Nature of Legendre foretold by Pascal

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(Submitted 11-08-2015)

Abstract

Divisibility of a non–zero coefficient of an integer polynomial, by the binomial coefficient associated with the degree of the polynomial and the corresponding index of the power of the argument, is established for a Legendre Polynomial, multiplied by the exact power of 2 which divides the factorial of its degree, and a *modified* Hermite polynomial. These two coefficients have the same parity and their ratio is always an odd integer. Together the coefficients of each one of these two integer polynomials produce a perfect palindrome with respect to their parity. In spite of divisibility, the parity of a coefficient of a Laguerre polynomial, multiplied by the factorial of its degree, and the parity of a coefficient of a Hermite polynomial, cannot be predicted perfectly by the associated binomial coefficient.

1 Introduction

The Legendre polynomials are extremely useful in Special Functions, Mathematical Physics, Numerical Methods, Electromagnetic Theory, Quantum Mechanics, Quantum Theory of Angular Momentum, and Nuclear Physics [1, 2, 3, 4]. The power series expansion of the Legendre polynomial of degree n in x is given by [1, 2, 3]

$$P_n(x) = \sum_{s=0}^{[n/2]} \frac{(-1)^s \, (2n-2s)! \, x^{n-2s}}{2^n \, s! \, (n-s)! \ \ (n-2s)!}. \quad (1)$$

Here $[\rho]$ is the greatest integer $\leq \rho$. The common denominator of all the coefficients of $P_n(x)$, when reduced to their lowest terms, is 2^B, the greatest power of 2 which divides $n!$ [5, p. 352] (Pearl # 1). Our Legendre polynomial, *from now onwards*, is a rational function (the symbol "$\triangleq$" standing for "is equal to by definition"),

$$P_n(x) \ \triangleq \ K_n(x)/2^B, \quad n+1 \in \mathbb{N},$$

$$n!/2^B = odd \ \#, \quad (2)$$

such that all the coefficients of $K_n(x)$, a polynomial of degree n in x, are integers. Thus, by the numerator of $P_n(x)$ we always mean $K_n(x)$, and the denominator of $P_n(x)$ is always 2^B. The coefficients of $K_n(x)$ are 2^B times the Legendre coefficients. It is nice to note that $K_n(x)$ is an *integer polynomial*, i.e., a polynomial whose coefficients are integers [6]. Let us remember that $K_n(x)$ has a definite parity $(-1)^n$ [1, 2, 3]. It is an even (odd) polynomial when the degree n is even (odd). See Eqs. (1) and (2). Focussing on x^{n-2s} in Eq. (1), let us note that both $n-2s$ and n have the same parity: odd (even) when n is odd (even), since our $2s$ is an even integer.

The purpose of our paper is to show that the coefficient of $x^{n-2s}, 0 \leq s \leq [n/2], n+1 \in \mathbb{N}$, in $K_n(x)$, is always divisible by the associated binomial coefficient $nn-2s$ (Pearl # 2), and that the ratio of these two coefficients is an odd integer forever (Pearl # 3):

$$nn-2s \ \Big| \ \text{coefficient of } x^{n-2s} \text{ in } K_n(x), \quad (3)$$

$$(-1)^s \times \text{coefficient of } x^{n-2s} \text{ in } K_n(x) = \textit{odd positiveinteger} \times nn-2s. \quad (4)$$

We also establish the *parity–palindromic* nature of the coefficients of $K_n(x)$ (Pearl # 4). Our paper is organized as follows: Section 2 deals with the contributions of Legendre and Kummer [7, 8, 9] on the $p-adic$ valuation of certain interesting integers. In Section 3, we prove Holt's result [5, p. 352] (Pearl # 1) concerning the common denominator of the Legendre coefficients, when reduced to their lowest terms. That Pascal (binomial coefficient) is able to predict perfectly the parity of Legendre (coefficient multiplied by 2^B), is established. We show that each coefficient in the numerator of a Legendre polynomial is divisible by the associated binomial coefficient, $DegreePower$, always resulting in an odd integer upon division. We also prove that the coefficients in the numerator of $P_n(x)$ produce a palindrome with respect to their parity (parity–palindrome) perfectly. Section 4 contains other lovely and pleasant results concerning the Legendre coefficients. In Section 5, we appreciate the beauty of the Pascal's Triangle and consider its application to the Legendre polynomials. Dealing with the Laguerre, Hermite, and *modified* Hermite polynomials [1, 2, 3, 10] in Section 6, we prove that the (non-zero) coefficient of x^s in $\Phi_n(x)$, an integer polynomial [6], is divisible by the binomial coefficient ns, when $\Phi_n(x) = n! L_n(x), H_n(x), He_n(x); n+1,\ s+1 \in \mathbb{N},\ 0 \le s \le n$. Here $He_n(x), n \ge 0$, are the *modified* Hermite Polynomials [1, p. 189], [10].

