = \gamma_v(c^2t - \vec{x}.\vec{v})$ , which equals  $-cr$  when  $\vec{v} = 0$ , and approximately equals  $-cr$  when  $v = ||\vec{v}||$  is small compared to  $c$ . That is,

$$a = -\frac{kc^3}{(X.V)^3} \approx \frac{k}{r^3}. \tag{3}$$

We now use the fact that the components of the 4-force are not independent, but must satisfy [11]

$$F.U = 0, \tag{4}$$

where  $U = \gamma_u(c, \vec{u})$  is the 4-velocity of the particle on which the force acts. This comes about simply since

the revised form of the equations of motion is now

$$m_0 \frac{d^2Y}{ds^2} = F, \tag{5}$$

where  $m_0$  is the rest mass and  $s$  is proper time along the world line,  $Y(s)$ , of the “attracted particle”. Since the 4-force  $F$  is parallel to the 4-acceleration of the particle on which it acts, it must be perpendicular to its 4-velocity  $U$  (which is a vector of constant norm). Accordingly, taking the dot product of  $U$  with both sides of (2), we obtain

$$0 = a(X.U) + b(V.U). \tag{6}$$

Now the dot products  $X.U$  and  $V.U$  are scalars, or Lorentz invariants, and the Lorentz invariant  $a$  is already determined. Hence, (6) determines  $b$  as a Lorentz invariant. Explicitly,

$$b = -\frac{a(X.U)}{(V.U)} \approx \frac{k}{cr^2}. \tag{7}$$

Note that we would not have been able to satisfy the requirement (4) had we already set  $b = 0$  to begin with. This shows that the Lorentz covariant gravitational force we seek *cannot* be purely position dependent but must depend also on velocity.

Substituting these values of  $a$  and  $b$  in (2), the force in RGT is explicitly given by

$$F = -\frac{kc^3}{(X.V)^3}X + \frac{kc^3}{(X.V)^3} \frac{(X.U)}{(V.U)}V. \tag{8}$$

Since the equations of motion (5), and the expression for the force (8) are Lorentz covariant, we can use these expressions in any Galilean frame, and are not tied to any special frame. Note, however, that RGT, unlike GRT, is restricted to Galilean frames.

In studying motions such as those of stars in the galaxy we can use the non-relativistic approximate expressions for  $a$  and  $b$  given in (3) and (7). This leads to

$$F \approx \frac{k}{r^2} \left( \frac{X}{r} + \frac{V}{c} \right), \tag{9}$$

which simple form exhibits clearly the departure from Newtonian gravitation.

Thus, we have made two changes to Newton's "inverse square law" of gravitation. First, the RGT gravitational force uses the retarded distance, not the instantaneous distance between the two bodies. Second, the gravitational force cannot be a pure "inverse square law" force (even with the retarded distance), but has a velocity-dependent ( $\frac{v}{c}$ ) component. This RGT modification of Newtonian gravitation is completely different from other earlier modifications of the "inverse square law". Furthermore, it is not *ad hoc* or speculative like earlier modifications; rather RGT is a logical consequence of an *essential* correction to a conceptual defect about time in Newtonian physics.

## 5 The solution for the galaxy

Now special relativistic effects, such as the  $\frac{v}{c}$  term in the above force, are believed to be relevant only when velocities approach that of light. However, this piece of text-book wisdom is true only for the *one* body problem, the only problem solved by texts in special relativity. For the galaxy, however, we need to do a many-body problem.

Now, stars in a spiral galaxy all *systematically* co-rotate in one direction. Could a tiny but systematic  $\frac{v}{c}$  effect become significant when summed over a large number of stars? Specifically, The observed rotation velocities of stars in spiral galaxies are of the order of a few hundred  $\text{km s}^{-1}$  corresponding to  $\frac{v}{c} \sim 10^{-3}$ , which is small. However, a systematic effect of this order must be summed over some  $10^{11}$  stars in a galaxy. Could the sum be significant?

