A Brief History of Gravitation: Copernicus to Newton *

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Abstract

The article traces the contributions of the pioneers of the theory of gravitation, in particular the heliocentric model of Nicolaus Copernicus, planetary data collected by Tycho Brahe, Johannes Kepler's analysis of Tycho's data leading to the formulation of his three laws of planetary motion, and the "falling of apple" episode that gave Newton a sudden flash of a larger vision that unified the orbital motion of the moon, the orbital motion of the planets and the falling of terrestrial objects downward, into one single law of universal gravitation. Based on the model of the heliocentric universe the geocentric paths of Venus and Mars have been constructed in two ways, using a geometrical method, and plotting the relevant parametric equations. The orbital radius R and the time period T of each planet's revolution around the sun has been calculated from the observed value of the angle of its maximum deviation from the sun and the measured value of its synodic period. Kepler's laws of planetary motion have been reviewed and the third law has been checked using the calculated values of T and R. The role played by the 3rd Law of planetary motion in shaping Newton's law of universal gravitation has been highlighted. How the inverse square nature of the law of gravitation relates the orbital motion of the moon to the falling of an apple has been worked out in mathematical details.

1 Newton and the Apple and the Moon

We have all heard the story that Newton was led to the law of universal gravitation while

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watching an apple fall to the ground. This story has become as much a landmark in the history of science as the discovery that followed it. We shall piece together relevant parts of the story to make a story of our own and make a picture of the mind of a genius that saw in such an insignificant event a glimpse of a much larger scheme of the universe that works behind the motion of the moon around the earth, the motion of the planets around the sun, as much as in the falling of an apple, unifying all such diverse and unrelated phenomena into one single law of nature[1, 2, 3, 4, 5, 6].

The period was 1665-1666, the years of plague, when many public institutions were closed and Isaac Newton, now 23 years old, had to leave Cambridge to take shelter in his mother's farm Woolsthrope Manor near Grantham in Lincolnshire. One moonlit evening he was sitting in the garden. His mind was immersed in a deep thought, seeking answer to the question, "what force makes the moon go round the earth?" He looked up to the sky and saw the moon and thought of the force of gravity (of the earth) extending to its orbit. An apple fell from a tree nearby. Spurred by this incident, an idea came to his mind that the same force that the earth exerted on the apple making it fall straight down might also be exerted on the moon making it go round. His conjecture:

Conjecture 1 *The moon is a falling body, just like an apple, falling under the earth's force of gravity.*

We shall explain how Newton calculated the rate of falling of the moon. Let us consider an apple which has been thrown into the air from the ground, as shown in Fig.1(a). The trajectory is a parabola. At the top of the trajectory its tangent is a horizontal line. The fall of the apple in time t is the distance y, measured vertically downward from the horizontal tangent, and is given by the well known formula

$$y = \frac{1}{2}gt^2. \quad (1)$$

Newton applied the same principle to the "falling moon". However, the trajectory of the moon in this case is not a parabola, but a circle. The surface of the earth, lying underneath the moon, curves into a sphere as the moon travels along its orbit, whereas for an apple or a cricket ball, whose range of flight is negligible compared to the radius of the earth, the ground underneath is nearly a flat surface. Nevertheless, Newton calculated the distance y by which the moon would be falling in a small time $t \ll T$, where T the period of one complete revolution of the moon around the Earth. He probably used simple geometry and Pythgoras's theorem for his calculations.

Fig. 1(b) depicts (part of) the circular trajectory of the moon around the earth. The speed of the moon in the orbit is $v = \omega R$ where ω is the angular velocity of revolution of the moon around the earth, and R is the radius of the moon's orbit.

Let us get some crucial data first. The radius of the moon's orbit, as calculated by the Greek astronomer Hipparchus in 130 B.C., was approximately 3.8×10^8 m, which Newton should have used for his calculations. The

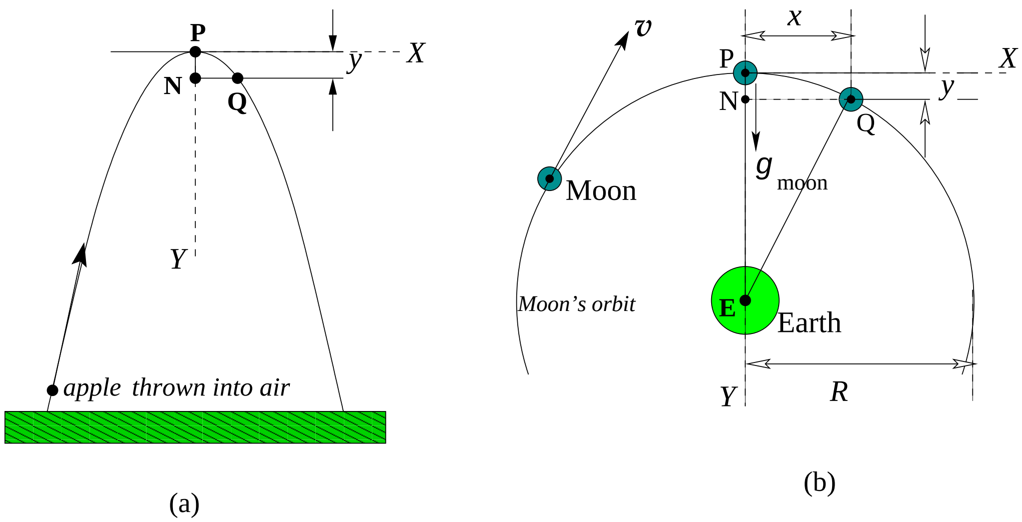

Figure 1: *Trajectories of a projected apple in (a), and the Moon in (b).*

period of the moon's revolution around the earth is $T = 27.3$ days $\approx 2.4 \times 10^6$ s. From these two figures we can calculate the orbital velocity v of the moon $v = \omega R = \frac{2\pi}{T} \times R = \frac{2\times\pi\times 3.8\times 10^8}{2.4\times 10^6} \approx 10^3$ m/s $= 1$ km/s.

At a certain time $t = 0$ the moon is at the point P. We have drawn the Cartesian X and Y axes through P such the the X axis is tangential to the circular path pointing in the direction of the moon's motion, and the Y axis is pointing down radially to the centre of the circle. If there had been no Earth, the moon would be following a straight path, along the X axis. The Earth's gravity bends the path downward.

Let P and Q be two locations of the moon differing by a very small time interval δt (say, 1 second.) The coordinates of Q are (x, y). $x \approx v\,\delta t = \omega R\,\delta t$. y is the distance by which the centre of the moon "drops" vertically downward in time δt from the straight line path along the X axis. For the very small time δt this drop y is very small compared to the distance x the moon travels. For example if we take $\delta t = 1$ s, then $x = v\,\delta t = 1000$ m, whereas $y \approx 1.3$ mm, as we shall soon find out. In the following estimate we shall therefore set $y \ll x$.

Note that $\overline{EN} = R - y$. By Pythagoras's theorem $\overline{EN}^2 + \overline{NQ}^2 = \overline{EQ}^2$. In terms of the coordinates,

$$(R-y)^2 + x^2 = R^2.$$

Cancelling R^2 on both sides,
$$-2Ry + y^2 + x^2 = 0.$$

Or, neglecting y^2 compared to x^2,
$$y \approx \frac{x^2}{2R} = \frac{(\omega R\,\delta t)^2}{2R} = \frac{1}{2}(\omega^2 R)\,\delta t^2. \quad (2)$$

Comparing Eq. (2) with (1) we find that in every tiny interval of time δt the moon drops perpendicular to the tangent drawn to its trajectory and the acceleration of this fall is:

$$g_{\text{moon}} = \omega^2 R = \frac{\omega^2 R^2}{R} = \frac{v^2}{R}. \quad (3)$$

This is also the familiar expression for what we often refer to as the "centripetal acceleration". Taking the values of v and R already given, Newton obtained the acceleration of the moon to be

$$g_{\text{moon}} = \frac{10^6}{3.8 \times 10^8} = 0.26 \times 10^{-2}\,\text{m/s}^2. \quad (4)$$

The distance through which the moon falls in the tiny interval δt, say 1 second, is then

$$y = \frac{1}{2}\,g_{\text{moon}}\,\delta t^2 = 0.13 \times 10^{-2}\,\text{m}. \quad (5)$$

This acceleration given in (4) is too small compared to the the acceleration due to gravity near the surface of the earth, given as $g = 9.8\,\text{m/s}^2$, which is also the acceleration of the apple. The fall distance given in (5) is also incredibly small compared to a fall distance of 4.9 m in one second for an apple. Such small values puzzled Newton. He left the problem of "falling moon" for the time being and diverted his mind to seek answer to a larger question "what forces are acting on the planets making them follow the orbits as described by Kepler's laws of planetary motion?"

2 Heliocentric Model of Copernicus

2.1 Motion of planets as seen from Earth - Geocentric view of the Greek school

The theory of universal gravitation did not descend on Newton's mind in one stroke with the falling of an apple. Newton arrived at this theory primarily by analyzing Kepler's laws of planetary motion. Kepler had earlier formulated these laws by a meticulous analysis of the data on the position of Mars and other planets observed and recorded over a period of thirty five years by Tycho Brahe. These historical anecdotes are important and should be part of one's understanding of gravitation. We shall try to present an elementary sketch of what had happened before Newton[7].

The ancients had keenly observed the pattern of the motion of the heavenly objects in the sky, and had invented a scheme of their motion around the Earth. The Greek philosophers, from Aristotle to Plato, from Ptolemy to his followers, had drawn up an earth-centric model of the universe, known as the *geocentric* universe, in which the Earth was considered to be fixed in space, and constituted the centre of the universe. The heavenly objects, namely the sun, the moon, the planets and the stars, were hypothesised to be moving in *perfect circles*, on crystal spheres, in perfect harmony, because a circle was considered to be a perfect curve, and because for the heavenly bodies only the perfection of circular motion was permitted.

This simplistic idea came into conflict with the apparent motion of the planets. Seen in the background of the stars the planets were moving non-uniformly. They moved eastward nearly straight for much of the time, in what is now referred to as the *direct motion*. However, at certain points they slowed down, reversed the motion westward, made a loop-the-loop, then preceeded eastward, as before. This reverse motion was called *retrograde motion* [8, 9].

We have shown this pattern in Fig.3a for Venus and in Fig.5a for Mars. In both figures we have labelled the background stars as "fixed stars".

The ancients had also observed that the brightness of the planets was not constant but changed with time. The retrograde motion and the varying brightness pointed to a complex motion of the planets, quite different from a simple minded uniform circular motion, in which the distances of the planets from the earth were continuously changing.

