Reinforcing Harmonic Oscillators In Electrostatics

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Abstract

We present a newly designed modulefor a General Physics course helping to reinforce the concept of harmonic motion through examples from electrostatics.

1. Introduction

Harmonic motion is one of the most important topics in general physics, particularly since understanding such motion is crucial for grasping a number of important processes in physical chemistry, such as absorption and emission of electromagnetic radiation, non-linear optics, or vibrotational molecular spectra [1]. Within the framework of a typical American university curriculum, harmonic oscillatorsare introduced in the first semester of a two-semester general physics sequence [2]. However, the amount of material and available choices for topics are so wide that occasionally introduction ofthe harmonic motion is delayed until the second semester of this sequence, the one traditionally devoted to electricity and magnetism. Below I describe a module(a series of activities and exercises) I designed for this specific purpose, namely for familiarizing students with harmonic motion through electrostatics-related problems and for reinforcing the relevant concepts.

Throughout this module I tried to keep the needed *mathematics* as simple as possible, thus allowing one to concentrate on the *physics* of things.

Less than a week into electrostatics [3], I start with a brief description of harmonic motion, both in terms of the definition and the transformations of energy during the oscillations of an object attached to a spring. I also present the main formulafor the period of such oscillations,

$$\tau = 2\pi\sqrt{m/k}, \tag{1}$$

with m being the mass of the oscillator and k the spring constant. Then I assign, for the class work, the following problem:

Problem 1.A positively charged bead (charge q, mass m) is placed between two identical positive point charges Q which are separated by a fixed distance $2a$ (Figure 1). The bead, confined to move along the line connecting charges Q, is displaced a certain distance ε along the axis and released.

Determine the condition upon which the subsequent bead oscillations are simple harmonic motion and determine the frequency ν of these oscillations. What is the significance of this result with regard to the general features of harmonic vs. non-harmonic oscillations in the nature?

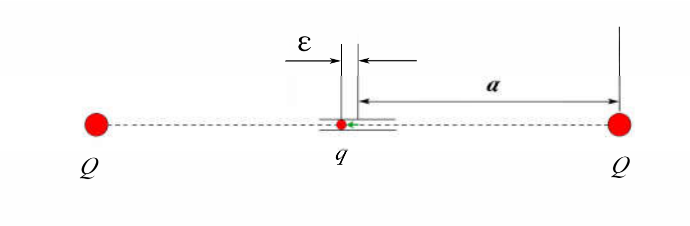


Figure 1. Geometry of Problem 1.

A number of students arrive at the solution successfully. The total force acting upon the bead is (see Fig. 1):

$$F_{tot} = F_{left} - F_{right} = k_e qQ\left[\frac{1}{(a-\varepsilon)^2} - \frac{1}{(a+\varepsilon)^2}\right] = 2k_e qQ\, a\,\frac{\varepsilon}{(a-\varepsilon)^2(a+\varepsilon)^2}, \tag{2}$$

where F_{right} , F_{left} are the magnitudes of the forces acting due to the right-hand-side and the left-hand-side point charges respectively. Clearly, when $\varepsilon \ll a$, Eq. (2) yields

$$F_{tot} = \frac{2k_e qQ}{a^3}\varepsilon \ \propto\ \varepsilon, \tag{3}$$

thus the oscillations in this case are harmonic, and according to Eqs. (1), (3),

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2k_e qQ}{ma^3}}. \tag{4}$$

In order to solve this problem, students needed to figure out that the force acting upon the bead is, for $\varepsilon \ll a$, directly proportional to its displacement ε from the position in the center of the gap between the charges *Q*. And, as Richard Feynman once said (and an article on mechanical harmonic motion [4] reminded), "the same equations have the same solutions" – in other words, electric forces in our system work exactly like a spring with a spring constant of $\frac{2k_e qQ}{a^3}$, thus yielding the solution Eq. (4).

Discussing the answer to the latter part of Problem 1, I describe to the students the analogy with oscillations of electronic clouds inatoms under the action of ordinary light, when the magnitude of oscillations is small and thus the oscillations are harmonic. However, when a powerful enough laser is used for irradiating the medium, deviations from harmonic behavior become significant, which gives rise to the whole new class of phenomena of non-linear optics, particularly to frequency doubling (see, for example, Ref. [1], p. 776).

As a homework, I assign students a problem on determining the field of a semi-infinite charge line (problem 24 [3]), along with the following problem dealing with a similar system:

<u>Problem 2.</u> A positively charged bead (charge *q*, mass *m*) is placed between two identical semi-infinite line charges extending one from negative infinity to $x=-x_0$, and another one from $x=x_0$ to positive infinityas shown inFigure 2. The lines carry positive charges with uniform charge density λ_0. The bead, confined to move along the *x* axis, is displaced by a small distance ε along the axis (where $\varepsilon \ll x_0$) and released (Figure 2). Show that the particle oscillates in simple harmonic motion and determine the frequency of these oscillations.

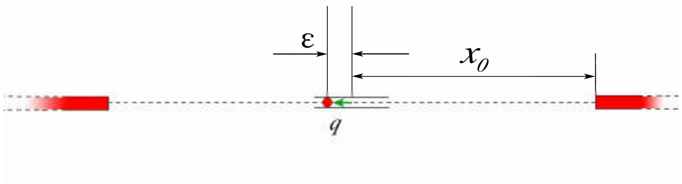


Figure 2. Geometry of Problem 2.