To make a quick check we can re-frame the question. Suppose we have numerous mass points spread in a disk rotating around a central mass. Suppose, now, we introduce a test mass into this configuration, and let us further suppose the test mass is moving with the Newtonian velocity required for equilibrium at a distance  $r$  from the centre. What will happen to the test mass?

For a test mass, the calculation is very simple, for the meaning of a "test" mass is that we neglect the effect of that mass on the remaining particles (the galaxy). That is we prescribe the motion of the other

particles not only in the past, but for all time. Under these circumstances of a one-body problem, where the motion of all remaining bodies is prescribed for all time, the FDEs of motion in RGT reduce to ODEs, which can be readily solved. However, the force still differs from that of Newtonian gravitation.

According to Newtonian gravitation the rotational *velocities* of the other masses are irrelevant, and only the total mass counts. Hence, our test mass which begins in Newtonian equilibrium, should continue in equilibrium. On RGT, however, the test particle is violently accelerated. Depending upon the total mass it may stay within the system, with a non-Newtonian velocity, or get thrown out. Therefore, on RGT, a large number of co-rotating particles can significantly increase the rotational velocity of a test particle in Newtonian equilibrium. Further, unlike Newtonian gravitation, if we consider a shell of rotating particles, the velocity effect acts on the test particle even inside the shell.(Fig. 2)

The story has a very important moral: tiny special relativistic effects, at non-relativistic velocities, can add up across a large number of particles, and become immense. The text book claim that special relativity matters only at relativistic velocities needs to be corrected: that claim is true only for the one body problem.

Thus, RGT predicts that stars in a spiral galaxy will have non-Newtonian velocities just because the velocity-dependent gravitational force adds up across a large number of co-rotating stars. It is not necessary to hypothesize dark matter just to explain non-Newtonian velocities in spiral galaxies, but if there is any dark matter, its effects would be in addition to those predicted by RGT.

What about the other feature of rotation curves that rotational velocities become constant at the edge of the galaxy? In principle, this feature too admits a simple explanation in RGT. Far from the centre of the galaxy, the gravitational pull of the central mass becomes weak, and the velocity effect becomes more prominent. Consider two nearby stars co-rotating at the edge of the galaxy. The velocity dependent component of the RGT gravitational force will tend to equalise their velocities. Thus, there is a simple and natural explanation for the approximate constancy of

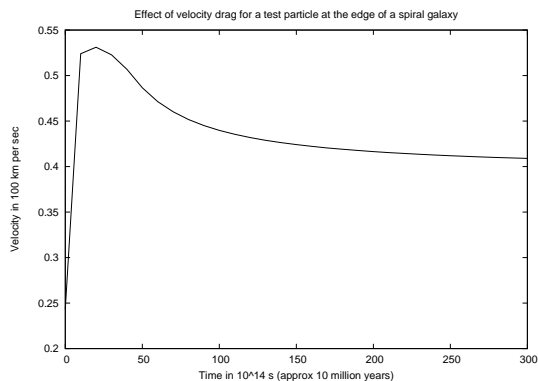


Figure 2: **The velocity effect of retarded gravitation:** The velocity of a test particle in our model galaxy increases due to velocity drag, and the particle escapes. The plot is velocity vs time. Time units are approximately 10 million years, while length units are 1 kpc. A central mass of  $1.5 \times 10^{10}$  solar masses is surrounded by 10,000 particles (each of  $10^5$  solar masses) in a rotating ring of radius 12 kpc. The test particle is initially in Newtonian equilibrium at 12.2 kpc.

star velocities at the edge of the galaxy, and there is no need to hypothesize halos of dark matter.

Thus, the hitherto mysterious qualitative features of the rotational velocities of stars in spiral galaxies are expected on RGT.

RGT, unlike MOND, involves no speculative hypothesis, but proceeds solely on the theoretically necessary principle of Lorentz covariance, and its origins in the problem of equal intervals of time in Newtonian physics. Therefore, even dark matter theorists, who set aside MOND, **MUST** take into account the special relativistic effects incorporated into RGT to estimate the amount and distribution of dark matter. *Specifically, all current estimates of dark matter and its distribution obtained by using only Newtonian gravitation are defective and unreliable, and must be recalculated using RGT.* Because RGT is a completely general theory, these remarks apply equally to the dy-

namics of clusters of galaxies. (To reiterate, science is NOT about “authorised knowledge”, or popularity among scientists, or “reputability”; it is about refutability.)

