Satellite in a Circular Orbit About a Rotating Spherical Planet and Calculation of the Velocity Change Along the Ground Track

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Abstract

An object in frictionless contact with and moving along a rotating spherical surface will experience Coriolis and centrifugal accelerations in the rotating frame. Although these are the only accelerations that appear explicitly in the equations of motion, assuming no other tangential physical forces are present, velocity changes of the object over a finite time interval cannot be correctly computed by integrating the sum of only these two accelerations over that time interval. This was proven in a recent publication [J.C. Piquette, "Velocity Change Calculation for an Object Moving on a Rotating Spherical Surface," *Phys. Educ.* 31(1) (2015), art. num. 3, pp. 1-12]. It was found there that an unexpected additional acceleration, therein termed the "kinematic" acceleration, was also required to be integrated over the finite time interval in order to deduce the correct velocity change. Interestingly, a satellite in circular orbit about a spherical rotating planet satisfies everything required for the results of this previous work to apply. Hence, for example, the change in velocity of such a satellite, as seen in the rotating frame, cannot be determined by integrating over only the sum of the Coriolis and centrifugal accelerations. It was also found in the earlier work that the influence of the kinematic acceleration is dominant for high initial object speeds. The kinematic acceleration dramatically dominates both the Coriolis and centrifugal accelerations in the case of a satellite in circular near-Earth orbit, since such a satellite has a speed of about 18000 miles/hour. These conclusions also apply to the calculation of velocity changes along the ground track. To permit detailed understanding of the satellite's motion along the ground track, the notion of a "shadow satellite" is introduced. The results and examples given here can be used in an undergraduate- or graduate-level classical mechanics course as modern space-age applications of classical mechanics that may be of high interest to students.

1. Introduction

The problem of computing the velocity change over a finite time interval of an object moving on the surface of a rotating sphere was considered in a recent publication [1]. In the problem considered there, the object was taken to be free of any applied tangential physical forces, and was constrained to remain on the sphere's surface. Naturally, from the perspective of the rotating frame, the object experiences both Coriolis and centrifugal accelerations. Indeed, these two accelerations are the only accelerations that appear explicitly in the equations of motion. But despite that fact, if one attempts to compute the change of velocity of the object over a finite time interval by integrating over only the sum of these two accelerations, an incorrect result is obtained. By integrating the two equations of motion over the finite time interval of interest, it was found that a third acceleration component appears. That third component was termed the "kinematic" acceleration.

Interestingly, a satellite in circular orbit obeys all the requirements considered in the previous work, and hence the same conclusions apply to a satellite in such an orbit. An example of such a satellite that is especially interesting is one that orbits the Earth and passes over the two poles [2]. The ground track of such a satellite carries it over a very large percentage of the Earth's surface.

The influence of the kinematic acceleration was found to become increasingly dominant over the Coriolis and centrifugal accelerations as the initial velocity of the object increases. For an object in near-Earth orbit, which travels at a speed of about 18000 miles/hour, the kinematic acceleration is by far the largest of the three acceleration terms that contribute to changes in velocity.

Here, applications of the results of Ref. 1 to satellites in circular orbit are considered. For simplicity, in example calculations involving the Earth, it is assumed the Earth is a perfect sphere of radius 4000 miles, and completes one rotation in exactly 24 hours. It is also assumed that the orbital velocity of a satellite at the approximate POES [2] altitude of 700 miles above the surface is exactly 18000 miles/hour.

It is hoped that the results and examples given here may be useful in either an undergraduate- or graduate-level classical mechanics course. As space-age applications of classical physics, these examples may be of high interest to students. Those who would like to see additional references related to non-inertial frames and relating the material to classroom teaching are directed to the larger list of references given in Ref. 1.

In Sec. 2, the two coordinate systems of interest are described, and the equations of motion are presented. A brief summary of the velocity-change calculation developed in Ref. 1 is given in Sec. 3. It is shown in Sec. 4 that the solution of an object given an initial velocity at the equator can actually be applied to cases with more generality than initially considered. The concept of the shadow satellite, which is useful for studying the satellite's ground track, is developed in Sec. 5. Numerical examples are given in Sec. 6. A summary and conclusion are given in Sec. 7.

2. Coordinate Systems and Equations of Motion

The approach used here applies the results of Ref. 1 to a satellite in a circular orbit about a spherical rotating planet. The results of that reference apply directly to the satellite problem if the sphere radius is simply replaced by the radius of the circular orbit. Of course, when applied in that way the rotating sphere is actually an imaginary mathematical surface, rotating with the same angular speed as the planet, over which the satellite is assumed to be moving. Also of interest is the ground track of the satellite, and the notion of a "shadow satellite" is introduced for studying the ground track. The shadow satellite is taken to be a physical object located on the rotating planet's surface, with the planet assumed airless and frictionless. The shadow satellite moves along the surface of the rotating planet, and always remains directly underneath the orbiting satellite.