This latter problem is usually solved by a majority of students. They learn upon solving the (simultaneously assigned) problem 24 [3] that the magnitude of the force F_{right} acting due to the right-hand-side line charge is

$$F_{right} = k_e q \int_{x_a}^{\infty} \frac{\lambda_0 dx}{x^2} = k_e q \frac{\lambda_0}{x_a} \qquad (5)$$

where x_a is the distance between the particle and the edge of the line charge. Thus, for a bead deviated from x=0 by ε, the total force acting on it would be:

$$F_{tot} = F_{left} - F_{right} = k_e q \frac{\lambda_0}{x_0 - \varepsilon} - k_e q \frac{\lambda_0}{x_0 + \varepsilon} = 2k_e q \lambda_0 \frac{\varepsilon}{(x_0 - \varepsilon)(x_0 + \varepsilon)}, \qquad (6)$$

and $F_{tot} \propto \varepsilon$ for $\varepsilon << x_0$. Therefore, as in Problem 1, electrical field acts as a spring with a constant of

$$k = \frac{2k_e q \lambda_0}{x_0^2}, \qquad (7)$$

and thus

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2k_e q \lambda_0}{m x_0^2}}. \qquad (8)$$

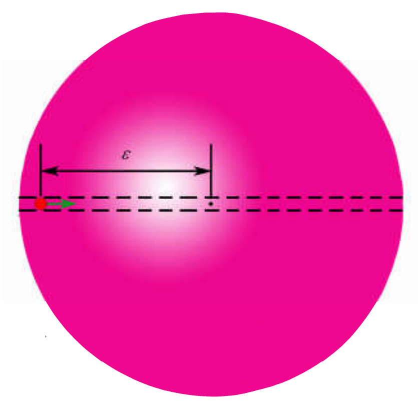
True to my usual routine [5,6], I assign a similar[i], but somewhat more open-ended, problem as a bonus problem for the test:

Problem 3. In a recent class we saw an example of an oscillator (pendulum) where the oscillations are harmonic only when their magnitude is small. Using the topics from electrostatics that we have just studied, design an oscillator for which the oscillations would remain harmonic for the finite, not just small, magnitude of oscillations. Express the frequency ν of these oscillations through the parameters of your designed system (size, etc).

Surprisingly, only very few students arrive at the correct solution this time – even though they all have studied the related system less than a week ago! In class, we consider the application of Gauss's law to an insulating,uniformly charged solid sphere (Ref. [3], pp. 641-642). For $r < a$ (where r is the distance from the sphere's center and a is its radius), it yields:

$$E = \frac{k_e Q}{a^3} r, \qquad (9)$$

where E is the electric field and Q is the total charge of this sphere. Thus, if we drill a narrow hole through the center of this sphere and let a small charged bead (with



a mass *m* and charge *q* opposite to Q) travel through that hole (see Figure 3), it will undergo harmonic oscillations for as long as their amplitude $\varepsilon < a$, i.e. for the ***finite*** ε. According to the above, the frequency of these oscillations is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_e qQ}{ma^3}} \quad .(10)$$

Figure 3. Geometry of Problem 3

In the author's opinion, the above activities/exercises allow one to "kill two rabbits with a single bullet", namely to reinforce the application of the principles of electrostatics in non-standard context and to teach students to recognize the harmonic motion in an unfamiliar setting, thereby providing the necessary reinforcement to the new concept. Besides, it allows for another reinforcement of the topic of frequency doubling ([1], p. 776), an example with which I always conclude the discussion of the solution to Problem 3 during the next class meeting, emphasizing the qualitative difference of the laser radiation's effect on the molecules from that of the traditional light source. Last, but not least, these exercises help the students to see one of the most beautiful aspects of physics – the universality of the characteristics of seemingly unrelated effects [4].

Remarkably, even though the module is fairly brief, it appears to do the trick – now, when students come to my physical chemistry class next year, absolute majority of them clearly recognize the relevant concept and easily transition to the study of non-linear optical phenomena [1].

References

1. Peter Atkins and Julio de Paula, *Physical Chemistry,* 10th Edition (WH Freeman, New York, 2014), pp. 503-522.
2. Raymond A. Serway and John W. Jewett, Jr., *Principles of Physics. A Calculus-Based Text,* 5th ed. (Thomson, Brooks/Cole, 2013), volume 1, pp 1- 618.
3. ibid., volume 2, Chapter 19 "Electric Forces and Electric Fields", pp. 619-655.
4. Iosif S. Bitensky, "Look at equations heedfully and you'll win," *Phys. Teach.* **35,** 540-541 (1997).
5. Michael A.Waxman, "Exploring Rotations Due to Radiation Pressure: 2D to 3D Transition Is Interesting!" *Phys. Teach.* **48**, 30-31 (2010).
6. Michael A.Waxman, "Using physics to investigate blood flow in arteries: A case study for pre-med students," *Am. J. Phys.* **78**, no. 9, pp. 970-973 (2010).

[i] Another similar problem that could be assigned here is the one dealing with oscillations of a particle near the center of a ring (problem 76 [3]). Unfortunately, in order to solve that problem, greater sophistication in mathematics (particularly, vector analysis) is needed. As a result, students are unable to solve the problem on their own, and the physics of the solution (when presented to them) gets buried under the math.