## 6 Laboratory tests of RGT

Is there any way to test RGT closer home? Indeed there is. RGT, unlike MOND, changes the gravitational force at all scales from the laboratory to the galaxy, and beyond. In a conventional Cavendish experiment, if the two attracting masses are rapidly rotated in opposite directions that would change the deflection of the suspended dumbbell, on RGT, though it would have no effect on Newtonian gravitation. In particular, if the two attracting and rotating masses are exactly lined up with the dumbbell, there would be a non-zero deflection on RGT, but zero deflection on Newtonian theory.

Of course, very high precision would be required to carry out such experiments. An unexpected difficulty here is that the value of the Newtonian gravitational constant  $G$  is not known sufficiently precisely. One reason for this is that there is an apparent discrepancy between static and dynamic ways of determining  $G$ .

According to RGT, in the dynamic way of determining  $G$ , velocity effects must be taken into account. If these are neglected, we could end up with a slightly different value of  $G$ . So, the existence of tiny discrepancies between different ways of measuring  $G$  constitutes an additional way of testing RGT. While discrepancies have indeed been noted, and it has even been speculated that these might be due to some fundamental issues,[12] the discrepancies are still within experimental error. Hopefully, these issues will be clarified in future, since it is anyway important to determine  $G$  to high precision.[13]

## 7 The flyby anomaly

One experiment which has already been carried out (and is likely to be repeated with greater precision) involves spacecraft when in near-earth orbit. On

RGT, one expects a tiny  $\frac{v}{c}$  effect due to the rotational velocity  $v$  of the earth. The rotation of the earth has no effect on Newtonian gravitation.

Between 1990 and 2005, six NASA spacecraft flew by earth, using the technique of earth gravity assist, to either gain or lose heliocentric orbital energy. Tiny anomalies were observed [14] corresponding to an unexplained velocity difference of the order of a few mm/s at perigee. However, the observations were very precise, with systematic experimental error ranging from 0.01 mm/s to 1 mm/s, so the observations could not be put down to experimental error. Of course, the tiny anomalies may have been potentially due to many causes because the perigee velocities of the spacecraft were of the order of a few km/s, but the causes could not be explained despite a careful audit and consideration of various possible factors including general relativistic effects.[15]

Further, Anderson et al[14] found an empirical formula which fitted all six flybys:

$$\frac{\Delta V_{\infty}}{V_{\infty}} = K(\cos \delta_i - \cos \delta_o), \quad (10)$$

where  $\Delta V_{\infty}$  was the difference between the incoming and outgoing asymptotic velocity in a geocentric frame. (Conceptually, this is the hyperbolic excess velocity at infinity of an osculating Keplerian trajectory, so the difference ought to have been zero on the Newtonian theory.) Further,  $\delta_i$  and  $\delta_o$  were the declinations of the incoming and outgoing asymptotic velocity vectors. The constant  $K = 3.099 \times 10^{-6}$  was expressed in terms of the Earth's angular rotational velocity as  $\omega_E$  ( $7.292115 \times 10^{-5}$  rad/s), its mean radius  $R_E$  (6371 km) and the speed of light  $c$  by

$$K = \frac{2\omega_E R_E}{c}.$$

Clearly, this expression for  $K$  shows that the flyby anomaly is an effect related to the rotation of the earth, a relation expected on RGT, but inexplicable on Newtonian gravitation. If we look for a  $\frac{v}{c}$  term, related to earth's rotation, using just dimensional analysis,  $K$  is the natural term that would arise. Clearly, also, if the incoming and outgoing declinations of the spacecraft are both zero (i.e., it enters and exits in

the equatorial plane), then the additional RGT force which accelerates it on entry will symmetrically equal the force which retards it on exit, so there will be no net gain or loss of asymptotic velocity. A net gain or loss will arise only in the event of a difference between the asymptotic incoming and outgoing declinations. Thus, the observed anomalous effect in the flybys is *prima facie* a systematic  $\frac{v}{c}$  effect depending on the rotational velocity of the earth, as expected on RGT.

