Physics Through Problem Solving - XXIII
Classical Lagrangian and Hamiltonian

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In this article we shall discuss some problems where we find the Hamiltonian of a system starting from its from Lagrangian using Legendre transformation. We also discuss when the Hamiltonian can be taken as $H = T + V$, that is, as the sum of kinetic and potential energy functions, with no need for Legendre transformation.

In classical mechanics we learn that Lagrangian and Hamiltonian equations of motion are two equivalent formulations of Newtonian equations of motion, though much more convenient in many ways, especially in advanced applications. This means the Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and the Hamiltonian function $H(\mathbf{q}, \mathbf{p}, t)$ of a system contain the same information. (Here $\mathbf{q} \equiv (q_1, q_2, \dots q_n)$, $\dot{\mathbf{q}} \equiv (\dot{q_1}, \dot{q_2}, \dots \dot{q_n})$, and $\mathbf{p} \equiv (p_1, p_2, \dots p_n)$; q_i's being generalized coordinates, $\dot{q}_i$'s generalized velocities and p_i's canonical conjugate momenta. n is the degrees of freedom for the system, and t is time) Thus we should be able to obtain the Hamiltonian from the Lagrangian and the the *vice versa*. The general method for doing the is through Legendre transformations. The functions $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and $H(\mathbf{q}, \mathbf{p}, t)$ form a Legendre transformation pair - that is, one is the Legendre transform of the other. Mathematically this means:

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{1}$$

where, the conjugate momenta p_i are *defined* by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2}$$

The transformation given by Eq.1 is used to obtain H when we have L. We see that the right hand side is a function of $q_i, \dot{q}_i$ and t, whereas the Hamiltonian on the left hand side is a function of q_i, p_i and t. But this is easily taken care of by eliminating $\dot{q}_i$'s using the Eq. 2, as we shall see in the examples to follow.

The reverse transformation (to obtain L from H) is given by simply rewriting eq. 1:

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{i=1}^{n} \dot{q}_i p_i - H(\mathbf{q}, \mathbf{p}, t) \tag{3}$$

In this case we need to eliminate p_i's on the right hand side, and we can do by using one set of Hamiltonian equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{4}$$

This also we shall demonstrate in the examples.

Problem 1: Let us begin with a simple case. The Lagrangian of a simple pendulum is given by $L = \frac{1}{2} ml^2\dot{\theta}^2 + mgl\cos\theta$, where m is the mass of the bob, l the length of the string and θ the angle made by the string with the downward vertical. Find the Hamiltonian, and from the Hamiltonian recover the Lagrangian.

Solution: Here we have only one generalized coordinate, θ, and therefore the sum in eq. 1 will consist of only one term. Also the Lagrangian has no explicit time dependence, and therefore Hamiltonian will have no explicit time dependence either. Thus

$$\begin{aligned} H(\theta, p_\theta) &= \dot{\theta} p_\theta - L \\ &= \dot{\theta} p_\theta - \frac{1}{2} ml^2\dot{\theta}^2 - mgl\cos\theta \end{aligned} \tag{5}$$

where p_θ is the conjugate momentum for θ, given by eq. 2. That gives us

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} \\ &= ml^2\dot{\theta} \end{aligned}$$

which can be readily inverted to get $\dot{\theta}$ in terms of p_θ:

$$\dot{\theta} = \frac{p_\theta}{ml^2}$$

which we use in eq. 5 to eliminate $\dot{\theta}$, and finally get Hamiltonian expressed as the function of θ and p_θ:

$$H(\theta, p_\theta) = \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl\cos\theta \tag{6}$$

This is a good point to discuss one important issue. You might know that usually Hamiltonian is taken as the total energy function $H = T + V$. And we have Lagrangian defined as $L = T - V$, T and V being the kinetic and potential energy functions . So to obtain H from L why not just reverse the sign (plus or minus) of V (when it can be identified), and then eliminate $\dot{\theta}$ for p_θ, instead of going to the trouble of performing Legendre transformation? The reason is that the Legendre transformation definition of H given in eq. 1 is general and always applies, where as the relation $H = T + V$ applies only when (1) the system is conservative, that is L is not an explicit function of time t, (2) The forces acting on the particles of the system can be obtained from a scalar potential V. These conditions are met in many important applications (examples are harmonic oscillator, central force driven motion), and therefore we very often use the relation $H = T + V$. You can easily check that we can follow this procedure to obtain the above Hamiltonian.

But there is one all important case where these conditions are not met - the motion of a charged particle in a magnetic field (even if the field is constant in time). We know that the magnetic force acting on a charged particle *cannot* be obtained as a spatial gradient of a scalar function (i.e. cannot be written as $-\nabla\phi$ where ϕ is some scalar function of coordinates), and it is a function of velocity of the particle. We shall not discuss this topic any further here, because it is a standard topic discussed in detail in every classical mechanics textbook. In the following problem we consider magnetic field to see the necessity of going through a Legendre transformation to obtain the correct Hamiltonian. But before that let us verify that we can get back the Lagrangian for the simple pendulum starting with the Hamiltonian in eq. 6. From eq. 3

$$\begin{aligned} L(\theta, \dot{\theta}) &= \dot{\theta} p_\theta - H(\theta, p_\theta) \\ &= \dot{\theta} p_\theta - \frac{1}{2}\frac{p_\theta^2}{ml^2} + mgl\cos\theta \end{aligned}$$

Now we need to eliminate p_θ for $\dot{\theta}$, which we do using eq. 4. That is, $\dot{\theta} = \partial H/\partial p_\theta = p_\theta/ml^2$, which gives us $p_\theta = ml^2\dot{\theta}$. Using this in the above

$$\begin{aligned} L &= \dot{\theta}\cdot ml^2\dot{\theta} - \frac{1}{2}\frac{(ml^2\dot{\theta})^2}{ml^2} + mgl\cos\theta \\ &= \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \end{aligned}$$

as expected. Now we move on to the next problem.

