Euclidean group E(2) and the quantum rotator in three dimensions

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(*Submitted* 13-07-2015)

Abstract

The paper considers the non-relativistic Schrodinger equation for a particle under the influence of a homogeneous field. The strategy of Hamilton-Jacobi method permits the solutions take the well known results from the classical mechanics. On the other hand, the generators of the Lie group SO(3) for the homogeneous quantum rotator in the large radius limit can be contracted to the Euclidean group E(2) for the motion in the homogeneous field.

Key words: Homogeneous field, Euclidean group E(2), Hamilton-Jacobi method, Contraction of SO(3), Quantum rotator.

Introduction

The motion of a system having two degrees of freedom is said to take place in two dimensions. The simplest motion is the motion of a particle in a homogeneous external field in Cartesian coordinates (x, y, say), with the force F_x, F_y acting on the particle. The potential energy of the particle in the homogeneous field is of the form: $(x, y) = -F_x x - F_y y + constant$; choosing the constant so that $V(x, y) = 0$ for $x = 0, y = 0,$ we have $V(x, y) = -F_x x - F_y y$.

Two interesting cases of the homogeneous fields are an electric field of intensity ε acting on the charge -e with the force $\vec{F} = -e\vec{\varepsilon}$, and the other is the gravitation field in the y coordinate with the force $F_x = 0, F_y = -mg$. Projectile motion constitutes a very elementary problem of classical mechanics since the works done by Galoleo Galilei and Sir Issac Newton. The projectile motion is one of the main problems used to teach elementary physics and not well known facts about it appear in physics literature by now. Many of these ideas are presented in a compelling paper by Groetsch [1], Rao [2] and [3]. Also, the motion of the charge in the homogeneous field ε is the elementary problem in the electromagnetic field too [4].

Applications of group theory in physics establish the standard framework for the application of geometric symmetry groups to the treatment of Quantum mechanics systems that possess some geometric symmetry. Contraction is a process to reparameterize the Lie group's parameter space in such a way that the group properties in the Lie algebra remain well defined. The parameter space for the contracted group, Casimir operators, matrix elements of operators are well defined. Also, contraction provides limiting relations among the special functions of mathematical physics.

One of the interesting group contractions is the group SO(3) that is contracted to the Euclidean group E(2). The internal space-time symmetries of massive and massless particles are isomorphic to SO(3) and E(2) respectively. It has been shown that [5] transverse rotational degrees of freedom for massive particles become contracted to gauge degrees of freedom for massless particles.

In this work we investigate the motion of the particle in the homogeneous field in two directions. In particular, it is shown that by using the Hamiltonian-Jacobi method we get the well known results from the classical mechanic. Then, we show that quantum rotator on the sphere S^2 with Lie group SO(3) can be contracted to the motion of the particle with Euclidean group E(2) on the plane R^2.

The paper is organized as follows: we begin by proposing the Schrodinger equation for the motion of a particle in the homogeneous field in two dimensions and the limiting behavior of it, is investigated which goes to the well known results from the classical mechanics. In section 3, we will see that Quantum projectile yields from the contraction of the Quantum rotator and its eigenvalues and eigenfunctions take the form of the motion in the homogeneous field. Finally, we end the paper by conclusion.

2-1: Schrodinger equation for a particle in the homogeneous two dimensional fields

In this section, a particle in the homogeneous field is considered as a point mass *M* that its equation of motion must be solved by the Schrodinger equation. Since, the latter has no spatial extensions, this amount to neglecting issues of shape, orientation and rotation altogether. Furthermore, we suppose lift forces to be negligible too.