Detailed modelling of the earth and exact calculations using RGT are still to be done. However, preliminary calculations already give a result which is very close. The figures below show a couple of calculations done for the Galileo (Fig. 3) and Cassini (Fig. 4) spacecraft. Past data on the orbits of these spacecraft was obtained using the NASA Horizons interface.

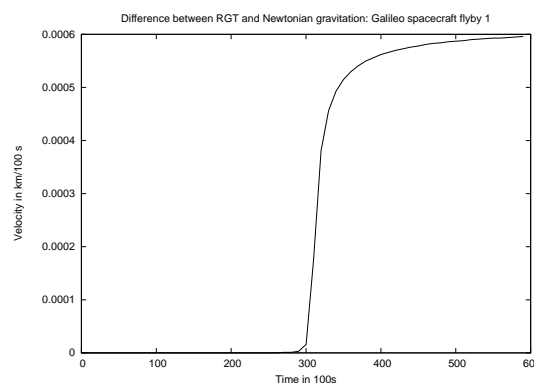


Figure 3: **Galileo.** The difference in the velocity between the solutions obtained using the new velocity-dependent RGT force and the Newtonian force, for the first flyby of the Galileo spacecraft. The  $x$ -axis is time (in units of 100 s) and the  $y$ -axis is difference of (scalar) velocity in units of km per 100 s.

The above calculations reproduce also the qualitative behaviour that most of the velocity gain or loss is close to the perigee. The computed increase or decrease in velocity is the right order of magnitude, and we expect greater accuracy with more sophisticated



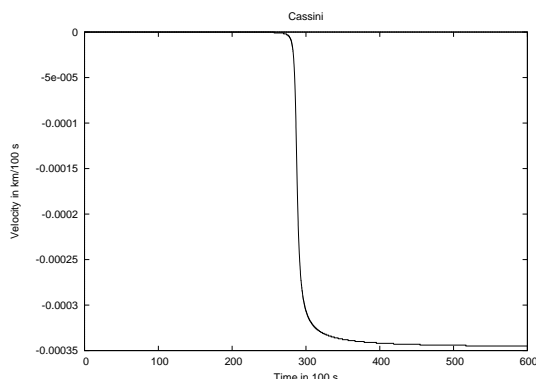


Figure 4: **Cassini**. The difference in the velocity between the solutions obtained using the new velocity-dependent RGT force and the Newtonian force for the earth flyby of the Cassini spacecraft. Same units as before. The calculated change in velocity is  $-3.4$  mm/s compared to the reported change of  $-2$  mm/s.

modelling of the earth.

## 8 Two body orbits

### 8.1 Laplace’s argument

The Lorentz covariant RGT described above should not be confounded with the naive theories of retarded gravitation proposed over a century ago. Those naive retarded gravitation (NRG) theories were pre-relativistic and aimed to explain the discrepancy between Newtonian gravitation and the observed anomalous advance of the perihelion of Mercury. While they succeeded in that aim, they suffered from a theoretical defect: two body orbits on those theories would be unstable, as pointed out by Laplace,[16] long ago.

Thus, NRG theories typically assumed that the gravitational force pointed towards the retarded position of the attracting body (and was equal to the inverse square of the retarded distance). Laplace’s objection to this was as follows. Consider two bodies

in circular motion around a common centre of mass— a typical problem of Newtonian gravitation. The line of action of the NRG force would not pass through the instantaneous centre of mass. Consequently, the system would be unstable (due to a delay torque).

Laplace’s argument does not apply to RGT for various reasons. First, relativistically, there is no such thing as “instantaneous centre of mass”. [17] That does not mean that all relativistic theories are unstable! Further, even if one somehow defines something which can be called the instantaneous centre of mass, as some people have attempted to do, no one has proved that the “centre of mass” so defined plays the same fundamental role in deciding stability as in Newtonian mechanics. Secondly, RGT involve FDEs which do not have the same theory of stability as ODEs.