The Greek philosophers, strongly rooted in the faith that only circular motion was permissible for heavenly bodies, had tried to explain away the above mentioned anomalous movement of the planets by hypothesizing the planetary motion to be a combination of two or more circular motions. We have illustrated this idea in Fig. 2. With the earth E as the centre, the planet P moved on a *smaller* circle, called *Epicycle*, the centre of which moved on a *bigger* circle called *Deferent* (fixed on the crystal sphere), as shown in (a). If this construction did not fit the observation, they added further epicycles, as shown in Fig.(b). [10, 11]

2.2 Motion of planets as seen from Earth - Heliocentric Explanation of Copernicus

A different model of the universe was suggested by a Polish astronomer Nicolaus Copernicus in 1543 in his book *De revolutionibus orbium coelestium (On the revolutions of the Celestial Spheres)*. In the Copernican system the Sun was the centre of the universe and assumed to be immovable. The stars were also fixed on the "immovable celestial sphere" with its centre on the sun. The planets, including the Earth, moved around the sun in a *uniform circular motion*. This model was called the *Heliocentric universe* (Helio=the Sun). Another feature of Copernicus's proposal was that the Earth rotated about its axis, once a day, as it moved along its orbit around the sun.

Copernicus had cited the Greek philoso-

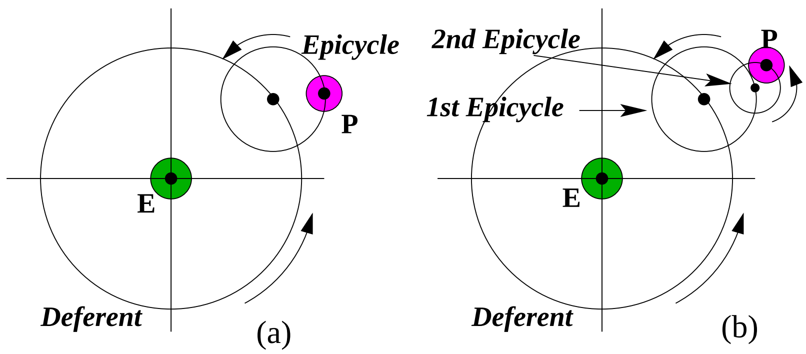

Figure 2: *Epicycle on Deferent*

pher Aristarchus as the source of his central heliocentric idea.

This new model came in direct conflict with the pre-existing Ptolemic view, the geocentric universe, which was also the most natural and obvious thing to believe, and which had the support of the Roman Catholic Church. Copernicus, himself a cleric under the Catholic church, in order to avoid any controversy, suggested that this model was for mathematical convenience only, and was not necessarily the truth. The mathematical calculations needed to predict the position of the planets in the sky, as in the Ptolemic scheme, could only be simplified if one used this model.

2.3 Geocentric path of Venus from the Copernican model

We shall follow the hint given by Copernicus, use the heliocentric model as a starting point, and reconstruct the Ptolemic paths of the planets Venus and Mars, each moving on its own epicycle around a deferent, and causing the "loop-the-loop" retrograde motion. We shall refer to this path as the *geocentric path*, or *g-path* for abbreviation.

For the geometric constructions that will follow, it will be convenient to distinguish between the *inferior planets*, or, the *inner planets*, having orbital radii less than that of the Earth, and the *superior planets* or the *outer planets*, having orbital radii larger than that of the Earth. Venus belongs to the first category, and Mars to the second. We shall measure planetary distances in Astronomical Unit (AU). *One AU is equal to the mean radius of the Earth's orbit around the Sun,* and is equal to 1.496×10^{11} m.

In this subsection we confine ourselves to Venus, and illustrate our construction of the g-path of the planet in Fig. 3. The Earth, the Sun and Venus have been represented by the letters E,S and V respectively.

Fig. 3(b) shows the Copernican picture of the motion of E and V around S. Here S is the centre of the universe. It is fixed and is the origin of the Cartesian coordinate system.

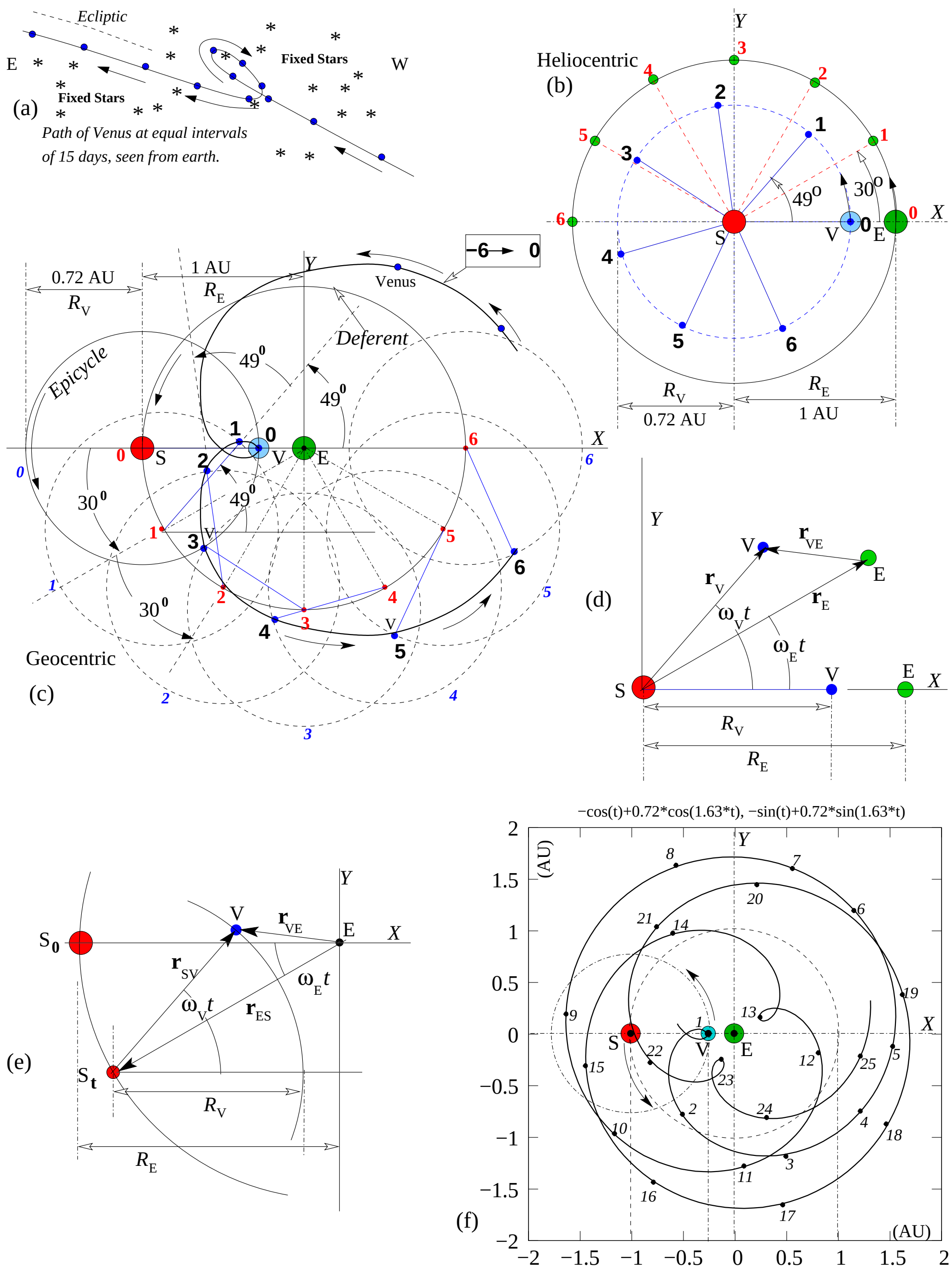

Figure 3: *Path of Venus. (a) Seen against background stars; (b) Heliocentric paths of V and E; (c) Geometrical construction of the g-path; (d) and (e) Radius vectors R_E, R_V, R_{VE}, etc; (f) Gnuplot of the g-path.*

Figs. 3(c),(f) show the g-path of V. Here E is the centre of the universe and the origin of the Cartesian axes X and Y. Seen from E, the Sun S is moving on the *Ecliptic*, which in this case is the *Deferent.* And V, which is actually moving around S, is seen to be moving on the *Epicycle* which is a moving circle with its centre always lying on the deferent.

We shall use the following data [12] for our calculations and plotting. The radii of the orbits of the Earth and Venus around the Sun are $R_E = 1$ AU and $R_V = 0.72$ AU respectively. The period of one complete revolution around the Sun is $T_E = 365.25$ days for the Earth and $T_V = 224.7$ days for Venus.

The orbital angular velocities of E and V follow from their periods. $\omega_E = \frac{2\pi}{12}$ rad/month, and $\omega_V = \frac{2\pi}{12} \times \frac{T_E}{T_V} = \frac{2\pi}{12} \times \frac{365.25}{224.7} \approx 1.63 \times \frac{2\pi}{12}$ rad/month. Converting into degrees, $\omega_E = 30^0$ per month, and $\omega_V \approx 1.63 \times 30^0 \approx 49^0$ per month. We shall adopt one month as the unit of time.

In Fig. 3b we have shown two concentric circles, of radii R_V and R_E on which E and V are revolving around the centre S in the *anticlockwise* direction. This is the Copernican view.

The locations of V at equal time intervals of *one month* are represented by large boldface sans serif numerals **0,1,2,3,4,...**. Similarly normal size numerals 0,1,2,3,4,... indicate the locations of E in Fig. (b) and the locations of S with respect to E in Fig (c) at the same time intervals.

Let us now get a *qualitative* understanding of the geocentric motion on the basis of the Copernican hypothesis (Fig. c). The point S is always located at a distance R_{E} from E. As seen from E, it is moving in a circle of radius R_{E} anticlockwise. This circle is called the *Ecliptic*. For our purpose this circle is the *Deferent*.

The point V is always moving around S in a circle of radius R_{V}, according to the Copernican model. In the geocentric picture S is moving, and therefore V is moving in a circle of the same radius about the moving point E. This moving circle is then identified with the *Epicycle*.

In Fig (c) we have shown the epicycle at seven instants of time $t = 0, 1, 2, 3, 4, 5, 6$, labelling them by slanted numerals *0,1,2,3,4,5,6*.

Let us now consider the instant $t = 0$. In Fig (b) V and E are both on the right side of the centre S. Therefore, in Fig (c) S and V are on the left of the centre E and (0, **0**) mark the locations of (S,V).

After one month, i.e., at $t = 1$, V moves to **1** and E moves to 1 in Fig. (b). In Fig (c) E remains stationary at the origin, whereas V is riding on the epicycle. The centre of the epicycle has moved by angle 30^0 to its new location 1, whereas V has moved by angle 49^0 to **1**.

Note that the line $\overrightarrow{1E}$ in (c) is parallel to the line $\overrightarrow{S1}$ in (b), and the line $\overrightarrow{1\mathbf{1}}$ in (c) is parallel to the line $\overrightarrow{S\mathbf{1}}$ in (b).