2 Remembering and Honouring Legendre and Kummer

Let $\mathbb{P}$ be the set of prime numbers, $p \in \mathbb{P}$, and μ be an integer ≥ 2. Then $\nu_p(\mu)$, the $p-adic$ valuation of μ, is the exponent of p in the canonical decomposition in prime factors of μ. It has been elegantly shown by Legendre [7, 8, 9] that

$$\begin{aligned} \nu_p(n!) &= (n - \sigma_p(n))/(p-1), \\ & \qquad n \ge 2,\ n \in \mathbb{N},\ p \in \mathbb{P}, \end{aligned} \tag{5}$$

where $\sigma_p(n)$ is the sum of the digits in the base–p expansion of n, with

$$\sigma_p(np) = \sigma_p(n),\ n \in \mathbb{N},\ p \in \mathbb{P}. \tag{6}$$

Example # 1: Since $2015 = 5 \times 13 \times 31$ and since $5, 13$, and 31 are prime numbers, $\nu_5(2015) = 1 = \nu_{13}(2015) = \nu_{31}(2015)$; $\nu_3(2015) = 0 = \nu_{11}(2015) = \nu_{23}(2015)$. $\checkmark cfc$ (=*Checked and found correct!!*). We are extremely grateful to our revered Guruji, Prof. Dr V Devanathan [4], who always insists: "*Check, Recheck, Cross-Check, Double-Check, Multi-Check!!*" Let us develop the culture of checking the correctness of whatever we do [11]!! As his faithful students, we always ask our students *to check the correctness of their own calculations* in various ways.

Let a be a positive integer and let $a_k, a_{k-1}, \ldots, a_0$ be the digits of a, when written in base p. Hence the base–p expansion of a is

$$\begin{aligned} a &\triangleq (a_k\, a_{k-1} \ldots\ a_0)_p \\ &= \sum_{s=0}^{k} a_{k-s}\, p^{k-s}, \\ & \qquad 0 \le a_{k-s} \le p-1,\ p \in \mathbb{P}. \end{aligned} \tag{7}$$

Similarly, let $b \in \mathbb{N}$ with $b \triangleq (b_k\, b_{k-1} \ldots\ b_0)_p$ and let $\epsilon_q = 1$ if there is a carry–over in the

q^{th} digit when a and b are added; otherwise $\epsilon_q = 0$. Then (see [8, p. 1113], [9, p. 7])

$$c = a + b = p^{k+1}\epsilon_k + p^k c_k + \cdots + pc_1 + c_0, \quad (8)$$

with

$$\begin{aligned} c_0 &= a_0 + b_0 - p\,\epsilon_0, & (9)\\ c_t &= a_t + b_t + \epsilon_{t-1} - \ p\,\epsilon_t, & \\ &\qquad 1 \le t \le k, & (10) \end{aligned}$$

$$\sigma_p(a) + \sigma_p(b) - \sigma_p(a+b) = (p-1)\sum_{r=0}^{k}\epsilon_r. \quad (11)$$

When we perform the addition of a and b (in base $p \ge 3$), let us remember that $a_r,\ b_r \ge (p+1)/2$ leads to $\epsilon_r = 1$ (i.e., a carry–over, irrespective of whether there is a carry–over in the previous digit or not (see Eqs. (7) − (10))); $a_r,\ b_r \le (p-3)/2$ leads to no carry–over (i.e., $\epsilon_r = 0$), even if there is a carry–over in the previous digit; $a_r,\ b_r = (p-1)/2$ leads to $\epsilon_r = 1$, only when there is a carry–over in the previous digit (i.e., $\epsilon_{r-1} = 1$; a conditional carry–over).

Kummer (1852) [7, 8, 9] beautifully and cheerfully unveiled the following result for the binomial coefficient: $\nu_p(nm)$ is equal to the number of carry–overs when m and $n-m$ are added in base p.