Finally, RGT differs from NRG in that the RGT force depends also upon velocity. In the above situation, of circular 2-body orbits, this means that the RGT force does NOT point directly to the retarded position of the other body, as Laplace assumed. An easy calculation shows that, in the non-relativistic case, the RGT force points closer to the instantaneous centre of mass up to  $\frac{v^2}{c^2}$  terms (Fig. 5).

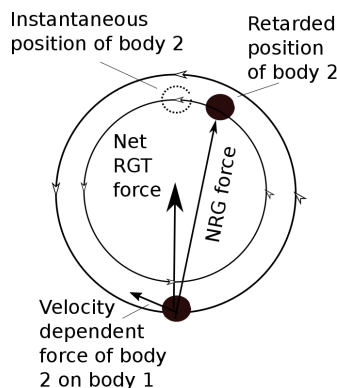


Figure 5: Difference between NRG and RGT force: Because the RGT force includes a velocity-dependent component, it points closer to the (non-relativistic) instantaneous centre of mass.

Departures from Newtonian gravitation at the  $\frac{v^2}{c^2}$  level are not undesirable. Thus, for the case of Mer-

cury, there is a long-known discrepancy with Newtonian gravitation. The classical GRT formula for the advance of perihelion  $\epsilon$ , based on geodesics of the Schwarzschild solution, is

$$\epsilon = 24\pi^3 \frac{a^2}{T^2 c^2 (1 - e^2)} \quad (11)$$

where  $a$  is the semi-major axis of the ellipse,  $e$  its eccentricity,  $T$  the time period, and  $c$  the speed of light. For  $e \sim 0$ , when the orbit is nearly circular,  $\frac{2\pi}{T}$  is an estimate of the angular velocity, and  $a$  is just the radius of the circle, so that  $\frac{4\pi^2 a^3}{T^2}$  is just  $v^2$ . As such, the anomalous perihelion advance of Mercury can be regarded as approximately a  $\frac{v^2}{c^2}$  effect.

Of course, since with RGT, unlike GRT, we can easily do many-body problems, the right way would be to do a many body problem, and not just linearly add up perihelia advances “due to” various causes.

## 8.2 Two body problem for Jupiter

We conclude with a solution of a planetary 2-body problem in RGT, as an example of how to do many body problems in RGT. The relevant equations are derived in the appendix. The equations initially involve two proper times. To solve them, we need to rewrite the equations in terms of a single coordinate time. The only non-obvious trick here is the particular 3+1 decomposition to use, as described in the appendix. (It is obvious, once we see it.)

Secondly, since these are FDEs, we need to prescribe past data. For the planetary 2-body problem, we took up the sun-Jupiter case, and prescribed past data as perfectly circular theoretical Newtonian orbits about a common centre of mass. In the Newtonian case, circular orbits remain circular, but with RGT the circle gets deformed into an ellipse, as shown in Fig. 6. However, there is no runaway instability. Had we used NRG instead, that would result in a runaway instability, as shown in Fig. 7

## 9 Conclusions

FDEs are an essential feature of post-relativity physics. RGT arises from modifying Newtonian grav-

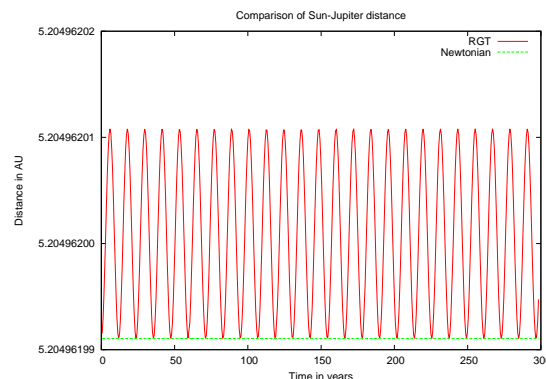


Figure 6: The relative distance of the two bodies  $(\sqrt{(\vec{y}_1(t) - \vec{y}_2(t))^2})$  stays stable. The average RGT distance is marginally larger, since RGT changes the prescribed Newtonian circular orbit to an ellipse.