At the end of two months, i.e., at $t = 2$, V moves to **2** and E moves to 2 in Fig. (b). In Fig (c) the centre of the epicycle has moved further by another angle of 30^0 to its new location 2, whereas V has moved further by another angle of 49^0 to the location **2**. The line $\overrightarrow{2E}$ in (c) is parallel to the line $\overrightarrow{S2}$ in (b), and the line $\overrightarrow{2\mathbf{2}}$ in (c) is parallel to the line $\overrightarrow{S\mathbf{2}}$ in (b).

Proceeding in this way we find the positions of V at $t = 3, 4, 5, 6$, shown as **3,4,5,6**. Joining the points **0,1, ... ,6** by a smooth curve we have completed the construction of the g-path **0-1-2-3-4-5-6** traced by Venus in six months, from $t = 0$ to $t = 6$.

We can extrapolate the path to the past, from $t = 0$ to $t = -6$, by making an inversion of the original curve **0-1-2-3-4-5-6** and then adding to it. This extrapolated curve is labelled as $\boxed{\textbf{-6}\rightarrow\textbf{0}}$.

We have thus obtained the g-path of Venus, spread over one full earth year, from $t = -6$ to $t = +6$ (time measured in months) on the basis of the heliocentric model of Copernicus. The loop-the-loop cusp is prominent at $t = 0$, i.e., around the point **0** in Fig.(c).

The same path can be obtained using coordinate geometry and “gnuplot”. For this we need the transformation equation that will convert the Copernican orbit into a Ptolemian orbit.

Referrring to Fig.(d) the radius vectors of V and E, are $\mathbf{r}_V$ and $\mathbf{r}_E$. The radius vector of V relative to E is $\mathbf{r}_{VE} = \mathbf{r}_V - \mathbf{r}_E$. Then

$$\begin{aligned}
\mathbf{r}_V &= R_V(\cos\omega_V t\,\mathbf{i} + \sin\omega_V t\,\mathbf{j}) & (a)\\
\mathbf{r}_E &= R_E(\cos\omega_E t\,\mathbf{i} + \sin\omega_E t\,\mathbf{j}) & (b)\\
\mathbf{r}_{VE} &= (R_V\cos\omega_V t - R_E\cos\omega_E t)\mathbf{i} + (R_V\sin\omega_V t - R_E\sin\omega_E t)\mathbf{j}. & (c)
\end{aligned} \qquad (6)$$

We have shown the graphical construction of the vector $\mathbf{r}_{VE}$ in two different ways. In Fig. (d) the vectors $\mathbf{r}_V$ and $\mathbf{r}_E$ are drawn according to Eq. (6 a,b). The straight line joining E to V is the relative displacement vector $\mathbf{r}_{VE}$.

In Fig. (e) $\mathbf{r}_{ES}$ is the displacement of the sun S with respect to the “fixed” earth E, and $\mathbf{r}_{SV} = \mathbf{r}_V$ is the displacement of Venus V with respect to the “moving Sun”. Adding these two vectors we get back $\mathbf{r}_{VE} = \mathbf{r}_{ES} + \mathbf{r}_{SV}$.

$$\begin{aligned}
\mathbf{r}_{ES} &= R_E\left[\cos(\pi + \omega_E t)\,\mathbf{i} + \sin(\pi + \omega_E t)\,\mathbf{j}\right] = -R_E(\cos\omega_E t\,\mathbf{i} + \sin\omega_E t\,\mathbf{j}) & (a)\\
\mathbf{r}_{SV} &= R_V(\cos\omega_V t\,\mathbf{i} + \sin\omega_V t\,\mathbf{j}) & (b)\\
\mathbf{r}_{VE} &= (R_V\cos\omega_V t - R_E\cos\omega_E t)\mathbf{i} + (R_V\sin\omega_V t - R_E\sin\omega_E t)\mathbf{j}. & (c)
\end{aligned} \qquad (7)$$

We can now plot the g-path of Venus, given by the radius vector $\mathbf{r}_{VE}$, at different times. Taking the values of R_E, R_V and ω_V, ω_E obtained at the beginning of this subsection, we write the parametric equation of the path, with t as the parameter.

$$\begin{aligned}
x &= R_V \cos\omega_V t - R_E \cos\omega_E t &&= 0.72\cos(1.63 \times \tfrac{2\pi t}{12}) - \cos\tfrac{2\pi t}{12} \\
& &&= 0.72\cos(1.63\tau) - \cos\tau. \\
y &= R_V \sin\omega_V t - R_E \sin\omega_E t &&= 0.72\sin(1.63 \times \tfrac{2\pi t}{12}) - \sin\tfrac{2\pi t}{12} \\
& &&= 0.72\sin(1.63\tau) - \sin\tau.
\end{aligned} \tag{8}$$

In the last equalities we have chosen a new parameter $\tau = \frac{2\pi t}{12}$. We have made a plot of the above curve in Fig. 3(f) from $\tau = -\pi/2$ to $\tau = 9\pi$, covering 4.75 years. The numerals 1,2,3,..., 25 written alongside the path are in increasing order of time, but not placed at equal time intervals. The "loop-the-loop cusps" appear at the points 1,13,23. These are the points where retrograde motion of the planet appears to take place.

The g-path we have just plotted is not what is seen from the Earth. It is seen by a stationary observer sitting on the Z axis (i.e., the axis passing through the origin E in Figs. c and f) above the plane of the Ecliptic . An earthbound observer sees the projection of the g-path on the "celestial sphere". Noting that the plane of the orbit of Venus makes an angle of 3.4^0 with the plane of the Ecliptic [16] (though in our drawing we have taken them to be coplanar) one should be able to show that this projection is similar to the path shown in Fig. (a).

2.4 Geocentric path of Mars from the Copernican model

Mars has a special place in our narrative. By a painstaking analysis of the observation data of Mars, taken earlier by Tycho Brahe, Kepler was able to obtain his laws of planetary motion.

Let us obtain the necessary data for Mars. Radius of the orbit $R_M = 1.524$ AU. Period of one complete revolution around the Sun $T_M =$ 686.98 days. The orbital angular velocity of Mars: $\omega_M = \frac{2\pi}{12} \times \frac{T_E}{T_M} = \frac{2\pi}{12} \times \frac{365.25}{686.98} \approx 0.53 \times \frac{2\pi}{12}$ rad/month. Converting into degrees, $\omega_M \approx 0.53 \times 30^0 \approx 16^0$ per month.

We have illustrated the construction of the g-path of Mars, both by geometrical construction and by plotting of the parametric equation, in Fig. 5. The Earth is represented by E and Mars by M. This construction is similar to the one for Venus with one important difference. Venus is an inner planet having its orbit inside that of the Earth, and Mars is an outer planet having orbit outside Earth's. For the interior planets the ecliptic (larger circle) is the Deferent and the orbit (smaller circle) is the Epicycle. The roles get interchanged when we go to the outer planets. Now the orbit of the planet (larger circle) is the Deferent, and the Ecliptic (smaller circle) is the epicycle. Since this requires some clarification, and can be confusing, we shall first show a simpler construction of a part of the orbit of Mars in Fig. 4.

Fig. 4(a) shows the heliocentric motion of the Earth and Mars, and their locations at equal intervals of one month, with the Sun S fixed at the origin, all drawn to scale. The Earth and Mars are moving on their respective circular orbits around the Sun, of radii R_E and R_M and covering angles 30^0 per month and 16^0 per month respectively. As in the case of Venus we adopt one month as the unit of time. At $t = 0$, E and M are in conjunction, i.e., they lie on one straight line passing through S. We denote their locations as E_0 and M_0 respectively. At $t = 1$, E has moved by 30^0 to E_1 and M has moved by 16^0 to M_1. Continuing this way we get the locations $(\mathrm{E}_2, \mathrm{M}_2), \cdots, (\mathrm{E}_6, \mathrm{M}_6)$, corresponding to $t = 2, \cdots, 6$.

Going backward in time we get the locations $(\mathrm{E}_{-1}, \mathrm{M}_{-1}), \cdots, (\mathrm{E}_{-6}, \mathrm{M}_{-6})$, corresponding to $t = -1, \cdots, -6$. Note that the Earth comes back to the same location after 12 months, and therefore, the point E_6 coincides with point E_{-6}, and we have merged the two points with the label $\mathrm{E}_{6,-6}$. The displacements of Mars relative to the Earth at the times $t = n$ are given by the vectors $\mathbf{r}_n$, stretching from $\mathrm{E_n}$ to $\mathrm{M_n}$; $n = -6, \cdots + 6$.

We have brought these vectors to a common staring point at the location E in Fig. 4(b) The displacements of M relative to E at equal intervals of one month are now clearly seen.

In Fig. 4(c) we have joined the tips of these vectors with a smooth curve. This curve is the g-path over a period of one year, spread equally before and after the conjunction time $t = 0$.

We now come to a more methodical construction and plot of the g-path in Fig. 5. Since this construction is similar to the one for Venus, we shall avoid some of the details. Fig. 5(b) gives the Copernican picture of the positions of E and M at equal intervals of 1 month, with the Sun at the centre, and E and M going around it.

Fig. 5(b) shows the Copernican picture of the motion of E and M around S. Here S is the centre of the universe. It is fixed and is the origin of the Cartesian coordinate system.

Fig. 5(c) has E fixed at the origin of the Cartesian coordinates, as we are analyzing the motion of M relative to E. S goes on the smaller circle of radius R_E labeled *Ecliptic* with angular velocity ω_E, and M on the bigger circle of radius R_M labeled *Orb* with angular velocity ω_M. At $t = 0$, S is at S_0 and M is at M_0.

After time t, S has moved on the Ecliptic, through an angle $\omega_E t$, to the location $\mathrm{S_t}$. And M, riding on the Orb, has moved through an angle $\omega_M t$ to $\mathrm{M_t}$. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ represent the displacements $\overrightarrow{\mathrm{ES_t}}, \overrightarrow{\mathrm{S_tM_t}}, \overrightarrow{\mathrm{EM_t}}$ respectively. Then $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$

Here $\boldsymbol{a}$ is the displacement of S relative to E, as the former moves through the angle $\omega_E t$ on a circle of smaller radius R_E. Similarly, the vector $\boldsymbol{b}$ is the displacement of M relative to S, as the former moves through the angle $\omega_M t$ on a circle of larger radius R_M.