This motion has 2 degrees of freedom in the plane and can be considered as a problem with the classical group E(2) [6]. This group consists of all the transformations R_θ on R^2 plane. In other words, the Euclidean group E(2) consists of matrices of the form $\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$. This group has three infinitesimal generators L_z, P_x, P_y as following:

$$L_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ P_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ P_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2\text{-}1)$$

With the commutation relations:

$$[P_x, P_y] = 0, [L_z, P_x] = 0, [L_z, P_y] = 0. \quad (2\text{-}2)$$

The time independent Schrodinger's equation for a particle M in the *X-Y* Cartesian plane is:

$$H\psi(x,y) = E\psi(x,y), H = \frac{P_x^2 + P_y^2}{2M} + V(x,y), \quad (2\text{-}3)$$

Where by replacing the momentum operators:

$P_x = -i\hbar\partial_x$ and $P_y = -i\hbar\partial_y$ we have:

$$-\frac{\hbar^2}{2M}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\psi(x,y) + V(x,y)\psi(x,y) = E\psi(x,y) \quad (2\text{-}4)$$

We consider the coordinates are completely separable as following:

$$\psi(x,y) = X(x)Y(y), V(x,y) = V(x) + V(y), E = E_x + E_y, \quad (2\text{-}5)$$

So, the Schrodinger equation separates into two terms:

$$\left(-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} + V(x)\right)X(x) = E_x X(x), \quad (2\text{-}6)$$

$$\left(-\frac{\hbar^2}{2M}\frac{d^2}{dy^2} + V(y)\right)Y(y) = E_y Y(y), \quad (2\text{-}7)$$

These equations determine the motion of a particle in the x- and y-coordinates completely. By choosing appropriate variables the equation (2-6), (2-7) are:

$$\left[\frac{d^2}{d\acute{q}^2} + (E_{\acute{q}} - \acute{q})\right] Q(\acute{q}) = 0, \quad q = 1,2. \quad (2\text{-}8)$$

Where $\acute{q} = \frac{q}{l}$ and $l = \left(\frac{\frac{\hbar^2}{2M}}{F}\right)^{\frac{1}{3}}$; also $E_{\acute{q}} = \left(\frac{\hbar^2}{2M}F^2\right)^{\frac{1}{3}}$

and F is the force acting on the particle. Since the potential energy depend on the coordinates, it is clear that the energy levels form a continuous spectrum occupying the whole range of energy values E from $-\infty$ to $+\infty$. None of these eigenvalues is degenerate, and they correspond to motion which is finite. The equation (2-6), (2-7) do not contain the energy parameter. Hence, if we obtain a solution of them, we at once have the eigenfunctions for arbitrary values of the energy. Their solutions, which are finite for all q coordinate is called the Airy function [7].

Since, we'll be using a superposition of several single energy solutions, we must go to the general time dependent Schrodinger's equation, which has solutions like the usual time independent Schrodinger equation. So, by using linear combinations of time dependent solutions, we have:

$$\psi = NAi(-\acute{q} + E_{\acute{q}})e^{-iE_{\acute{q}}\frac{t}{\hbar}}, \quad (2\text{-}9)$$

Where N could be determined from the normalization condition.

Two special cases for the potential energy are the gravitational field and the electric field. For gravitational field in the y-direction we have $(x) = 0, V(y) = Mgy$. In this case the differential equation (2-6) is a familiar equation for us, with sinusoidal solution like $sin(kx)$ or $exp(ikx)$, where $k^2 = E_x$ and the complete solution is:

$$\psi = NAi\left(-\acute{y} + E_{\acute{y}}\right)e^{-iE_{\acute{y}}\frac{t}{\hbar}}\, exp(ikx)e^{-iE_x\frac{t}{\hbar}}, \tag{2-10}$$

The other case is the electric field in two directions, and the potential energy are: $(x) = -e\mathcal{E}_x x, V(y) = -e\mathcal{E}_y y$, where $\mathcal{E}$ is the electric field and we have two Airy function for the solution in the x- and y-directions.

2-2: The Hamiltonian-Jacobi method and the well known classical results

In this sub section we see that previous solution is completely compatible with the classical solution for semi classical wave function by using the Hamilton-Jacobi method [8]. Our Hamiltonian is:

$$H = \frac{P_x^2 + P_y^2}{2M} + V(x) + V(y) = E_x + E_y. \tag{2-11}$$

We obtain the Hamilton-Jacobi equations for S (action) by replacing P_x by $\frac{\partial S_x}{\partial x}$ and P_y by $\frac{\partial S_y}{\partial y}$ in the Hamiltonians (2-7), (2-8). Since our Hamiltonian is separable we obtain two equations:

$$\frac{1}{2M}\left[\left(\frac{\partial S_q}{\partial q}\right)^2 + V(q)\right] + \frac{\partial S_q}{\partial t} = E_q, \quad q = 1,2. \tag{2-12}$$