Two coordinate systems are used in the analysis. These are termed the "unprimed" and "primed" coordinate systems. The systems are depicted in Fig. 1.The unprimed coordinate system is an inertial frame at rest with respect to the fixed stars. The Cartesian coordinates of this system are denoted (x, y, z). Not shown is a related unprimed spherical coordinate system (r, θ, ϕ). However, the angle ϕ of this system is depicted, and expresses the angle between the x and x' axes, and between the y and y' axes. The primed system also consists of Cartesian and spherical coordinates as shown. However, the primed system rotates about the common z, z' axes at constant angular speed $\dot{\phi} = \omega$.

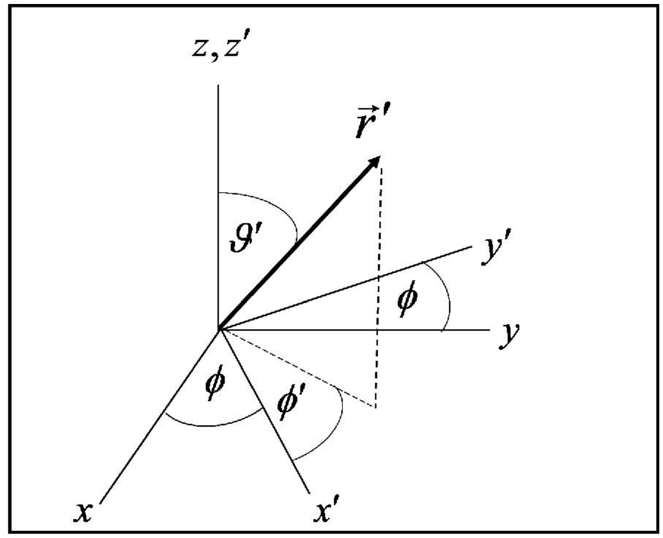

Fig 1 Primed and unprimed coordinate systems

In the primed coordinate system, the well-known equations of motion of an object moving on the surface with no physical tangential forces can be expressed as [3]

$$r'\ddot{\theta}' - r'\dot{\phi}'^2 \sin\theta' \cos\theta' = [a_{COR}]_{\theta'} + [a_{CENT}]_{\theta'} \qquad \textbf{(1)}$$

and,

$$r'\ddot{\phi}' \sin\theta' + 2r'\dot{\theta}'\dot{\phi}' \cos\theta' = [a_{COR}]_{\phi'}. \qquad \textbf{(2)}$$

Here r' is assumed constant, and for the cases of interest either $r' = R$, where R is the radius of the rotating spherical planet, or $r' = R + h$, where h is the height of the satellite above the planet's surface. But when $r' = R + h$, the spherical surface in question is an imaginary, mathematical surface having this radius and rotating at the same rate ω as the planet. The notations $[a_{COR}]$ and $[a_{CENT}]$ denote the Coriolis and centrifugal accelerations, respectively. These are considered in more detail in the next section.

3. Summary of the Velocity-Change Calculation

The Coriolis acceleration is $[\vec{a}_{COR}] = -2\vec{\omega} \times \vec{\mathrm{v}}'$ and the centrifugal acceleration is $[\vec{a}_{CENT}] = -\vec{\omega} \times (\vec{\omega} \times \vec{r}')$, with $\vec{\mathrm{v}}'$ denoting velocity of the object of interest in the primed frame. These are expressed in component forms in Eq. (3) through Eq. (5)

$$\left[a_{COR}\right]_{\theta'} = 2\omega\ r'\dot{\phi}'\sin\ \theta'\cos\ \theta', \qquad (3)$$

$$\left[a_{COR}\right]_{\phi'} = -2\omega\ r'\dot{\theta}'\cos\ \theta', \qquad (4)$$

and,

$$\left[a_{CENT}\right]_{\theta'} = r'\omega^2\sin\ \theta'\cos\ \theta'. \qquad (5)$$

It should be noted that $\left[a_{CENT}\right]_{\phi'} = 0$, always.