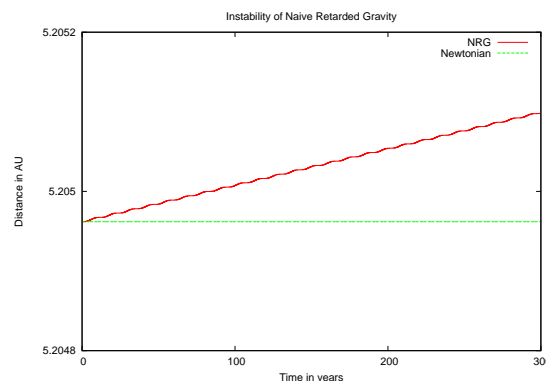


Figure 7: In naive retarded gravity (NRG) the force varies inversely as the square of the retarded distance and points in the direction of the retarded position of the other body. The plot brings out the instability of NRG: the distance between the two bodies  $(\sqrt{(\vec{y}_1(t) - \vec{y}_2(t))^2})$  systematically increases over time in NRG.

itation to make it Lorentz covariant. This modifies the Newtonian gravitational force to include a velocity-dependent term, and the basic equations of motion of RGT are FDEs. The velocity dependent term suggests a natural explanation of the flyby anomaly as due to the rotation of the earth. Though tiny it may add up significantly across a billion stars in a spiral galaxy, which are all co-rotating. Because RGT is a theoretically essential correction to Newtonian gravitation, all galactic simulations and calculations of dark-matter must be redone using RGT. RGT does not suffer from the instability problem of pre-relativistic naive theories of retarded gravitation.

## Appendix: Equations of motion in RGT

Consider two particles, with world lines given by  $Y_1(s_1)$  and  $Y_2(s_2)$ , where  $Y_1$  and  $Y_2$  are 4-vectors, and  $s_1$  and  $s_2$  are the respective proper times. The equations of motion in RGT are

$$m_1 \frac{d^2 Y_1}{ds_1^2} = F_{12}, \quad m_2 \frac{d^2 Y_2}{ds_2^2} = F_{21}, \quad (12)$$

where  $m_1$  and  $m_2$  are the respective rest masses of the two particles, and  $F_{12}$  the 4-force exerted by particle 2 on particle 1 is given by the Lorentz covariant expression

$$\begin{aligned} F_{12} = & -\frac{kc^3}{(R_{2\text{ret}} \cdot V_{2\text{ret}})^3} R_{2\text{ret}} \\ & + \frac{kc^3}{(R_{2\text{ret}} \cdot V_{2\text{ret}})^3} \frac{(R_{2\text{ret}} \cdot V_1)}{(V_{2\text{ret}} \cdot V_1)} V_{2\text{ret}} \\ \equiv & \left[ -\frac{kc^3}{(R_2 \cdot V_2)^3} R_2 + \frac{kc^3}{(R_2 \cdot V_2)^3} \frac{(R_2 \cdot V_1)}{(V_2 \cdot V_1)} V_2 \right]_{2\text{ret}}. \end{aligned} \quad (13)$$

Here,  $k = Gm_1m_2$ ,  $G$  is the Newtonian gravitational constant,  $c$  is the speed of light,  $R_{2\text{ret}} = Y_{2\text{ret}} - Y_1$  is the retardation vector,  $V_1 = \frac{dY_1}{ds_1}$  and  $V_2 = \frac{dY_2}{ds_2}$  denote the respective 4-velocities, and, in (14),  $[\ ]_{2\text{ret}}$  indicates that the quantities with subscript 2 are to be evaluated at the corresponding retarded proper time, as explicitly indicated in (13). The other force  $F_{21}$  is given by interchanging 1 and 2 in (14).