The explicit dependence of S_x and S_y on t is present only in the last terms, so, for S_q we have:

$$S_q = \pm\int\sqrt{2M\left(E_q - V(q)\right)}\,dq - E_q t, \tag{2-13}$$

Now, if we choose the particle is in the gravitational field in the y-direction we have for S_x:

(2-14) $S_x = \pm\sqrt{2ME_x}x - E_x t,$

And for S_y we also have:

$$S_y = \pm \int \sqrt{2M\left(E_y - Mgy\right)}\, dy - E_y t,$$

$$= \mp \tfrac{2}{3g}\sqrt{\tfrac{2}{M}}\left(E_y - Mgy\right)^{\frac{3}{2}} - E_y t.$$

(2-15)

To get out the explicit solutions of x and y coordinates we have:

$$\frac{\partial S_x}{\partial E_x} = \frac{\sqrt{2M}}{2} E_x^{-\frac{1}{2}} x - t = \alpha_1 \Rightarrow$$

$$x = \sqrt{\tfrac{2E_x}{M}}(\alpha_1 + t),$$ (2-16)

Where α_1 is an arbitrary constant, and we can find it by using initial conditions of the problem. For simplicity at initial time $t = 0$ we take it $\alpha_1 = 0$ and E_x is the kinetic energy $\frac{1}{2}mv_{0x}^2$, so for x equation we have:

$$x = v_{0x} t,$$ (2-17)

And for y coordinate:

$$\frac{\partial S_y}{\partial E_y} = -\frac{1}{Mg}\sqrt{2ME_y - 2M^2 gy} - t = \alpha_2 \Rightarrow$$

$$y = \tfrac{2E_y}{2Mg} - \tfrac{1}{2}(\alpha_2 + t)^2,$$

(2-18)

Again α_2 at initial time $t = 0$ is a constant and we take it equal zero, and $E_y = \frac{1}{2}Mv_{0y}^2 + Mgy_0$, so, we obtain:

$$y = -\tfrac{1}{2}gt^2 + v_{0y}t + y_0.$$ (2-19)

On the other hand the action is:

$$S = S_x + S_y =$$

$$\pm\sqrt{2ME_x}x \mp \tfrac{2}{3g}\sqrt{\tfrac{2}{M}}\left(E_y - Mgy\right)^{\frac{3}{2}} - Et,$$

(2-20)

And semi classical wave function is:

$$\psi = ae^{i\frac{S}{\hbar}} = ae^{\frac{i}{\hbar}\left[\pm\sqrt{2ME_x}x \mp \frac{2}{3g}\sqrt{\frac{2}{M}}\left(E_y - Mgy\right)^{\frac{3}{2}} - Et\right]},$$

(2-21)

Where a is the normalization parameter. Now, we want to compare the relation (2-21) with the wave function (2-10). So, by replacing the asymptotic relation for Airy function, i. e. $Ai(z) = e^{-\frac{2}{3}z^{\frac{3}{2}}}$ and ignoring the normalizing parameters $"N"$ and $"a"$, the right hand side of two relations (2-10) and (2-21) are the same.

For the electric field in two directions we obtain:

$$S_q = \pm \int \sqrt{2M\left(E_q - e\mathcal{E}_q q\right)}\, dq - E_q t,$$

$$= \mp \frac{2}{3}\sqrt{\frac{2M}{e\mathcal{E}_q}}\left(E_q - e\mathcal{E}_q q\right)^{\frac{3}{2}} - E_q t. \qquad (2\text{-}22)$$

By choosing appropriate conditions, we could obtain the well known results for this case too.

3: Quantum rotator and its relation to the Euclidean motion in the gravitational field

A rigid body may be defined in mechanics by the angular momentum operator J, which has the Lie group SO(3) on the sphere S^2. The generators of this group are:

$$J_i = x_j\partial_k - x_k\partial_j \qquad i,j,k = 1,2,3. \qquad (3\text{-}1)$$

The commutation relations are $[J_i, J_j] = \epsilon_{ijk} J_k$.