The components of the kinematic acceleration, as introduced in Ref. 1, are

$$\left[a_{KIN}\right]_{\theta'} = r'\dot{\varphi}'^2\sin\theta'\cos\theta', \qquad (6)$$

and

$$\left[a_{KIN}\right]_{\phi'} = -r'\dot{\theta}'\dot{\varphi}'\cos\theta'. \qquad (7)$$

(The need to consider more than just the Coriolis and centrifugal accelerations was also discussed in Ref. 4.) In the primed frame, the tangential velocity $\left[\vec{\mathrm{v}}'\right]_{\theta',\phi'}$ is expressed as

$$\left[\vec{\mathrm{v}}'\right]_{\theta',\phi'} = r'\dot{\theta}'\hat{\theta}' + r'\dot{\phi}'\sin\theta'\hat{\phi}', \qquad (8)$$

where $\hat{\theta}'$ and $\hat{\phi}'$ are the usual spherical unit vectors, and the components of the velocity changes over the finite time interval $(0,t)$ are expressed as

$$\Delta\left[\vec{\mathrm{v}}'\right]_{\theta'} = \Delta\left[r'\dot{\theta}'\right] = \left[r'\dot{\theta}'\right]_0^t, \qquad (9)$$

and,

$$\Delta\left[\vec{\mathrm{v}}'\right]_{\phi'} = \Delta\left[r'\dot{\phi}'\sin\theta'\right] = \left[r'\dot{\phi}'\sin\theta'\right]_0^t. \qquad (10)$$

Notice from Eq. (9) and Eq. (10) that the velocity changes of interest here are the velocity changes in a given compass direction. That is, it is the changes in the coefficients of the unit vectors of Eq. (8) that are of interest, not the changes in the unit vectors themselves. The symbol Δ has the usual meaning of “change.”

As shown in Ref. 1, the components of the velocity change over a finite time interval $(0,t)$ in the primed frame are computed from the components of the acceleration as

$$\left[\Delta\vec{\mathrm{v}}'\right]_{\theta'} = \int_0^t \left[a_{COR}\right]_{\theta'} dt + \int_0^t \left[a_{CENT}\right]_{\theta'} dt + \int_0^t \left[a_{KIN}\right]_{\theta'} dt\,, \tag{11}$$

and,

$$\left[\Delta\vec{\mathrm{v}}'\right]_{\phi'} = \int_0^t \left[a_{COR}\right]_{\phi'} dt + \int_0^t \left[a_{KIN}\right]_{\phi'} dt\,. \tag{12}$$

There is no loss in generality by taking the time interval to start at $t=0$.

4. Generalized Solution

The problem of an object initially located at the equator and given an initial tangential velocity was also considered in Ref. 1. The problem is depicted in Fig. 2.

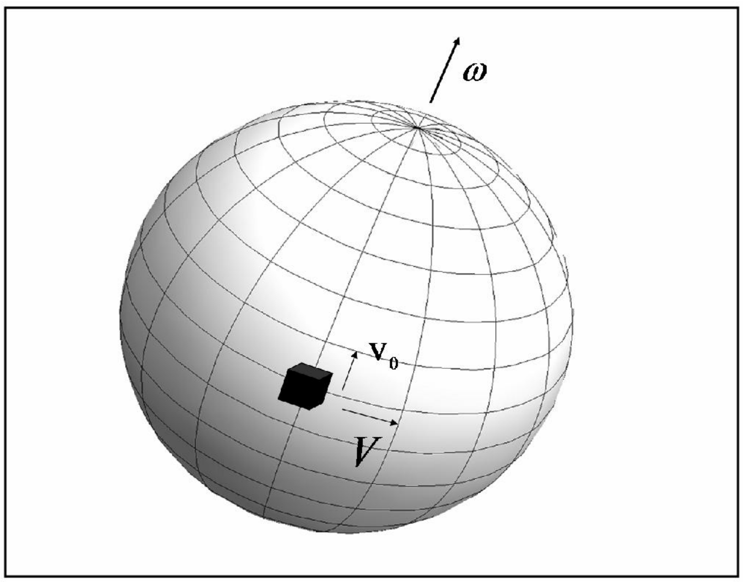

Fig 2 A sphere of radius *R* rotates uniformly at angular speed ω.

Here, V_0 is the initial tangential velocity in the northern direction and V is the initial tangential velocity in the eastern direction as specified in the unprimed, or inertial, frame.