Example # 2: We want to establish that the number 9060 is *not* divisible by 5. Now the representation of 60 in base 5 is $(220)_5$; that of 30 is $(110)_5$. Since no carry–over occurs when we add 220 and 110 (in base 5), Kummer [7, 8, 9] is pleased to tell us that the exponent of 5 in 9060, a 24–digit number ending in 4, is just zero. See also [8, p. 1114] and **Example # 5.**

3 Pascal predicts the parity of Legendre perfectly!

It follows from Eqs. (5) and (6) that

$$\nu_2\left(2n - 2sn \times ns\right) = \sigma_2(n) + \nu_2\left(nn - 2s\right). \quad (12)$$

Since a factor $2^{\sigma_2(n)}$ is common to all $2n - 2sn \times ns$, $0 \le s \le [n/2]$, $n - 1 \in \mathbb{N}$, we have

$$\begin{aligned} (2n-2s)! \quad / \quad \{2^n\, s!\,(n-s)!\,(n-2s)!\} \ &= 2n - 2sn \times ns \Big/ 2^n \\ &= \ 2^{\sigma_2(n)} \times \text{Integer}/2^n = \text{Integer}/2^{n-\sigma_2(n)} \ = A_{n-2s}/2^B, \quad (13) \end{aligned}$$

where (see Eqs. (5), (12), and (13))

$$\begin{aligned} B &\triangleq n - \sigma_2(n) \equiv \nu_2(n!), & (14)\\ A_{n-2s} &\triangleq (2n-2s)!\,/\{2^{\sigma_2(n)} s!\,(n-s)!\,(n-2s)!\}, & (15)\\ \nu_2(A_{n-2s}) &= \nu_2\left(nn - 2s\right). & (16) \end{aligned}$$

Equations (1), (2), (13), and (14) demand that the denominator of $P_n(x)$ is $2^B = 2^{\nu_2(n!)}$. Thus, we have proved Holt's result [5, p. 352; Pearl # 1]. It is clear from Eqs. (1), (2), and (13)−(15) that $(-1)^s A_{n-2s}$ is the coefficient of x^{n-2s} in $K_n(x)$. Equation (16) shows that the (non-zero) coefficient of x^{Power} in $K_{Degree}(x)$ and the corresponding binomial coefficient $DegreePower$ have the *same* parity!! These nice results lead to more such nice results, as shown below. The *same* exponent of 2 appears in the canonical decomposition in prime factors of the coefficient of x^{n-2s} in $K_n(x)$ and that of the corresponding binomial coefficient $nn-2s$. We will not forget that in the case of this binomial coefficient, both *Degree* and *Power* have the same parity ($\because n-(n-2s) = 2s = even \; \# \geq 0$).

It follows from Eqs. (15) and (16) that

$$\begin{aligned} L(n,s) &\triangleq A_{n-2s} \Big/ nn-2s \\ &= \frac{2n-2sn-s \times 2ss}{2^{\sigma_2(n)} \times ns} \\ &= L(n, n-s), \qquad (17) \\ \nu_2(L(n,s)) &= 0. \qquad (18) \end{aligned}$$

Equation (17) exhibits a nice symmetry with respect to an interchange of s and $n-s$.

From Eqs. (5) and (17), we have

$$\begin{aligned} (p-1)\,\nu_p(L(n,s)) = \sigma_p(n) \; &+ \; \sigma_p(n-s) + \sigma_p(s) \\ &- \; \{\sigma_p(2n-2s) + \sigma_p(2s)\} \\ &- \; (p-1)\,\nu_p(2^{\sigma_2(n)}), \quad p \in \mathbb{P}. \qquad (19) \end{aligned}$$

Application of Kummer's Theorem [7, 8, 9] to Eq. (17) leads to the following result (p is an *odd* prime; $\epsilon_r^{\beta,\eta} = 0, 1$; see also Eqs. (11) and (19)):

$$\nu_p(L(n,s)) = \sum_{r=0}^{r_{max}} \{(\epsilon_r^{n-s,n-s} \;+\; \epsilon_r^{s,s}) - \epsilon_r^{n-s,s}\} \in \mathbb{Z}, \quad p \geq 3, \quad p \in \mathbb{P}. \qquad (20)$$

Equations (19) and (20) reveal a nice symmetry: $\nu_p(L(n,s)) = \nu_p(L(n,n-s))$. See Eq. (17).