Problem 2: The Lagrangian for a charged particle with charge q, mass m moving in a uniform, time-independent magnetic field B in z-direction is given by

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{qB}{2}(x\dot{y} - y\dot{x}) \tag{7}$$

Find the Hamiltonian

Solution: Now if we try using the relation $H = T + V$, identifying as "kinetic energy" $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$, and "potential energy" $V = -\frac{qB}{2}(x\dot{y} - y\dot{x})$ we come up with the Hamiltonian

$$H = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{qB}{2}(x\dot{y} - y\dot{x})$$

This is wrong! The right answer is

$$H = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Its only the "kinetic energy"!

You might worry that there is no magnetic field in this expression, which cannot be quite right, because we do know that the motion of the charged particle is affected by the magnet is field. But we will see that the field does appear as soon as we put the Hamiltonian in the standard form, that is, by eliminating velocities for respective conjugate momenta. So us work this out. From eq. 1 we get

$$\begin{aligned} H &= \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L \\ &= \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{qB}{2}\left(x\dot{y} - y\dot{x}\right) \end{aligned}$$

At this point we normally eliminate the velocities $\dot{x}, \dot{y}, \dot{z}$ for respective conjugate momenta p_x, p_y, p_z. But we can save some algebra by persisting with the velocities for a while. Using $p_x = \partial L/\partial\dot{x} = m\dot{x} - qBy/2$, $p_y = \partial L/\partial\dot{y} = m\dot{x} - qBy/2$, and $p_z = \partial L/\partial\dot{z} = m\dot{z}$, the expression readily simplifies to, as promised

$$H = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{8}$$

Inverting the expressions for p_x, p_y, p_z above we have

$$\dot{x} = \frac{1}{m}\left(p_x + \frac{qB}{2}y\right) \tag{9}$$

$$\dot{y} = \frac{1}{m}\left(p_y - \frac{qB}{2}x\right) \tag{10}$$

$$\dot{z} = \frac{1}{m}p_z \tag{11}$$

using eqs. 9, 10 and 11 in eq. 8 and we have

$$H = \frac{1}{2m}\left[\left(p_x + \frac{qB}{2}y\right)^2 + \left(p_y - \frac{qB}{2}x\right)^2 + p_z^2\right]$$

Thus the magnetic field does appear in the Hamiltonian.

Our final problem is one involving Lagrangian with explicit time dependence.

Problem 3: Consider a Lagrangian whose length is l at time $t = 0$ and is gradually shortening, so that at time t the length is given by $l - y(t)$, and $y(0) = 0$. Its Lagrangian is given by

$$L(\theta, \dot{\theta}, t) = \frac{1}{2}m\left[(l - y(t))^2\,\dot{\theta}^2 + \dot{y}(t)^2\right] + mg\,[l - y(t)]\cos\theta \tag{12}$$

Find the Hamiltonian.

Solution: Note that y is *not* a coordinate of the pendulum, but some known function of time. A coordinate of the a system is determined by the equation of motion, and the initial conditions for that coordinate. Here that is not the case with y, whose time dependence does not depend on the forces acting on the mass. Also, the Lagrangian is explicitly time dependent, so this is not, in general, a conservative system (though it could be so for some specific function $y(t)$). Thus once again the Hamiltonian is not given by $H = T + V$, and we have to use the Legendre Transformation of eq. 1.

$$\begin{aligned} H(\theta, p_\theta, t) &= \dot{\theta}p_\theta - L(\theta, \dot{\theta}, t) \\ &= \dot{\theta}p_\theta - \frac{1}{2}m\left[(l - y(t))^2\,\dot{\theta}^2 + \dot{y}(t)^2\right] - mg\,[l - y(t)]\cos\theta \end{aligned} \tag{13}$$

The conjugate momentum $p_\theta = \partial L/\partial\dot{\theta} = m(l - y)^2\dot{\theta}$, which gives $\dot{\theta} = p_\theta/m(l - y)^2$. Using this eq. 13 above

$$\begin{aligned} H(\theta, p_\theta, t) &= \frac{p_\theta}{m\,(l - y(t))^2}\cdot p_\theta - \frac{1}{2}m\left[(l - y(t))^2\left(\frac{p_\theta}{m\,(l - y(t))^2}\right)^2 + \dot{y}(t)^2\right] - mg\,[l - y(t)]\cos\theta \\ &= \frac{p_\theta^2}{2m\,(l - y(t))^2} - \frac{1}{2}m\dot{y}(t)^2 - mg\,[l - y(t)]\cos\theta \end{aligned}$$

You can easily check in this case also the relation $H = T + V$ (identifying $T = \frac{1}{2}m\left[(l - y(t))^2\,\dot{\theta}^2 + \dot{y}(t)^2\right]$, and $V = -mg\,[l - y(t)]\cos\theta$) yields the wrong answer:

$$H(\theta, p_\theta, t) = \frac{p_\theta^2}{2m\,(l - y(t))^2} + \frac{1}{2}m\dot{y}(t)^2 - mg\,[l - y(t)]\cos\theta$$

But this reduces to correct Hamiltonian if $\dot{y} = 0$, as it should, because in that case the length of pendulum is fixed and the system is conservative.