In coordinates, if  $Y_1 = (ct, \vec{y}_1(t))$ , and  $Y_2 = (ct, \vec{y}_2(t))$ , the retarded coordinate time  $t_{12}$ , in the force  $F_{12}$  acting on  $Y_1$  at time  $t_0$ , is the root of the equation

$$c^2(t - t_0)^2 = r_{12}^2 \equiv (\vec{y}_2(t) - \vec{y}_1(t_0))^2, \quad (15)$$

satisfying  $t < t_0$ . That is, it is the value of  $t$  at the spacetime point where the backward null cone from  $Y_1(t_0)$  intersects the world line  $Y_2$ . The corresponding distance  $r_{12}$  is the retarded distance from particle 1 to particle 2. A similar equation holds for  $t_{21}$ , the retarded coordinate time in  $F_{21}$ , the asymmetry being only in the arguments of  $\vec{y}_1$  and  $\vec{y}_2$ .

Since the two equations (12) have to be solved simultaneously, it is convenient to use a common time parameter, which we take to be the coordinate time  $t$ . We assume that the functions  $t = t_1(s_1)$  and  $t = t_2(s_2)$  are suitably invertible and (at least) twice continuously differentiable, and will not explicitly indicate them further. Thus, we have  $\frac{dt}{ds_1} = \gamma_1$ , and  $\frac{dt}{ds_2} = \gamma_2$ , where  $\gamma_1$  and  $\gamma_2$  are the respective Lorentz factors. Using an overdot to denote derivatives with respect to  $t$ , we have, by the chain rule,  $V_1 = \frac{dY_1}{ds_1} = \frac{dY_1}{dt} \frac{dt}{ds_1} = \gamma_1 \dot{Y}_1$ . Similarly,  $\frac{dV_1}{ds_1} = \frac{dV_1}{dt} \frac{dt}{ds_1} = \gamma_1 \dot{V}_1$ .

Hence, (12) can be rewritten

$$\begin{aligned} \dot{Y}_1 &= \frac{1}{\gamma_1} V_1, \\ \dot{V}_1 &= \frac{1}{\gamma_1} \frac{F_{12}}{m_1}, \end{aligned} \quad (16)$$

with similar equations for particle 2.

Since the zeroth component of these equations is not independent, we can write them in 3-vector notation using  $Y_1 = (ct, \vec{y}_1(t))$ ,  $Y_2 = (ct, \vec{y}_2(t))$ , so that  $\dot{Y}_1 = (c, \vec{v}_1)$ ,  $\dot{Y}_2 = (c, \vec{v}_2)$ . Let  $\vec{u}_1$  and  $\vec{u}_2$  denote the space components of the velocity 4-vectors  $V_1$ , and  $V_2$ , so that  $\vec{u}_1 = \gamma_1 \vec{v}_1$ ,  $\vec{u}_2 = \gamma_2 \vec{v}_2$ . Further, we let  $\vec{r}_{2\text{ret}} = \vec{y}_2(t_{12}) - \vec{y}_1(t)$ , denote the 3-vector corresponding to  $R_{2\text{ret}}$ . Then the final equations are

$$\begin{aligned} \dot{\vec{y}}_1 &= \frac{1}{\gamma_1} \vec{u}_1 \\ \dot{\vec{u}}_1 &= \frac{1}{m_1 \gamma_1} \vec{f}_{12} \end{aligned} \quad (17)$$

where

$$\vec{f}_{12} = a \vec{r}_{2\text{ret}} + b \vec{u}_{2\text{ret}} \quad (18)$$

$b = a\tilde{b}$ , and

$$a = - \left[ \frac{kc^3}{(R_2 \cdot V_2)^3} \right]_{2\text{ret}} \quad (19a)$$

$$\tilde{b} = - \left[ \frac{(R_2 \cdot V_1)}{(V_2 \cdot V_1)} \right]_{2\text{ret}}$$

or

$$a = \left[ \frac{k}{r_2^3} \right]_{2\text{ret}} \quad \tilde{b} = \left[ \frac{r_2}{c} \right]_{2\text{ret}} \quad (19b)$$

Here, (19b) is the non-relativistic limit of (19a).

The equations of motion (17) (accompanied by (18), (19a) or (19b)), together with the corresponding equations for particle 2 are the four 3-vector equations (or 12 equations in all) we actually solved for the sun-Jupiter problem.

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