In Eq. (20) $\epsilon_r^{n-s,n-s}, \epsilon_r^{s,s}$, and $\epsilon_r^{n-s,s}$ are respectively the number of carry–overs (zero or one, in the r^{th} digit), when $n-s$ and $n-s$ are added, s and s are added, $n-s$ and s are added (see [7, pp. 63–65], [8, p. 1113], [9, p. 7]), all additions done in base p. Using Eqs. (7) − (10) and the discussions following Eq. (11), we can show that there can be a carry–over in $n - s$ added to s, only if there is a carry–over either from $n - s$ added to itself or from s added to itself (or from both). It is now clear that the right–hand side of Eq. (20), an integer, is never negative for all values of $p \in \mathbb{P}$. Hence $\nu_p(L(n, s)) + 1 \in \mathbb{N}, p \in \mathbb{P}$, and thus the coefficient of $x^{n-2s}, 0 \leq s \leq [n/2], n + 1 \in \mathbb{N}$, in $K_n(x)$, is an integer divisible by $nn - 2s$ (Pearl # 2). As $\nu_2(L(n, s)) = 0$ (see Eq. (18)), $L(n, s)$ is now an odd integer (Pearl # 3). Both 2^B times a Legendre coefficient and the associated binomial coefficient, *DegreePower*, sing joyfully in unison thus: "You are an integer, so am I! If you are odd, I am also odd!! (The contrapositive [12] of this true statement is also true: If I am even, you are also even!!) When I am odd, you too are odd! Truly, we are lovingly made for each other!! Whether we are odd or even, our ratio ("*Legendre / Pascal*") is always odd! Are we not the Adam and Eve of the Paradise of the Legendre Polynomials!?!" Yes, Pascal (*1623, †1662) predicts the nature of Legendre (*1752, †1833)!! "*Tell me who your friends are and I'll tell you who you are.*" So goes a Mexican Proverb. Ask any coefficient of $P_n(x)$. The reply will be the (corresponding) Binomial Coefficient (Pascal) and the Largest Power of 2 which divides the factorial of the degree n (Holt [5, p. 352])!! (*LP* for Legendre Polynomial, *LP* for Largest Power, *P* for Pascal.) See Eqs. (2) – (4).

Example # 3: $p = 3, n = 10 = (101)_3, s = 4 = (11)_3$. $\therefore\ n - s = 6 = (20)_3, 2(n - s) = 12 = (110)_3, 2s = 8 = (22)_3$. Now $\sigma_3(n) + \sigma_3(n - s) + \sigma_3(s) = 2 + 2 + 2 = 6$, $\sigma_3(2n - 2s) + \sigma_3(2s) = 2 + 4 = 6$, $\therefore \nu_3(L(10, 4)) = (6 - 6)/2 = 0$. See Eq. (19). Moreover, $\sum_r \epsilon_r^{n-s,n-s}$, $\sum_r \epsilon_r^{s,s}$, $\sum_r \epsilon_r^{n-s,s}$ are respectively $1, 0, 1$. $\therefore\ \sum_r(\epsilon_r^{n-s,n-s} + \epsilon_r^{s,s} - \epsilon_r^{n-s,s}) = 0$. These two numbers must be equal. ✓*cfc*. Therefore, 3 cannot be a factor of $L(10, 4)$. *Mathematica* says (see also Table 2) that the coefficient of x^2 in $K_{10}(x)$ is 3465. $\because 102 = 45, L(10, 4) = 3465/45 = 77$, an odd integer. Lo! Behold! 3| 77. ✓*cfc*. Yes, 3 does not divide $L(10, 4)$. Since $104 = 210 = even$, the coefficient of x^4 must be even and divisible by this binomial coefficient, the quotient being an odd integer. Actually (see Table 2), $-30030/210 = -143$, an odd integer. ✓*cfc*.

Example # 4: $p = 5, n = 13 = (23)_5, s = 3 = (3)_5$. $\therefore\ n - s = 10 = (20)_5, 2(n - s) = 20 = (40)_5, 2s = 6 = (11)_5$. Now $\sigma_5(n)+\sigma_5(n-s)+\sigma_5(s) = 5+2+3 = 10$, $\sigma_5(2n - 2s) + \sigma_5(2s) = 4 + 2 = 6$. $\therefore\ \nu_5(L(13, 3)) = (10 - 6)/4 = 1$. See Eq. (19). Moreover, $\sum_r \epsilon_r^{n-s,n-s}$, $\sum_r \epsilon_r^{s,s}$, $\sum_r \epsilon_r^{n-s,s}$ are respectively $0, 1, 0$. $\therefore \sum_r(\epsilon_r^{n-s,n-s} + \epsilon_r^{s,s} - \epsilon_r^{n-s,s}) = 1$. These two numbers must be equal. ✓*cfc*. Using *Mathematica*, we have found that the coefficient of x^7 in the numerator of $P_{13}(x)$ is -2771340. $\because\ 137 = 1716, L(13, 3) = 2771340/1716 = 1615 = 5 \times 17 \times 19 \to \nu_5(L(13, 3)) = 1$. ✓*cfc*. Yes, the exponent of 5, in the canonical decomposition in prime factors of $L(13, 3)$, an odd integer (Pearl # 3), is unity! One of its factors is 5.