The solutions of Eq. (1) and Eq. (2) for this problem are [1]

$$\theta'(t) = \arccos\left[\frac{v_0}{\sqrt{v_0^2 + V^2}}\sin\left(\frac{t\sqrt{v_0^2+V^2}}{r'}\right)\right], \quad \textbf{(13)}$$

and

$$\phi'(t) = \arctan\left[\frac{V}{\sqrt{v_0^2 + V^2}}\tan\left(\frac{t\sqrt{v_0^2+V^2}}{r'}\right)\right] - \omega\, t. \quad \textbf{(14)}$$

(This problem has also been solved numerically[5].) In Ref. 1, the case of interest was the object having the same initial tangential speed in the eastern direction as the tangential speed of the sphere at the equator. However, Eq. (13) and Eq. (14) remain valid for arbitrary speeds v_0 and V. Hence the restriction to a specific value V can be removed, and these solutions are valid for this more general case. It should be understood that Eq. (13) and Eq. (14) are the solutions to Eq. (1) and Eq. (2) subject to the initial conditions $(\vartheta'(0), \phi'(0)) = \left(\pi/2, 0\right)$ and

$$\left(\dot{\vartheta}'(0), \dot{\phi}'(0)\right) = \left(V/r' - \omega, {}^{-}v_0/r'\right).$$

Here, v_0, V, and ω again have the same meaning as in Fig. 2, and $r' = R$ for an object on the surface of the rotating planet, and $r' = R + h$ for the orbiting satellite, with R being the sphere radius and h being the height of the orbiting satellite above the sphere's surface.

5. The Shadow Satellite

Although Eq. (13) and Eq. (14) are valid for an orbiting satellite, where $r' = R + h$, the equations assume the satellite would be viewed from an imaginary spherical surface located at the same height as the satellite, and rotating at the angular speed of the planet. Since such a surface is a purely mathematical construction, it is more helpful to transform the solution to the planet's surface.

The path directly under the satellite along the planet's surface is the satellite's ground track. To find the ground track, the idea of the "shadow" satellite is now introduced. The shadow satellite is taken to be an object located on the planet's surface directly underneath the orbiting satellite. The planet is assumed airless and the shadow satellite is assumed to be in frictionless contact with the surface. It is first assumed, and then proven, that it is possible to impart an initial velocity to the shadow satellite such that the shadow satellite remains directly underneath the orbiting satellite at all times.

We now consider the calculation of the initial velocity of the shadow satellite that in fact produces the behavior of always remaining directly underneath the orbiting satellite. For definiteness, we will also assume the planet of interest is the Earth, having the idealized physical properties previously mentioned, although the analysis applies to any spherical planet. To find the required initial velocity of the shadow satellite, we consider first a special case: The geosynchronous satellite. A satellite in geosynchronous orbit appears to an observer on the Earth's surface to remain stationary above a given point on the equator. Clearly, any object located on the equator directly underneath the geosynchronous satellite will serve as its shadow satellite, as defined here. For both the geosynchronous satellite and for the shadow satellite, the tangential velocities obey the equation

$$\mathrm{v} = \mathrm{r}\,\omega\,. \qquad \textbf{(15)}$$

For the shadow satellite, $r = R$, the Earth's radius, and for the orbiting satellite $r = R + h$, where h is the height of the geosynchronous satellite above the surface. Writing out Eq. (15) for both the shadow and the orbiting satellite, and forming the ratio of these two equations, gives

$$\frac{\mathrm{v}_{SHADOW}}{\mathrm{v}_{ORBIT}} = \frac{R}{R+h}. \qquad \textbf{(16)}$$

Solving Eq.(16) for v_{SHADOW} gives

$$\mathrm{v}_{SHADOW} = \frac{R}{R+h}\ \mathrm{v}_{ORBIT}\,. \qquad \textbf{(17)}$$

Although there is no *a priori* reason to believe it will work, at this point we use Eq. (17) as a guide, and assume that each of the components of the tangential velocity of the shadow and orbiting satellite will obey an equation of the same form as Eq. (17), or

$$\left(\mathrm{v}_0\right)_{SHADOW} = \frac{R}{R+h}\ \left(\mathrm{v}_0\right)_{ORBIT}, \qquad \textbf{(18)}$$

and,

$$\left(\mathrm{V}\right)_{SHADOW} = \frac{R}{R+h}\ \left(\mathrm{V}\right)_{ORBIT}, \qquad \textbf{(19)}$$

where v_0 and V again are the northward and eastward tangential velocity components, respectively, as viewed in the inertial, or unprimed, frame. In this case, it is the shadow satellite that is depicted in Fig. 2, and the orbiting satellite, which is directly above it, is not shown.

In order to prove that Eq. (18) and Eq. (19) are valid for the general case, one starts by writing out Eq. (13) and Eq. (14) for both the shadow satellite and for the orbiting satellite. Naturally, the initial tangential velocities appearing in these equations should be replaced in each case with the appropriate subscript depending upon which of the two satellites the equations are being written for. Also, in the case of the shadow satellite, r' is replaced by R, and in the case of the orbiting satellite r' is replaced by $R + h$.