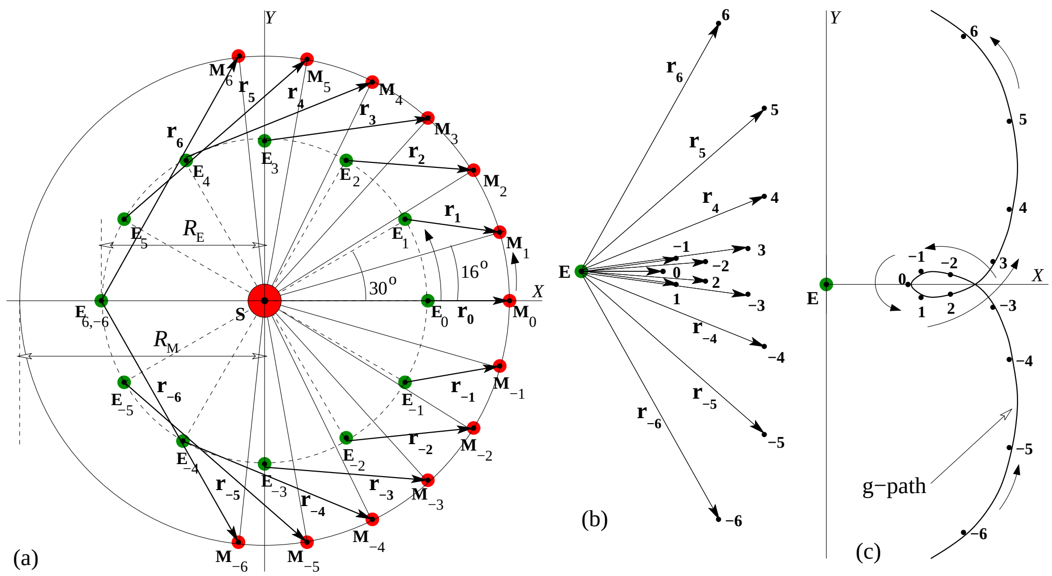

Figure 4: *Simple construction the g-path of Mars. (a) Heliocentric motion of E and M; and the relative displacemnt* $\{\mathbf{r}_n; n = -6, -5, \ldots, -1, 0, 1, \ldots, 5, 6\}$ *of M with respective to E over one year at 1 month intervals; (b) Relative displacement vectors* $\{\mathbf{r}_n\}$ *drawn from E; (c) Joining the tips of the vectors* $\{\mathbf{r}_n\}$ *with a smooth curve to get the g-path.*

Now, independent of S,E and M, the vector $\boldsymbol{b}$ can be looked upon as a displacement, in the anticlockwise direction, on a circle of larger radius R_M, and $\boldsymbol{a}$ as a displacement, in the same direction, on a circle of smaller radius R_E. Also $\boldsymbol{c} = \boldsymbol{b} + \boldsymbol{a}$. We have illustrated this in Fig 5(d).

We come to the *conclusion* that the *net displacement* of M, represented by the vector $\boldsymbol{c}$ is *same as a displacement on a bigger circle (deferent)*, represented by the vector $\boldsymbol{b}$, *followed by a displacement on a smaller circle (epicycle)*, represented by the vector $\boldsymbol{a}$.

In summary, E is fixed. An imaginary point I (shown in Fig 5d) is moving on the deferent of radius R_M *around E. Around this point I the planet M is moving on an epicycle of radius equal to the radius of the Earth* R_E.

Using the same graphical method employed for Venus we have done a graphical construction of the g-path of Mars in Fig 5(e).

To obtain the parametric equation for the g-path let us first note that:

$$
\begin{array}{lclcll}
\boldsymbol{a} & = & \mathbf{r}_{ES} & = & -R_E(\cos\omega_E t\,\mathbf{i} + \sin\omega_E t\,\mathbf{j}) \qquad \text{(see Eq.7a)} & (a) \\
\boldsymbol{b} & = & \mathbf{r}_{SM} & = & R_M(\cos\omega_M t\,\mathbf{i} + \sin\omega_M t\,\mathbf{j}) & (b) \qquad (9) \\
\boldsymbol{c} & = & \mathbf{r}_{EM} & = & (R_M\cos\omega_M t - R_E\cos\omega_E t)\mathbf{i} + (R_M\sin\omega_M t - R_E\sin\omega_E t)\mathbf{j}. & (c)
\end{array}
$$

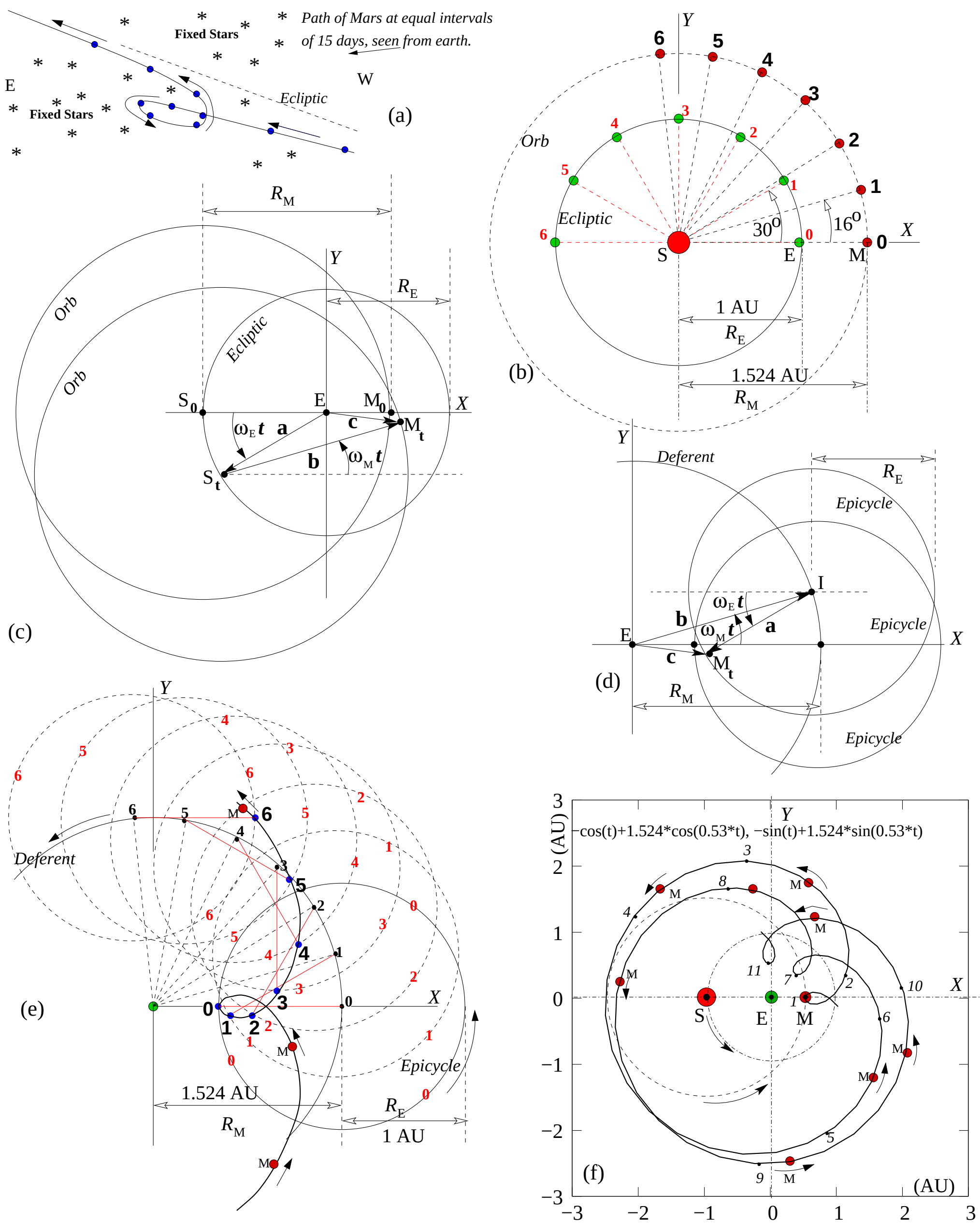

Figure 5: *Path of Mars. (a) Seen against background stars; (b) Heliocentric paths of M and E; (c) and (d) Identifying the deferent and the Epicycle; (e) Geometrical construction of the g-path; (f) Gnuplot of the g-path.*

We now write the parametric equation of the g-path as

$$\begin{aligned} x &= -R_E\cos\omega_E t + R_M\cos\omega_M t &&= -\cos\tfrac{2\pi t}{12} + 1.524\cos(.53\times\tfrac{2\pi t}{12}) \\ & &&= -\cos\tau + 1.524\cos(.53\tau). \\ y &= -R_E\sin\omega_E t + R_M\sin\omega_M t &&= -\sin\tfrac{2\pi t}{12} + 1.524\sin(.53\times\tfrac{2\pi t}{12}) \\ & &&= -\sin\tau + 1.524\sin(.53\tau). \end{aligned} \tag{10}$$

The parametric gnuplot of g-path is shown in Fig. 5(f). It has been drawn over the range $\tau = \pi/2$ to $\tau = 9\pi$, i.e., covering 4.75 years. However, (as in the case of Venus) this is not what is seen from Earth. An earthbound observer sees the projection of the g-path on the "celestial sphere". Noting that the plane of the orbit of Mars makes an angle of 1.9^0 with the plane of the Ecliptic [16] one should be able to show that this projection is similar to the path shown in Fig 5(a).

2.5 Calculation of the periods of the planets by Copernicus

Copernicus had also obtained the distances of the planets from the Sun and the time periods of their orbital motion. The ancient astronomers, starting from Ptolemy, had obtained the same or similar data. This should not be surprising since in the ancient world astrology, rather than astronomy, held sway, and it was of paramount importance to predict the time of appearance of a planet at a specified location in the sky, for astrological predictions.

We shall take a quick look at the trigonometrical methods which might have been applied by Copernicus to obtain the periods and the orbital radii of the planets.

First the period. The interval of time T_P in which a planet completes one full orbit around the Sun is the period of revolution of the planet. From our observation station, the Earth, we cannot determine T_P directly. We can, however, determine the *Synodic Period* τ of a planet by direct observation, from which one can obtain T_P.

We have explained synodic period in Fig. 6.

Let us take the planet P to be an *outer planet*, say Jupiter. There is a moment when when the Sun S and the planet P are in *opposition* with respect to the Earth. This means that they lie on the same straight line as E, but located on opposite sides of E, as seen in the configuration SE_1P_1 in Fig. 6(a). We can identify this moment by noting the date when P crosses the celestial *meridian* at *midnight*.