Example # 5: Consider the binomial coefficient 2311. Kummer [7, 8, 9] gently reminds us that there is only one carry–over when we add 12 and 11 in base 2, since $12 = (1100)_2$, $11 = (1011)_2$. Therefore, the exponent of 2 in the canonical decomposition in prime factors of this binomial coefficient is simply unity. (See also **Example # 2.**) It is just even, not even divisible by 4. Hence the coefficient of x^{11} in the numerator of $P_{23}(x)$ must be simply even, not at all divisible by 4. *Mathematica* tells us that this coefficient, the above binomial coefficient, and their ratio are respectively 1 805 044 411 170 (a 13–digit number; even, but not divisible by 4 since the last two digits are 70), 1 352 078 (7–digits; not divisible by 4, though even, because of the last two digits: 78), and 1 335 015. Hence their structure is: (Even, Even, Odd)!!. ✓cfc.

Using the elementary result,

$$nr = n!/\{r!(n-r)!\} = nn - r, \qquad (21)$$

and Eq. (5), we have

$$\begin{aligned} \nu_2\left(2n+12n+1-2s\right) &= \nu_2\left(2n+12s\right) \\ &= \nu_2\left(2n+12s+1\right). \end{aligned} \qquad (22)$$

Hence, due to the (odd or even) *parity* of the binomial coefficients (see [13, p. 156], [14, pp. 18–19]), the coefficients of $x^{2n+1-2s}$ and x^{2s+1} in $K_{2n+1}(x)$ have the *same* parity (Sum of Powers = Degree + 1; remember *DegreePower*). Similarly, since $2n2s = 2n2n-2s$, the coefficient of x^{2s} in $K_{2n}(x)$ has the *same* parity as that of the coefficient of x^{2n-2s} (Sum of Powers = Degree). This then is the reason for the *palindromic* behaviour of the coefficients of $K_n(x)$ with respect to their (odd or even) parity (Pearl # 4). The coefficients of $K_n(x)$ nicely produce a parity–palindrome!! Together the coefficients in the numerator of $P_n(x)$ generate a perfect *parity–palindrome* (P for *Pascal,* $P_n(x)$, $p-adic$, *Parity, Palindrome, Polynomial, Power, Prime, Pearl, Product, Play*)!! When we play with the number of letters in the words in the title, we get the *palindromic* number 628826, pointing at the *palindromic* behaviour of the coefficients of $K_n(x)$, with respect to their parity (odd or even).

4 More Pearls

As $nn = 1 = odd$, the leading coefficient of $K_n(x)$ is always *odd*, irrespective of the degree n. When n is odd, $K_n(x)$ is an odd polynomial in x [1, 2, 3]. Since the coefficient of x corresponds to $n-2s=1$ (see Eq. (1)) and since $n1 = n = odd$, this coefficient is also odd. If n is even, $K_n(x)$ has an even parity [1, 2, 3]. As the constant term comes from $2s = n$ (see Eq. (1)) and as $n0 = 1 = odd$, the constant term of $K_n(x)$ is also odd. Not only the leading coefficient but also the last coefficient of $K_n(x)$ are always *odd*, independent of the (odd or even) nature of n (Pearl # 5). Since the denominator of $P_n(x)$ is $2^{\nu_2(n!)}$ (see Eqs. (2) and (14)), you can now prove that $P_{2k}(x)$ and $P_{2k+1}(x)$, $k \in \mathbb{N}$, have the *same* denominator (Pearl # 6; see Eq. (2)). As $P_n(1) = 1$ [1, 2, 3], the sum of the coefficients of $K_n(x), n \geq 2$, is exactly the denominator of $P_n(x)$ (Pearl # 7; see Eq. (2)). As this sum is even when $n \geq 2$ (see Eq. (5)), the odd–valued coefficients in the numerator of $P_n(x), n \geq 2$, must occur an even number of times (Pearl # 8; see Eq. (2))!