Next, the tangential velocities $\left(\mathrm{v}_0\right)_{SHADOW}$ and $\left(\mathrm{V}\right)_{SHADOW}$ are replaced in the two equations for the shadow satellite by the expressions for these quantities as given by Eq. (18) and Eq. (19), respectively. If the resulting equations are then simplified algebraically, the resulting pair of equations will be found to be identical to the two equations for the orbiting satellite.

This exercise proves that the solutions $\theta'(t)$ and $\phi'(t)$ are identical for both the shadow and the orbiting satellite. This therefore proves that if the shadow satellite is given the initial tangential velocity components as specified by Eq. (18) and Eq. (19), the shadow satellite will remain perpetually directly underneath the orbiting satellite. It thus also proves that the transformations given by these two equations are correct. And since the shadow satellite always remains directly underneath the orbiting satellite, it follows that the shadow satellite will trace out the ground track of the orbiting satellite, as desired.

6. Examples

Two numerical examples will now be considered. In the first example, the satellite is considered to have been launched from the equator and no attempt has been made to counter the effect of the Earth's rotation. We also assume the satellite orbits at a height of about 700 miles. These conditions approximate those of the POES program [2]. However, rather than considering the orbiting satellite in the example, we will consider instead the shadow satellite, in order to see the properties of the ground track.

The initial eastward and northward speeds of the shadow satellite are calculated by first computing these quantities for the orbiting satellite, and then transforming the results down to the shadow using Eq. (18) and Eq. (19). The initial eastward speed of the orbiting satellite is taken to be $V_{ORBIT} = (R+h)\,\omega$, where $R = 4000$ miles and $h = 700$ miles. The quantity ω is calculated assuming the Earth rotates in exactly 24 hours, giving $V_{ORBIT} \approx 1230.46$ miles/hour. This result is then projected to the surface using Eq.(19). Carrying out these calculations gives $V_{SHADOW} \approx$ 1047.2 miles/hour. Not surprisingly, this is exactly the speed of the (idealized) Earth's rotation at the equator as seen from the unprimed, or inertial, frame.

The initial northward speed V_0 of the orbiting satellite is computed from the assumption that the orbital speed at the height of 700 miles is exactly 18000 miles/hour. Thus, $(\mathrm{v}_0)_{ORBIT} = \sqrt{18000^2 - V_{ORBIT}^2}$, or $(\mathrm{v}_0)_{ORBIT} \approx 17957.9$ miles/hour. Projecting this result to the surface using Eq. (18) then gives $(\mathrm{v}_0)_{SHADOW} \approx 15283.3$ miles/hour. Using these speeds together with $r' = R = 4000$ miles in Eq. (13) and Eq. (14) produces the $(\vartheta'(t), \phi'(t))$ values of the ground track. Taking the complement of $\theta'(t)$ to be the latitude, and $\phi'(t)$ to be the effective longitude, the ground track produced by these calculations is as shown in Fig. 3. The result is clearly similar to the ground track of a POES-type satellite [2].