The Hamiltonian of a homogeneous rigid body is [8]:

$$H = \frac{J_x^2 + J_y^2 + J_z^2}{2I} + V(\theta) = \frac{J^2}{2I} + V(\theta), V(\theta) = MgRsin\theta = J_z \dot{\varphi} sin\theta \qquad (3\text{-}2)$$

Where I is the moment of inertia, the parameter θ is the polar angle and φ is the azimuthal angle. We are familiar with the corresponding eigenvalues and eigenfunctions as follows:

$$|\psi\rangle \propto P_m^j\left(cos\theta\left(=\frac{y}{R}\right)\right) = \left|\begin{matrix} j \\ m \end{matrix}\right\rangle, \quad J^2\left|\begin{matrix} j \\ m \end{matrix}\right\rangle = \hbar^2 j(j+1)$$

$$J_z\left|\begin{matrix} j \\ m \end{matrix}\right\rangle = m\hbar\left|\begin{matrix} j \\ m \end{matrix}\right\rangle. \qquad (3\text{-}3)$$

Under contraction with respect to the subalgebra of rotations about the Z-axis and considering the limit $R \to \infty$, the operators J_x, J_y transform to the operators $-P_y, P_x$ respectively. So, the commutation relations of the contracted group, i. e., the group E(2) are:

$$[J_z, P_x] = -P_y, \; [J_z, P_y] = P_x, \; [P_x, P_y] = 0. \qquad (3\text{-}4)$$

This group consists of rotations about the Z-axis, and displacements of the origin in the x- and y-directions. So, we have:

$$J^2 \to P_x^2 + P_y^2,\ J_z \to P_z,\ \ j(j+1)\hbar^2 \to p^2,\ \left|\begin{matrix} j \\ m \end{matrix}\right\rangle \to \left|\begin{matrix} p \\ m \end{matrix}\right\rangle,\ I \to M, \tag{3-5}$$

And its wave function $P_m^j(cos\theta) = \sqrt{2\pi}e^{im\varphi}Y_m^j(\theta,\varphi)$ is transformed to the $\sqrt{2\pi}e^{im\varphi}J_m(y)$, where $J_m(y)$ is the Bessel function [6]. This answer is equal to the wave function (2-10), because the Airy function could be written in terms of the Bessel functions [7]. For the potential part of the Hamiltonian under contraction we have:

$$V(\theta) \to P_z\dot{\varphi}sin\theta = Mgt\dot{\varphi}sin\theta, \tag{3-6}$$

Where we use the classical mechanics to compute P_z. If we replace $\theta = y/R$ and take the limit $R \to \infty$ for the relation (3-6), we conclude that:

$$V(\theta) \to V(y) = Mgy, \tag{3-7}$$

So, the contracted Hamiltonian is:

$$\acute{H} = \frac{P^2}{2M} + V(y). \tag{3-8}$$

This means that by changing the corresponding space of the motion from S^2 to R^2 by the contraction, the rigid Quantum rotator transformed to the projectile motion.

Lorentz group with respect to the time coordinate yields the homogeneous Galilei group. Contraction of the de Sitter group yields the inhomogeneous Lorentz group and these are because of the great magic power of the contraction!

4: Conclusion

Many physical systems exhibit symmetry. So, it is possible to use group theory and algebra to simplify both the treatment and the understanding of the problem. Central two-body problems, such as the gravitational and coulomb interactions, give rise to systems exhibiting spherical symmetry (two particles) with the Lie algebra so(3) or broken symmetry (planetary systems) with the Lie algebra e(2).

In this paper the motion of a particle in the homogeneous field, such as the gravitational or electric field, is analyzed with the standard Quantum and semi-classical method. It has also indicated that the Quantum mechanics behavior of the eigenfunctions and eigenvalues transformed to the classical results in the limiting case.

We have shown that the contraction of the group SO(3) takes our rotator on the sphere S^2 to the projectile motion on R^2 plane, i. e., Euler rotations for our rotator under contraction take it to the simple motion on a flat space R^2.

We would like to end our paper with some words from Galileo:

The great book of Nature lies ever open before our eyes and the true philosophy is written in it, but we cannot read it unless we have first learned the language in which it is written. It is written in mathematical language and the characters are triangles, circles and other geometrical figures [9].

5: References

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