Now, both E and P are revolving around the Sun, but E is revolving faster than P (it should have been a common knowledge of the ancient astronomers that the angular motion of the sun around the ecliptic was faster than the angular motion of the outer planets on the celestial sphere.) After some time τ, S and P will come back to opposition once

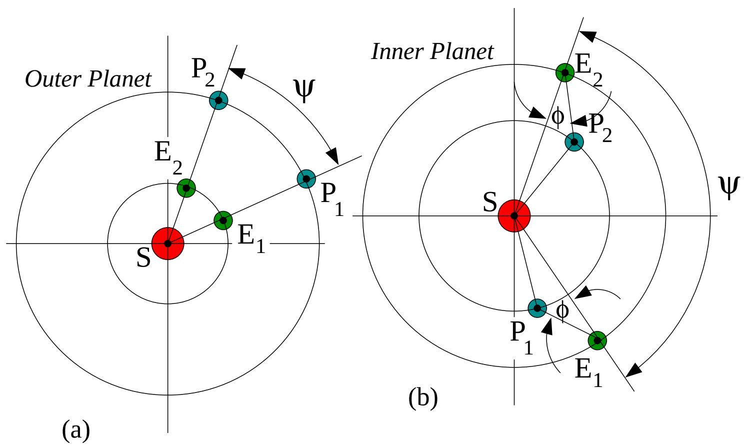

Figure 6: *Explaining Synodic Period. (a) Inner planet; (b) Outer planet.*

again, to the new configuration SE_2P_2. This time τ is the synodic period of the planet.

This will happen because in this time τ the planet will move through an angular displacement ψ, whereas E will go through an extra angle 2π, i.e., go through a total angular displacement $2\pi + \psi$. (This is similar to the movement of the minute hand over the hour hand in a clock.)

The angular velocity of E relative to P is $\omega_{rel} = \omega_E - \omega_P$. Then

$$\begin{aligned} \omega_{rel}\tau = 2\pi. \quad &\Rightarrow \quad \frac{2\pi}{T_E} - \frac{2\pi}{T_P} = \frac{2\pi}{\tau}. \\ &\Rightarrow \quad \frac{1}{T_E} - \frac{1}{T_P} = \frac{1}{\tau}. \end{aligned} \tag{11}$$

Inserting the vaue of τ obtained from measurement, and $T_E = 365.26$ days in Eq. (11) we get the required period T_P.

In the case of the *inner planets*, they will never be in opposition. We could have used their conjunction instead, i.e., position of the planet in the direction of the Sun along the line joining E and S. However, it is not possible to view the planet when it is in conjunction. Therefore we can offset the planet from the ES line by a certain angle ϕ. For example we view Venus some day when it makes angle $\phi = 30^0$ with the ES line (which will be, say 2 hours after sunset), and wait till it again makes the *same* angle $\phi = 30^0$ with the ES line. The period of waiting, τ, is the synodic period of the planet.

In this case $\omega_P > \omega_E$, and $\omega_{rel} = \omega_P - \omega_E$. Therefore,

$$\begin{aligned} \omega_{rel}\tau = 2\pi. \quad &\Rightarrow \quad \frac{2\pi}{T_P} - \frac{2\pi}{T_E} = \frac{2\pi}{\tau}. \\ &\Rightarrow \quad \frac{1}{T_P} - \frac{1}{T_E} = \frac{1}{\tau}. \end{aligned} \tag{12}$$

As in the previous case we obtain T_P from the known values of T_E and τ.

We now list below the Synodic Periods and the Time Periods of the planets as recorded by Copernicus.

Table 1: Copernican estimate of Time Periods of Planets (in years) and comparison with modern values*.

	0	1	2	3	4	5	6
1	Category →	inner	inner		outer	outer	outer
2	Planet →	**Mercury**	**Venus**	**Earth**	**Mars**	**Jupiter**	**Saturn**
3	τ (days) →	115.88	538.92	-	779.04	398.96	378.09
	τ (years) →	0.317	1.475	-	2.133	1.092	1.035
4	Formula:	Eq.(12) ↓	Eq.(12) ↓		Eq.(11) ↓	Eq.(11) ↓	Eq.(11) ↓
		0.24	0.596		1.885	11.869	29.57
5	**Copernican →**	**0.24**	**0.615**	**1**	**1.882**	**11.87**	**29.44**
6	Modern →	0.24	0.615	1.00	1.881	11.862	29.457

*Rows 3, 5 and 6 are taken from A.P.French, *op. cit.*

2.6 Calculation of the orbital radii of the planets by Copernicus

Copernicus calculated the orbital radii of the five planets (other than the Earth) known to the ancients. We shall obtain these values in Astronomical Units (AU), from direct observation of what one may call the *angle of maximum deviation*, denoted by θ_m.

We first take up the case of the *inner* planets. There are two planets in this category, Mercury and Venus. *Mercury* is the innermost planet. *Venus* comes next, its orbit placed between those of Mercury and the Earth.

We assume that to a first approximation the motion of the inner planet, as seen from the Earth, is along an epicycle of radius R_P, riding on a deferent of radius R_E. Here the subscripts $_P$ and $_S$ stand for the Planet and the Earth respectively. We have illustrated this in Fig. 7(a). The Sun S is at the centre of the epicycle, moving around E on the Deferent.

The planet P makes an angle θ with the Sun when it is at any arbitrary point P. However, there are two locations on the deferent, A and B, at which this angle has the maximum value θ_m. This angle is the angle of

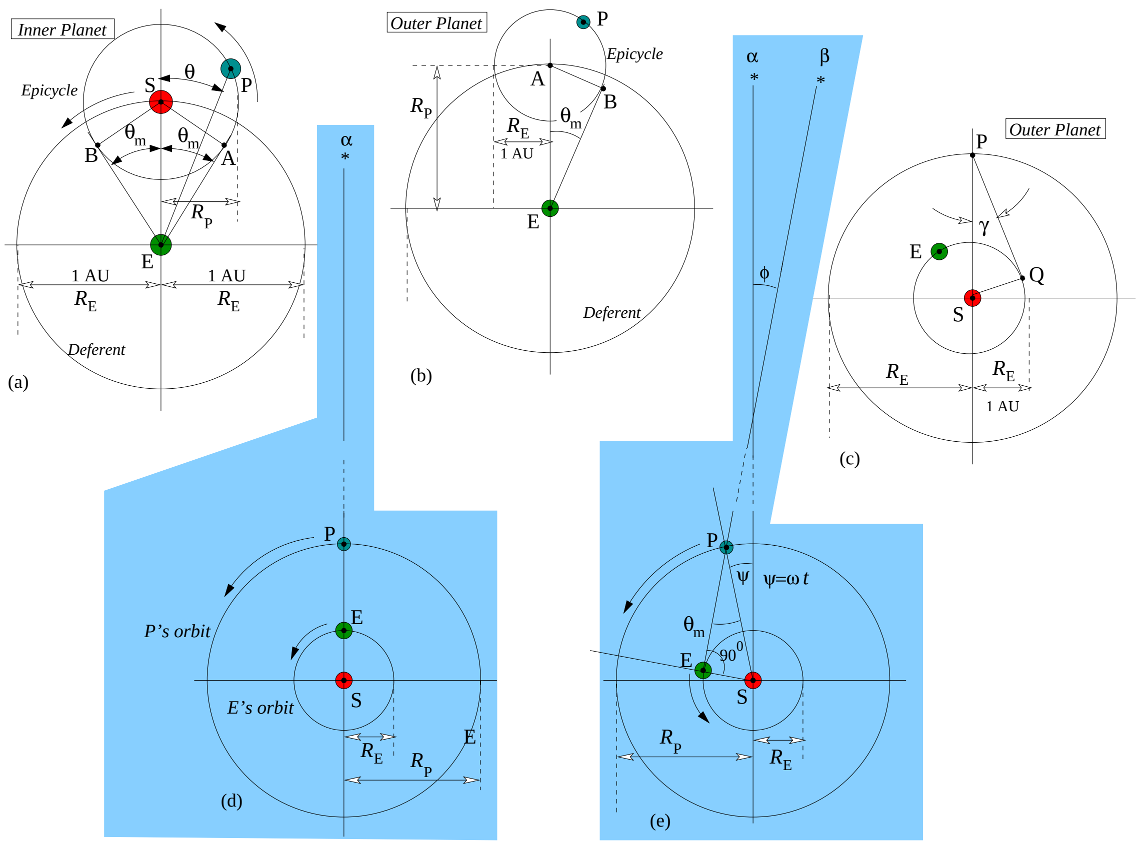

Figure 7: *Determination of the orbit radii of planets. (a) Inner planet; (b)-(e) Outer planet.*

maximum deviation.

Imagine that the planet is located at A. From the geometry of Fig. 7(a) the radius of the planet's orbit is

$$\begin{aligned} R_P &= \overline{\text{SA}} = \overline{\text{ES}} \sin\theta_m = R_E \sin\theta_m \\ &= \sin\theta_m, \ \text{(in AU)}, \end{aligned} \tag{13}$$

since $R_E = 1$ AU.

Let us turn to Venus as a special case. Venus is called the the *morning star* if seen in the morning and *evening star* [1] [14] , if seen in the evening (say, about 50 days later). Let us watch Venus in the evening. It can be seen as a bright object in the sky. Venus will set some time after Sunset. The time gap between the Sunset and Venus-set changes with time, but at a certain time of the year it reaches a maximum value, say τ_m (in hours). This time gap can be translated into θ_m It is obvious that $\theta_m = \frac{T_m}{24} \times 360^0$.

Alternatively, we can measure θ_m by mea-

[1] "The Greeks thought of the two as separate stars, Phosphorus and Hesperus untill the time of Pythagorus in the sixth century BC". See Wikipedia.

suring the maximum time lag τ_m between Venus-rise followed by Sunrise and convert it into θ_m applying the same formula.

The angle θ_m is *half of the angle that the angle $\angle AEB$ that the whole epicycle subtends at the centre of the deferent.* This should be clear from from Fig. 7(a). We shall apply this principle for the outer planets.

Let us now come to one of the *outer* planets (Mars, Jupiter, Saturn.) We have shown the planet in Fig. 7(b). In this case the deferent is the planet's orbit, of radius R_P, and the epicycle is the Earth's orbit of radius R_E. (See *conclusion* on page 12.) As in the case of the inner planets, we take θ_m as *half of the angle that the whole epicycle subtends at the centre of the deferent.* How to find the angle θ_m from observation?

The angle θ_m is half of the angle that the Earth's orbit subtends at the centre of P. To show this we have drawn a heliocentric (Copernican) view of S,E and P in Fig. 7(c). The outer circle is the orbit of P, the inner circle is the orbit of E, the Sun is at the centre S. The straight line PQ is a tangent to the Earth's orbit, so that the angle γ is half of the angle that the Earth's orbit subtends at the centre of P. Now compare the triangles $\triangle$ABE and $\triangle$SQP appearing in Figures (b) and (c) respectively. They are congruent, because they are both right angled triangles, AE = SP = R_P, and AB = SQ = R_E. Hence, $\angle$AEB = $\angle$SPQ = γ. Q.E.D.

One can now think of the following procedure. Let us find the location of P against a marker α on the celestial sphere (it can be a star, a nebulae, or some other astronomical object many light years away so that it can be taken to be permanently fixed on the celestial sphere) when S and P are *in opposition* (see page 14), as shown in Fig. 7(d).