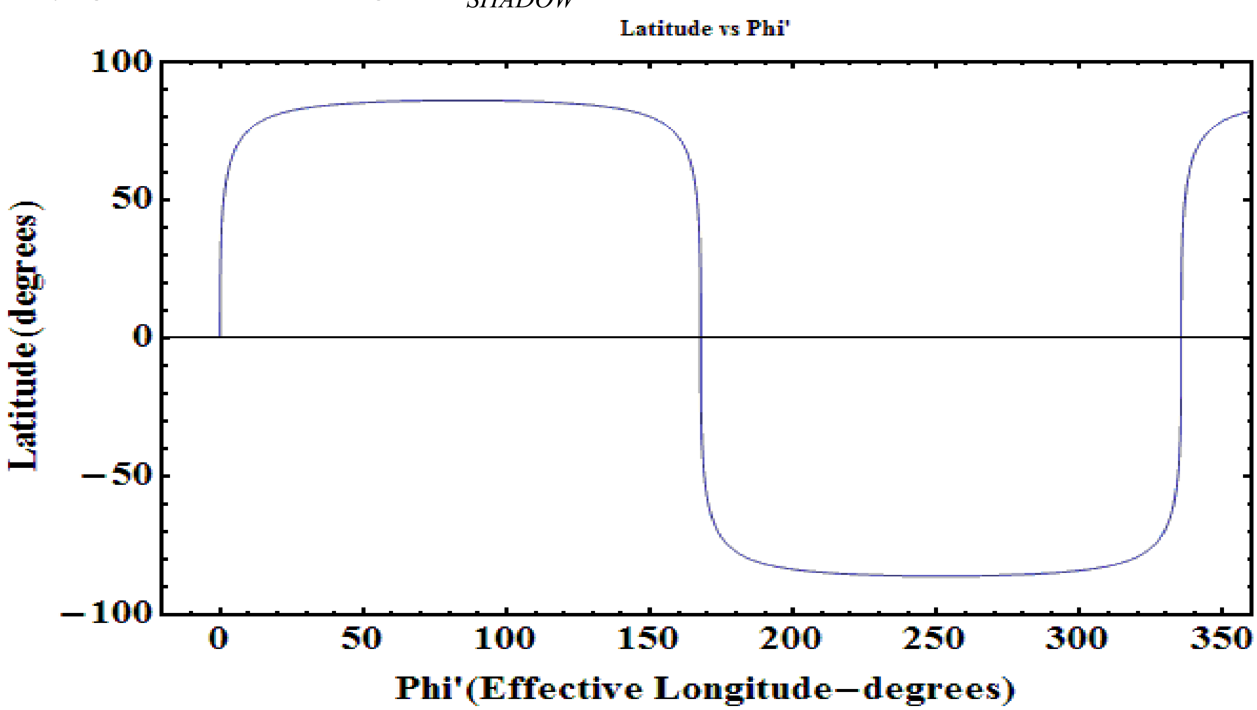

Fig. 3 - Ground track latitude vs. longitude for a POES-type satellite.

Examining Fig. 3, it is evident that that path is not a closed path. This is why a satellite in polar orbit can be used to observe the majority of the Earth's surface, since such a satellite passes over the equator at a different longitude on each orbit.

It is also interesting to consider specific velocity changes that occur for this satellite along the ground track, and to identify how much of the velocity change is attributable to each of the three accelerations involved in the problem.

Again examining Fig.3, it is clear that after starting at the equator $\left(\text{latitude} = 0 \text{ deg}\right)$, the shadow satellite closely approaches the northern pole $\left(\text{latitude} = 90 \text{ deg}\right)$, then rapidly reverses its northern motion and heads south. The distance of closest approach to the northern pole is about 273.6 miles. At the moment that the northern velocity reverses direction, it is clear that the initial northern velocity of 15283.3 miles/hour has been reduced to zero. The individual contributions to the change in northern velocity of the shadow satellite contributed by each of the three accelerations are:
$\left(\Delta \mathrm{v}\right)_{COR} =$ - 410.647 miles/hour,
$\left(\Delta \mathrm{v}\right)_{CENT} =$ - 36.273 miles/hour,
and $\left(\Delta \mathrm{v}\right)_{KIN} =$ - 14836.4 miles/hour. It is evident that the overwhelming contributor to the northern velocity change is the kinematic acceleration. These results were computed by separately evaluating each of the corresponding integrals in Eq. (11).

It is also interesting to consider the velocity change in the eastern direction and the contributions of the individual accelerations to it. The shadow satellite initially has zero velocity in the eastern direction as seen in the rotating frame. At the moment the velocity in the northern direction reduces to zero, that is, at the moment of closest approach to the northern pole, the velocity in the eastern direction reaches its maximum value. (This was proven in Ref. 1.) At that moment, the eastern velocity of the shadow is about 15247.56 miles/hour. The contributions of the individual accelerations to producing this velocity are:
$\left(\Delta \mathrm{v}\right)_{COR} =$ 1951.22miles/hour,
$\left(\Delta \mathrm{v}\right)_{CENT} =$ 0, and $\left(\Delta \mathrm{v}\right)_{KIN} =$ 13296.3 miles/hour.

These results were computed by evaluating the corresponding integrals in Eq. (12). Again we note that the null centrifugal contribution results from the fact that the centrifugal acceleration in the ϕ' direction is always identically zero. Again, it is apparent that the kinematic contribution strongly dominates the easterly velocity change, although not as significantly as in the northern direction.

We next consider a second example, but in this case it is assumed that the satellite has been launched in a way that almost cancels the velocity component due to the Earth's rotation at the equator as seen in the inertial frame. For the case of interest, it is assumed that the satellite that is in orbit at a height of 700 miles above the surface has been launched so that the initial velocity component that is directed toward the east has been reduced to just 10 miles/hour as seen from the unprimed, or inertial, frame.

Again, we are interested in the shadow satellite for the current example. Applying the same procedures as were described for the first example gives for the initial velocity components of the shadow satellite as seen in the unprimed, or inertial, frame $V_{SHADOW} \approx$ 8.51 miles/hour and $\left(\mathrm{v}_0\right)_{SHADOW} \approx$ 15319.15 miles/hour. The initial eastward velocity seen in primed, or rotating, frame is -1038.69 miles/hour. The minus sign signifies that the shadow satellite is actually moving westward as seen in the rotating frame. The initial northward velocity in the rotating frame is the same as that seen in the inertial frame.

Again using the initial shadow speeds for this case together with $r' = R =$ 4000 miles in Eq. (13) and Eq. (14) produces the $\left(\vartheta'(t), \phi'(t)\right)$ values of the ground track. Plotting these in the same way as was done to produce Fig. 3 results in the ground track as shown in Fig. 4. The fact that the ground satellite has an initially westward velocity can be seen by noting the negative slope of the initial ground-path curve as seen in Fig. 4. Again it is clear that when the shadow satellite reaches the northern pole it reverses its direction of travel and then heads southward. Hence the satellite again reaches a point where its northward velocity becomes zero as seen in the rotating frame. This happens at the point of closest approach to the northern pole, which happens when the shadow satellite is just 2.22 miles distant from the pole.

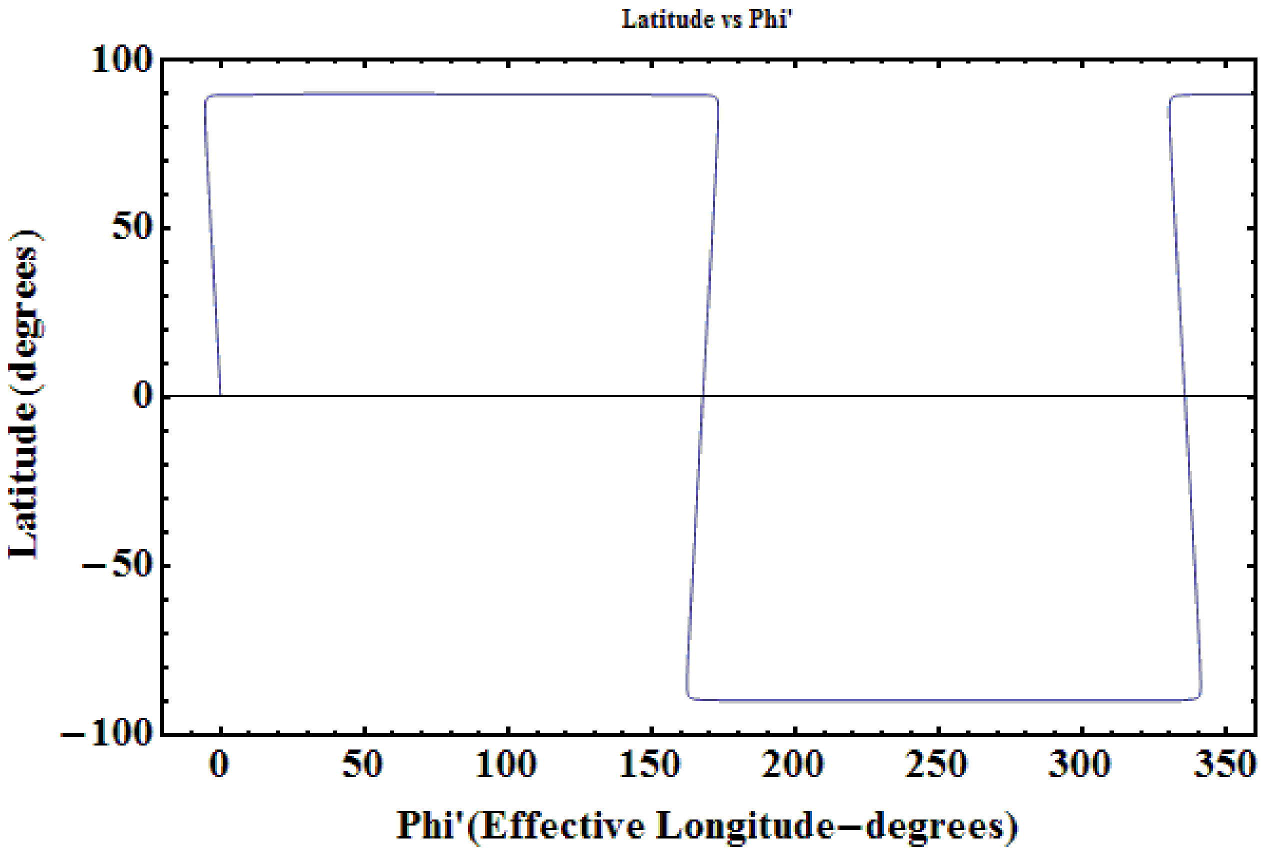

Fig. 4 - Ground track latitude vs. longitude for a satellite launched to almost cancel the Earth's rotational velocity.