We have drawn Figs. (d) and (e) on a light shaded background to demarcate their upper parts from the domains of Figs. (a)-(c).

A look at Table 1 shows that the time period of a planet's revolution around the Sun increases (and its angular velocity decreases) with increasing radius of its orbit. This means that the angular velocity ω_P of P is less than the angular velocity ω_E of E (the latter is 30^0 per month.) After some time t, E and P go to new locations as shown in Fig.(e), such that the angle $\angle$PES is a *right angle*, so that PE is tangent to the Earth's orbit and the angle $\angle$EPS is same as θ_m. (This will happen, for instance, on the day the planet crosses the meridian at sunset.) At this time, when viewed from Earth, P is seen against another marker β on the celestial sphere.

From the time periods of the planets listed in Table 1 we can find out ω_P. We have also measured t. Therefore we can find the angle $\psi = \omega_P t$ by which P has moved from the straight line $\widehat{\mathrm{S}\alpha}$, as shown in Fig.(e). Let ϕ be the angle between the straight lines $\widehat{\mathrm{S}\alpha}$ and $\widehat{\mathrm{S}\beta}$, as measured from Earth. Then $\theta_m = \phi + \psi = \phi + \omega_P t$. We now return to Fig.(b). The radius of the orbit we want to measure is given as

$$\begin{aligned} R_P &= \overline{\mathrm{EA}} = \overline{\mathrm{AB}} \csc\theta_m = R_E \csc\theta_m \\ &= \csc\theta_m, \text{ (in AU)}. \end{aligned} \tag{14}$$

We have tabulated in Table 2, the values of θ_m that were probably known to Copernicus, and the values of orbital radii he had obtained.

TABLE 2: COPERNICAN ESTIMATE OF ORBITAL RADII OF PLANETS* (IN A.U.), AND COMPARISON WITH MODERN VALUES.

	0	1	2	3	4	5	6
1	Category →	inner	inner		outer	outer	outer
2	Planet →	**Mercury**	**Venus**	**Earth**	**Mars**	**Jupiter**	**Saturn**
3	θ_m (deg) →	22.5	46		41	11	6
4	Formula: $R_P =$	$\sin\theta_m \downarrow$ 0.382	$\sin\theta_m \downarrow$ 0.719		$\csc\theta_m \downarrow$ 1.524	$\csc\theta_m \downarrow$ 5.24	$\csc\theta_m \downarrow$ 9.57
5	**Copernican →**	**0.376**	**0.719**	**1.000**	**1.520**	**5.219**	**9.174**
6	Modern →	0.3871	0.7233	1.0000	1.5237	5.2028	9.5389

*Data in rows 3, 5 and 6 are taken from A.P.French, *op. cit.*. The values of θ_m listed in Row 3 are attributed to the ancient Greek Philosopher Ptolemy.

3 Kepler's Struggle with Mars

One important landmark in the path to the discovery of the law of universal gravitation was the arrival of a Danish astronomer named Tycho Brahe (1546-1601) who had made his observatory near Copenhagen with the patronage of the king of Denmark, but later moved to Prague to continue his study of the planets. Paradoxically the model of the universe as propagated by Tycho (the Tychonic model) was similar to the Platonic model with suitable modifications.

Johannes Kepler, a German astronomer (1571-1630) with extraordinary mathematical skills, was invited by Tycho to work with him in Prague. However, Kepler could not fall in line with the Tychonic model, but had faith in the Copernican system. After Tycho's death Kepler dedicated much of his life in analyzing the tables of planetary positions Tycho had left behind, after obtaining them with difficulty from the unwilling hands of his heirs.

Kepler's life is a saga of the indomitable human spirit, of the difficult battle a single individual fights against all odds and adversities, to follow the star of his conviction with the power of a superlative mathematical mind, and comes out unvanquished. He lived at a time when religious dogma reigned over reason, and any independent thinking differing from the official tenets of the church was met with religious persecution, humiliation and death. Elderly women living alone were charged with witchcraft and burnt alive at the stake. It is said that Kepler's discoveries prompted a war, in which thousands of innocent people died, including Kepler's wife and son.

Kepler had a dream, described in his science fiction *Somnium*, in which he told the story of space travelers going to the Moon, and watching Earth-rise from the lunar surface. Such an imagination was considered to be outrageous and heretic (as it came into conflict with the Earth-centred universe). It is said that his mother had to pay a heavy price for this heresy. She was carried away in a laundry basket in the middle of night.

Kepler's earlier work, when he was teaching secondary school mathematics, in Graz, Austria, was a discovery, which he called *Cosmic Mystery*. When teaching a class of bored students, his mind drifted to a different world, trying to find an order among the orbits of the six planets known to the world at that time, the radii of which had been found out by Copernicus. He was visited by a revelation that the orbits of the planets could be fitted into the geometrical solids of Pythagorus, in which he also found the answer why there existed only six planets. His geometrical construction is an outstanding artwork of sheer delight, and should be seen and studied by students of mathematics, physics and art, by all those who love to discover beauty in geometrical shapes.

We quote Feynman. "He quickly devised a model in which the six invisible spheres that regulated the orbits of the six planets then known were fitted on either side of each of the five perfect solids of antiquity (solids having all sides the same: the tetrahedron, cube, octahedron, dodecahedron, and isocahedron), nested one inside the other. Sure enough, by arranging the solids in the right order, the diameters of the spheres came out to be in almost the same ratios as those of the orbits of the planets.

"Kepler's model explained why there were six, and only six planets - because there were five and only five perfect solids"

Kepler's discovery was published in his book *Mysterium cosmographicum* in 1596. However, this discovery lost part of its relevance after the discovery of extra planets Uranus, Neptune and Pluto.

Kepler left Graz, anticipating the horror of religious persecutions that would follow, and came to work under Tycho in Prague as we have already mentioned. Here he concentrated his study on the position of Mars based on thirty five years of data collected by Tycho. "What real motion of the Earth and Mars about the Sun could explain the apparent motion of Mars in the sky, including its retrograde loop through the background constellations, as recorded by Tycho?". This was the raging question in his mind and he left no stone unturned to seek its answer. Tycho had earlier wanted Kepler to study the motion of Mars, because compared to other planets Mars appeared to move in the most anomalous manner, difficult to explain with the help of circle over circle.

Tycho's data consisted of measurements of the angle between Mars and the "fixed stars" at different times. These measurements were valid with respect to an observatory fixed on the earth. Like his predecessors Kepler had faith in circular paths for the planets, because a circle was considered to possess the perfect geometrical shape, and such perfections only deserved to be the attributes of heavenly bodies. Therefore Kepler, transformed the angles from Tycho's table of data into another set of angles that would be valid with respect to a frame of reference fixed at the centre of the sun. Assuming that the earth was spinning about its axis, and was also revolving around the sun in a circle, he applied his superb mathematical skill to perform the required transformation of the angles [2] and made "seventy attempts" to fit them into the assumed circles. However, somewhere there was an error of "eight minutes" of arc (there are sixty minutes of arc in one degree), which could not be patched up with the imagined circular motion.

When all such attempts failed Kepler could see the light at the end of the tunnel, with the realization that an ellipse rather than a circle will fit Tycho's data beautifully. That was the greatest revolution in the history of astronomy, and this discovery is known as Kepler's First Law of Planetary motion.

Kepler also realized that the prevailing notion that a planet moved with uniform speed along the orbit will be inconsistent with the available data. The data indicated that the planet moved faster when it is near the Sun, and slower as its distance from the Sun increased. In this process of varying speed, one thing remained constant. It is the areal velocity, by which we mean the rate at which the planet sweeps out area around the Sun. This realization had occurred to Kepler earlier but came to be known as Kepler's Second Law of Planetary motion.

We shall explain these two laws with the help of Fig.8.

In Fig. (a) we have shown an imaginary object P moving around the Sun. We call such an object a Planet. The path that the planet follows is an ellipse. The ellipse has a characteristic point F called its Focus, where the Sun will be always residing. In other words, the planet always moves in an ellipse in such a way that the Sun is always at the focus F. This is the *first law.*

Suppose the planet P moves from K to L in a given time interval τ. After some time P goes to another point M, and moves from M to N in the *same* time interval τ. The sector KL makes a certain area $\mathcal{A}$ at the focus F. Then the sector MN will make the *same* area $\mathcal{A}$ at the focus F. In other words the planet will sweep out equal areas at the focus F at equal intervals of time. This is the *second law.*

Let us introduce the term "eccentricity" e

[2] This is the reverse of we have done in the previous subsection, in which we drew the g-path of Mars on the basis of heliocentric circular orbits. Tycho's data gave the projection of the true g-path on the celestial sphere, as in Fig. 5a. From this, we guess, Kepler constructed the true g-path using his genius, and then "transformed" that into the heliocentric paths of the Earth and Mars.