As can be seen in Fig 4, as the shadow gets close to the northern pole (latitude 90 degrees), it undergoes an extremely rapid change in longitude. This is a consequence of the very close approach of the shadow satellite to the northern pole, where there is a discontinuity in the value of ϕ'. That is, if the satellite were approaching the northern pole along the zero-degree longitude line, there would be a discontinuous change in longitude as it passed the pole, because that line abruptly changes in longitude from 0 degrees to 180 degrees at the location of the pole. The longitude values will vary in a similar way for the shadow satellite in this example, owing to its very close approach to the northern pole.

Again it is of interest how much each of the acceleration terms contributes to reducing the shadow's initial northward velocity from $\left(V_0\right)_{SHADOW} \approx 15319.15$ miles/hour to zero. The results are:

$(\Delta v)_{COR} =$ +62.06 miles/hour, $(\Delta v)_{CENT} =$ - 35.79 miles/hour, and $(\Delta v)_{KIN} =$ - 15345.40 miles/hour.

It is again evident that the overwhelming contributor to the northern velocity change is the kinematic acceleration. It is also interesting to notice that the Coriolis contribution is positive, which occurs due to the westward-directed initial velocity component, and hence the Coriolis force actually acts to increase the northward velocity during the trip from the equator to the northern pole. As before, these results were computed by separately evaluating each of the corresponding integrals in Eq. (11).

Again considering the motion in the easterly direction, it is evident from Fig. 4 that the shadow suffers a huge apparent eastern acceleration as it approaches the northern pole. It then just as rapidly decreases its eastward velocity, and shortly past the northern pole the satellite again has a westward-directed velocity component, as can be seen from the fact that the longitude values begin to decrease as the shadow begins to move southward. The maximum eastward velocity of the shadow in the rotating frame, which again occurs at the point of closest approach to the northern pole, is 15318.57 miles/hour. Keeping in mind that the "eastern" velocity was initially -1038.69 miles/hour in the rotating frame (actually westward), the overall *change* in eastern velocity is 16357.3 miles/hour ($\approx$ 15318.57 miles/hour+1038.69 miles/hour, to within rounding error). Breaking down the overall eastward velocity change by acceleration term gives:

$(\Delta v)_{COR} =$ 2093.23 miles/hour,

$(\Delta v)_{CENT} =$ 0,

and $(\Delta v)_{KIN} =$ 14264.00 miles/hour.

These results were again computed by evaluating the corresponding integrals in Eq. (12), and once again the ϕ' component of the centrifugal acceleration is identically zero.

It may be of concern as to why the shadow acquires such a high easterly directed velocity at the moment of closest approach to the northern pole. This again happens due to the fact that the shadow approaches so closely to the pole. If one considers the latitude circle at the point of closest approach to the pole, that circle has a radius of just over 2.2 miles. As with all latitude circles, motion along their length is either purely eastward or purely westward. At the point of closest approach to the northern pole, the shadow is moving tangent to this latitude circle, and thus is moving purely eastward as seen from the surface. Since the northward movement is null at this point, it is clear that the entire surface velocity of the satellite must appear in the eastward direction. To do this, the eastward velocity must suffer a sharp increase in the vicinity of this latitude circle. This very high eastward velocity, however, is only present while the satellite is in close proximity to this latitude circle, an event which is of extremely short duration, and this occurs while the shadow moves over a very small area of the surface.

7. Summary and Conclusion

A satellite in a circular orbit obeys all the requirements for the methods of Ref. 1 to be applicable. Thus velocity changes for such satellites over a finite time interval cannot be computed correctly by integrating only the sum of the Coriolis and centrifugal accelerations over that time interval. It is also necessary to include the contributions from the kinematic acceleration. Interestingly, in the case of a satellite in circular orbit, the kinematic contribution is dramatically dominant over the Coriolis and centrifugal accelerations, despite the fact that those are the only two accelerations that appear in the equations of motion in the rotating frame.

The notion of a shadow satellite was introduced to allow detailed study of the ground track of the orbiting satellite. With suitable transformations applied, all the results that apply to the orbiting satellite also apply to the shadow satellite.

The results and examples presented here should be useful to those who teach graduate- or undergraduate-level classical mechanics courses. These space-age applications of classical mechanics are potentially of high interest to students.

ACKNOWLEDGMENT

The drawing of the sphere shown in Fig. 2 was produced using *Mathematica*, from Wolfram Research.

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[1] J.C.Piquette, "Velocity Change Calculation for an Object Moving on a Rotating Spherical Surface," *Phys. Educ.* 31(1) (2015), art. num. 3, pp. 1-12.

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