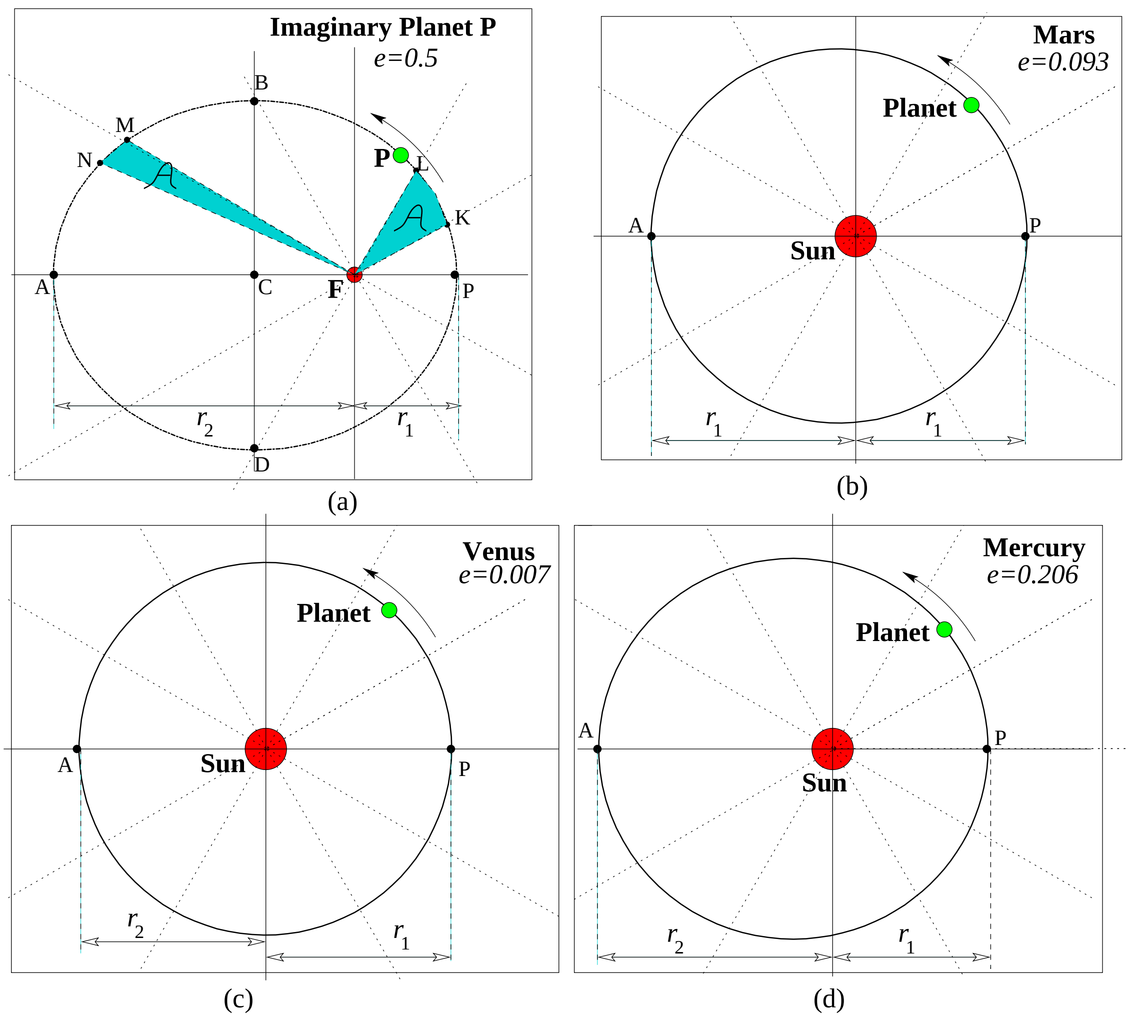

Figure 8: *Orbits of some planets around the Sun*

at this point without going into its definition. For an ellipse e is a number that gives a measure of how far the rounded figure is elongated in one direction (flattend in the other) compared to a circle. For an ellipse e is less than one. A circle is a very special case of an ellipse with $e = 0$.

Let us take another look at the elliptical path of the hypothetical planet P in Fig. (a). As the planet moves around the Sun, which is fixed permanently at F, its distance from F is continuously changing. This distance is a minimum, and equal to r_1 when it is the point P, called the *perihelion* of the planet, and maximum, and equal to r_2 when it is at the point A, called the *aphelion* of the planet.

Its dimension in the X direction is equal to $2a$, and is referred to as its *major axis* (the distance a is called the *semi-major axis*). And its dimension in the Y direction is equal to $2b$, referred to as its *minor axis* (the distance b is called the *semi-minor axis*).

The relationship between e and the other parameters is given as

$$r_1 = (1-e)a; \quad r_2 = (1+e)a; \quad \frac{b}{a} = \sqrt{1-e^2}. \tag{15}$$

It is seen that when $e \to 0$ the ellipse returns to a circle with $a = b$, $r_1 = r_2 = a$.

The hypothetical planet in Fig (a) moves in an ellipse of eccentricity $e = 0.5$. In Fig (b), (c) and (d) we have shown *true plots* of the orbits of Mars, Venus and Mercury (using gnuplot), having $e = 0.093, 0.007, 0.206$ respectively. It is seen that the orbit of Venus looks almost like a circle[15]. There is a marked departure from circle of the orbit of Mars and further departure for the orbit of Mercury. In both cases, even if the orbit looks nearly circular, there is a marked shift in the position of the Sun from the "centre". It is this anomaly, in the case of Mars, that showed up in the form of an error of eight minuites of arc when Kepler tried to patch up Tycho's data with a circle.

4 Kepler's Third Law - Key to Inverse Square

Let us make a formal statement of the three laws of planetary motion discovered by Johannes Kepler. We have already explained with diagram what the first two laws are. We shall now write all the laws together to summarize Kepler's most important work on planetary motion.

Kepler's Laws of Planetary Motion

1st Law. All planets move in elliptical paths.
2nd Law. A straight line drawn from the Sun to a planet sweeps out equal areas in equal times.
3rd Law. Let T represent the time period of one complete revolution of a planet around the Sun, and let a represent the semi-major axis of its (elliptical) orbit. Then the ratio $\frac{T^2}{a^3}$ is the same for all planets. In other words $T^2 \propto a^3$.

Let us spend a little time understanding the 3rd law, which says

$$T^2 = c\, a^3 \tag{16}$$

where c is a constant, same for all planets. For convenience we shall assume that the planetary orbit is (approximately) a circle, and replace a by the average distance of the planet from the Sun, which we shall call the average radius and represent by R. For Earth $T = 1$ Earth-year, and $R = 1$ AU. Therefore if time is measured in Earth-year, and radius in AU, then the constant c in Eq. (16) should come out to be equal to 1. We shall now make a table for T^2 and R^3 for some planets, to test Kepler's third law. We shall take the values of T and R, as obtained by Copernicus, and as tabulated in Tables 1 and 2, although Kepler had used different values, and had obtained better agreement.

Table 3 : R and T for the planets, and the value of c.

	0	1	2	3	4	5
	Planet ↓	R	R^3	T	T^2	$c = \frac{T^2}{R^3}$
		AU		E-year		
1	**Mercury**	0.376	0.53	0.24	.0576	1.087
2	**Venus**	0.719	.372	0.615	.378	1.016
3	**Earth**	1	1	1	1	1
4	**Mars**	1.520	3.51	1.882	3.54	1.008
5	**Jupiter**	5.219	142.15	11.87	140.90	0.991
6	**Saturn**	9.174	772.10	29.44	866.71	1.123

The average of the numbers given in column 5 comes out to be c=1.037. The variation of T with R can now be represented the empirical relation

$$T = \sqrt{c}\, R^{\frac{3}{2}} = 1.02\, R^{\frac{3}{2}}, \tag{17}$$

where R is given in AU and T in Earth-year. We have plotted the above relationship in Fig. 9, and shown the approximate locations of the six planets on the plot. We have represented the inner planets Mercury and Venus by the lower case letters m and v, the Earth by E, and the outer planets Mars, Jupter and Saturn by the upper case letters M, J and S respectively.

We shall review how Kepler's 3rd law led Newton to his discovery. As before, we shall approximate the planetary orbit by a circle of (average) radius R. Now we introduce a different constant K and write the third law in the form of the following relationship.

$$R^3 = KT^2, \tag{18}$$

Then (18) means that

$$\frac{R^3}{T^2} = K. \tag{19}$$

Instead of T we shall use the angular velocity ω.

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}. \tag{20}$$

Then from (19) and (20)

$$\frac{R^3\omega^2}{4\pi^2} = K \tag{21}$$

Now, the centripetal acceleration, as calculated by Newton, is given by, using the symbol a to denote this acceleration (this a should not be confused with the semi-major axis).

$$a = \omega^2 R. \tag{22}$$

Then from (21) and (22)

$$\frac{R^2 a}{4\pi^2} = K_s. \quad \text{Or, } a = \frac{4\pi^2 K_s}{R^2} = \frac{\mathcal{K}_s}{R^2}. \tag{23}$$

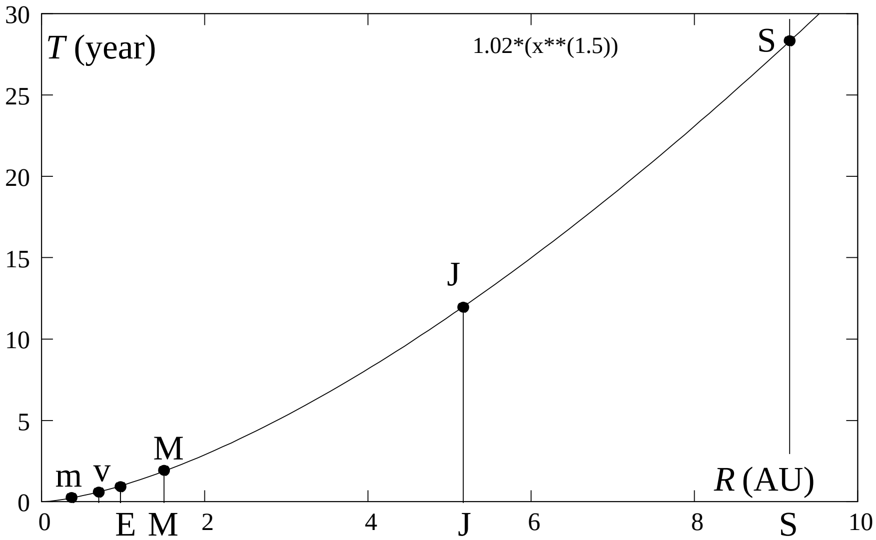

Figure 9: *Relationship between R and T for the six planets*

In the above we have "normalized" constant K_s to $\mathcal{K}_s \equiv 4\pi^2 K_s$. We have also added subscript s to mean that the proportionality constant is now associated with the force of gravitation emanating from the Sun.

Note the important inference we have derived from Kepler's third law: *The acceleration of a planet "freely falling" under the gravitational pull of the Sun, is inversely proportional to the square its distance from the Sun.*

The gravitational force F_s of the Sun on the planet whose mass is m is then

$$F_s = ma = \frac{\mathcal{K}_s m}{R^2}. \qquad (24)$$

The force of the Sun's gravity F_s acting on a planet, like its acceleration a, is inversely proportional to the square of its distance of the planet.

If the above force law is truly universal, then the same relation should also apply to objects moving under the gravitational pull of the Earth, as Newton surmised. Hence Newton conjectured

Conjecture 2 *The force of gravity F_e on a particle of mass m under the Earth's gravitation is*

$$F_e = mg = \frac{\mathcal{K}_e m}{R^2}. \qquad (25)$$

where g and $\mathcal{K}_e$ replace a and $\mathcal{K}_s$ respectively. The subscript e now implies "Earth", and g stands for the acceleration due to Earth's gravity. This acceleration is then

$$g = \frac{\mathcal{K}_e}{R^2}. \qquad (26)$$

Now,

$$\begin{aligned} R_{\text{apple}} &= \text{radius of the earth} \\ &= 6,371\,\text{km}. \\ R_{\text{moon}} &= \text{radius of the moon's orbit} \\ &= 384,000\,\text{km}. \end{aligned} \tag{27}$$

so that

$$\frac{R_{\text{moon}}}{R_{\text{apple}}} = 60.3 \tag{28}$$

Hence, by Eqs. (26) and (28), and noting that g_{apple} = acceleration due to gravity near the surface of the earth = 9.81 m/s^2, we get

$$\frac{g_{\text{moon}}}{g_{\text{apple}}} = \left(\frac{R_{\text{apple}}}{R_{\text{moon}}}\right)^2 = \left(\frac{1}{60.3}\right)^2 \approx \frac{1}{3636} \tag{29}$$

It now follows that

$$g_{\text{moon}} = \frac{9.81}{3636} \approx 2.7 \times 10^{-3}\text{m/s}^2. \tag{30}$$

Newton now found complete agreement with his estimate given in Eq. (3). It was a historic triumph of Newton's uncanny vision and crowning of the Law of Universal Gravitation.

5 The Law of Universal Gravitation

There is another element in Newton's discovery. This is about the constant $\mathcal{K}$. The gravitaional force F_e between the Earth and the Moon is proportional to the (inertial) mass m of the Moon (see Eq. 25). Similarly, the gravitational force F_s between the Sun and the Earth is proportional to the (inertial) mass m of the Earth (see Eq. 24). It made sense to Newton that the gravitational force between two objects A and B must be reciprocal. If A pulls B with a force F then B should also pull A with the same force F. (This comes under a wider principle called Newton's Third Law of Motion.) And this force must be proportional to the "material content" of A and also proportional to the material content of B. One may call this unspecified "material" quantity the *gravitational masses* of A and B. Call them $m_g(A)$ and $m_g(B)$ respectively.

If A is the Sun, and B the Earth, then Eq. (24) shows that $m_g(B)$ is to be identified with the (inertial) mass m of the Earth.

Similarly, if A is the Earth, and B the Moon (or the apple), then Eq. (25) shows that $m_g(B)$ is to be identified with the (inertial) mass m of the Moon (or the apple).

Therefore let us recognize the *gravitational mass and the inertial mass to be identical*, and when we say mass, we may mean either of them. In the following we shall represent the mass by a capital italic, e.g., M.

Newton therefore formulated his *Theory of Universal Gravitation* as follows.

Conjecture 3 *Two particles A and B, having masses M_A and M_B, when separated by a distance r, attract each other along the line AB joining them with a gravitational force F_g which is proportional to M_A and M_B, and inversely proportional to the square of the distance r between them.*

$$F_g \propto \frac{M_A\,M_B}{r^2}. \tag{31}$$

There is a constant of proportionality G, called *Gravitational constant*. We can now write the above equation more completely as

$$F_g = G\frac{M_A M_B}{r^2}. \qquad (32)$$

In the SI system mass is measured in kg, distance in meters and force in newtons. In that case

$$G = 6.67 \times 10^{-11}\,\mathrm{N.m^2/kg^2}. \qquad (33)$$

The value og G was first measured by Henry Cavendish many years after Newton.

In writing the force law, either in (31)or in (32), we have assumed the objects A and B to be *point particles*, as illustrated in Fig. 10a. In Fig. 10b we have shown two large size objects A and B, of which A can represent the sun and B a planet. We would like to find the gravitational force between these two objects on the basis of the force law given in (32). This can be achieved by treating A and B to be composed of a very large number of particles. The force between every particle in A and every particle in B is given by Eq. (32). We *add such forces between pairs of particles*, and obtain the force F_g between A and B. This force takes a very special and simple form when the distribution of matter in each of A and B is spherically symmetrical, as Newton proved himself.

Lemma: 1 *Let there be two spheres with centres at A and B containing masses M_A and M_B respectively distributed in a spherically symmetric manner, and let r be the distance between their centres. Then the force between these two spheres is given by Eq. (32). In other words, the gravitational force between two spherically symmetric distributions of matter A and B is exactly same as the force between two point particles coinciding with the centres of these spheres, and carrying the entire masses of A and B respectively, provided that there is no overlapping of these spheres.*

In order to prove the above theorem, we will need to introduce the concept of gravitational field, which we will defer for another occasion.

Let us now consider two LARGE non-symmetrical massive objects A and B (for example, A can be the Himalaya mountain, and B the Alps mountain) as shown in Fig. 10c we have shown. What is the force between them?

We can again begin with Eq. (32), add forces between pairs of particles, as in the previous example, to calculate the force between A and B. However, in this case the answer is not so simple. One may be tempted to say that the force is still given by Eq. (32) where r is the distance between the centres of mass of A and B. But that would be WRONG!! The force in this case will be very complicated. The force here will NOT be a pure inverse square force [17].

However, if the distance r between the "centres" of A and B is *very large* compared to the size of the objects, the force is almost the same as given by the inverse-square-law.

Consider, for example an asteroid at a distance of $r = 400$ million kilometers from the sun. It may be just a rocky mountain of irregular shape having a maximum dimension of, say 100 km. The diameter of the sun is about 1.4 million km. Therefore the force

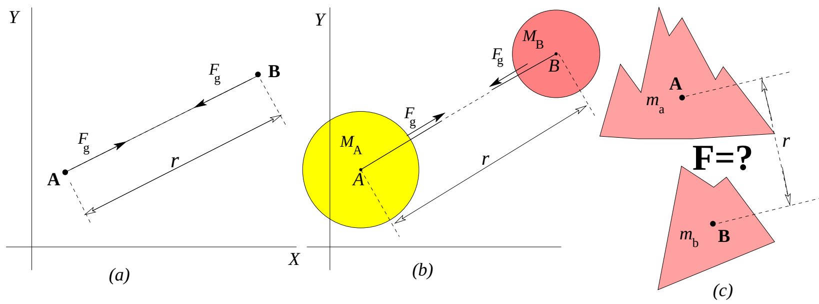

Figure 10: *Mutual forces of gravitation between (a) two point masses, (b) between two spherically symmetrical masses, (c) between two non symmetrical masses?*

with which the sun pulls the asteroid is almost exactly the same as given by the inverse square law. The same is true for the gravitational force of the earth acting on an artificial satellite, like the sky lab with its elaborate solar panels and housing colonies.

From the account we have given it may appear that Newton's role in the discovery of the inverse-square-law of gravitation may have been partly eclipsed by the pioneering work of Kepler. Newton himself acknowledged his debt to his predecessors when he said, "If I have seen further than others, it is because I was standing on the shoulders of giants".

We should pay due respect to the greatest genius of physics by recounting how he contributed to our understanding of the theory of gravitation in another way, characteristic of his greatness. Newton was the inventor of calculus (it is said that the German mathematician Gottfried Leibniz also invented calculus about the same time.) Newton had realized that it is impossible to absorb and apply the principles of mechanics without calculus. He showed us that all the laws of planetary motion, discovered by Kepler by analyzing Tycho's data, can be reconstructed from the inverse-square-law of gravitation and his second law of motion, with the use of calculus. Also it is his genius that recognized the common thread between the planetary motion and the motion of objects near and far from Earth under its own gravitational influence, making the inverse-square-law truly universal.

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References

[1] The author is not an expert in the history of science. He has collected materials from the resources listed below and organized them and expressed them in his own style and language. A serious reader should read the historical accounts first-hand in the mentioned references to get the facts more reliably and authentically.

[2] There are several websites where the "apple anecdote" has been narrated in some details. Two of them are (1) "*Newton, the Apple, and Gravity, Gravitation*" http://mb-soft.com/public2/newton.html; (2) "*The core of truth behind Sir Isaac Newton's apple*" http://www.independent.co.uk/news /science/the-core-of-truth-behind- ....

[3] *P.S.S.C.Physics*, 2nd ed, D.C.Heath and Co, Indian Edn, NCERT (1967), pp.368-371

[4] A.P.French, *Newtonian Mechanics*, Norton & Co., London (1971). Best resource about the early history. Gives mathematical details and facts on the measurements of planetary distances and planetary time periods. Read Ch. 2 (pp. 74-78) and Ch. 8 (pp. 245-258).

[5] George Gamow, *Gravitation*, Doubleday & Co. New York, pp.37-43. Gives an account of how Newton found out the free fall acceleration of the moon.

[6] D. Goodstein and J. Goodstein, *Feynman's Lost Lecture, The Motion of Planets around the Sun*, Vintage (1997). See Ch 1: *From Copernicus to Newton*, pp. 21-43. We have borrowed the phrase "war on Mars" from this book.

[7] Carl Sagan, *Cosmos*, Random House, New York (1980). Best narration of the history. Chapter III, pp.51-68, has the most vivid accounts of the works of Copernicus, Tycho and Kepler and a poignant narration of the trials and tribulations of Kepler.

[8] http://www.lasalle.edu/-smithsc/Astronomy/retrograd.html. This site gives a finest animation of the retrograde motion of Mars against the background constellations, and another animation of the epicycle-deferent motion of an unspecified planet. Also gives a Copernian explanation of the retrograde motion on the basis of his heliocentric model.

[9] http://csep10.phys.utk.edu/astr161 /lect/retrograde/retrograde.html. This site also gives an animation of the apparent motion of planets on the celestial sphere.

[10] http://csep10.phys.utk.edu/astr161 /lect/retrograde/aristotle.html. This site gives an animation of the hypothesised planetary motion on an Epicycle riding on a Deferent. Also gives an account of the early model of Ptolemy visualizing planetary motion in terms of

two epicycles, i.e., "Epicycle on Epicycle on Deferent". We have illustrated a similar motion in Fig. 2b. .

[11] This is similar to the familar problem in calculus. Construct any function $f(x)$ of x by making a power series of $1, x, x^2, x^3$, called Taylor series. Alternatively, construct any odd function $f(x)$ of x of periodicity L by making a Fourier series of sine functions: $\sin(n\pi x/L),\ n = 0, 1, 2, \cdots$.

[12] See Wikipedia. Also p. 582 of A.P.French, *op.cit.*

[13] This is the reverse of what we have done in the previous subsection, in which we drew the g-path of Mars on the basis of heliocentric circular orbits. Tycho's data gave the projection of the true g-path on the celestial sphere, as in Fig. 5a. From this, we guess, Kepler constructed the true g-path using his genius, and then "transformed" that into the heliocentric paths of the Earth and Mars.

[14] "The Greeks thought of the two as separate stars, Phosphorus and Hesperus untill the time of Pythagorus in the sixth century BC". See Wikipedia.

[15] It is said that Kepler could not have discovered his laws if Tycho had asked him to analyze the motion of Venus, instead of Mars

[16] *National Geographic Picture Atlas of Our Universe* (1986). See pp. 39,43

[17] The expression for the gravitational force F with which A and B are pulling each other will be in the form of an infinite series: $F = \frac{\kappa}{r^2} + \frac{\alpha}{r^3} + \frac{\beta}{r^4} + \ldots$, called *multipole expansion*. The coefficients $\alpha, \beta, \ldots$ will depend on the orientations of A and B. When r is very large compared to the dimensions of A and B, the 2nd term becomes very small compared to the first, the 3rd term very small compared to the second,..., etc. In that case $F \approx \frac{\kappa}{r^2}$, i.e., the force becomes almost the same as given by the inverse-square-law. The multipole expansion for the electrostatic field originating from a non-spherical charge distribution, which is also based on inverse square law, is discussed in D.J.Griffiths, *Introduction to Classical Electrodynamics*, 3rd Edn, Pearson Education/Prentce Hall, New Delhi (2006). See pp.164ff.