5 Beauty of the Pascal's Triangle and Binomial Coefficients

In 1899 Glaisher [13, p. 156] proved the following very interesting result (see also [8, Corollary 2.8, p. 1118], [9, p. 4], [14, p. 24]): *Each one of the Binomial Coefficients* $n0, n1, n2, \ldots, nn$ *is odd iff* n *has the form* $2^Q - 1, Q \in \mathbb{N}$. Thus we have four "*lovely lines*" *from Pascal's Poem / Triangle* (see Table 1)!! It follows from Table 1 that when $n = 2^Q$, $Q - 1 \in \mathbb{N}$, but for the leading coefficient and the constant term of $K_n(x)$, all the other coefficients (consistent with parity and degree) must be even only (Pearl # 9). In the case of $K_L(x)$, $L = 2^Q - 2,\ 2^Q - 1$, $Q \geq 2, Q \in \mathbb{N}$, all the non-zero coefficients of $K_L(x)$ must be odd only (Pearl # 10)!

Table 1. The structure of the beautiful binomial coefficients nr [8, 9, 13, 14], with $Q \geq 2, Q \in \mathbb{N}$, belonging to $n = 2^Q - 2$ is: (1, even, odd, ..., even, 1). Alternately, odd and even! The form of the pleasing binomial coefficients corresponding to $n = 2^Q - 1$ is (thanks to Glaisher [13, p. 156]): (1, odd odd, ..., odd, 1). Always odd!! The arrangement of the nice binomial coefficients corresponding to $n = 2^Q$ is: (1, even, even, ..., even, 1). They are even when they are greater than unity. These results follow from the way the Pascal's Triangle is constructed (3^{rd} Column onwards; $r \geq 1$), "Down (Bottom Row) = (Immediate Up + Immediate Left) (Immediate Top Row)" ($n + 1r = nr + nr - 1$) and the elementary fact that odd # + even # = odd #. Since the binomial coefficient $nn - 2s = nn - (n - 2s) = n2s$, with $2s = even\ \# \geq 0$, divides the coefficient of x^{n-2s} in $K_n(x)$, yielding an odd integer upon division, Pascal's prediction of the parity of Legendre is perfect!! Note: G for Glaisher's result [13, p. 156]; I for *Inference* from G; C for *Corollary* to G; E for *Even* #; Φ for *Odd* #; $Q \geq 2; Q, n - 1, r + 1 \in \mathbb{N}$; $0 \leq r \leq n$.

$n \Downarrow \setminus r \Rightarrow$	0	1	2	...	...	...	...	...
$(I)\, 2^Q - 2$	1	E	Φ	...	E	1		
$(\Downarrow G \Uparrow)\, 2^Q - 1$	1	$\Downarrow \Phi \Uparrow$	$\Downarrow \Phi \Uparrow$	...	$\Downarrow \Phi \Uparrow$	$\Downarrow \Phi$	$\Downarrow 1$	
$(C)\, 2^Q$	1	E	E	...	E	E	E	1

Example # 6: Let us gladly check [11] Glaisher's beautiful result [13, p. 156] by playing with $Q = 3, n = 2^Q - 1 = 7$. The eight binomial coefficients in this case are $1, 7, 21, 35,\ 35, 21, 7, 1$, and all of them are odd, honouring Glaisher! $\checkmark cfc$. Let us next check our inference from Glaisher (see Table 1). In the case of $n = 2^Q - 2 = 6$, the seven binomial coefficients are $1, 6, 15, 20, 15, 6, 1$. They are alternately odd, even!! Once again Glaisher is honoured!! $\checkmark cfc$. As far as our Corollary is concerned (see Table 1), in the case of $n = 2^Q = 8$, the nine binomial coefficients are $1, 8, 28, 56, 70, 56, 28, 8, 1$. As long as they are not unity, they are even only. $\checkmark cfc$. Hail Glaisher [13, p. 156]!!

6 Divisibility by the Binomial Coefficient extended to Laguerre, Hermite, and *modified* Hermite Polynomials!

In the case of the Laguerre Polynomials [1, 2, 3], we have

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^{n-r}\, n!\, x^{n-r}}{r!\,(n-r)!(n-r)!}, \tag{23}$$

$$n!\, L_n(x) = \sum_{s=0}^{n} (-1)^s\, nsns(n-s)!\, x^s. \tag{24}$$

Hence the coefficient of $x^s, 0 \le s \le n, n+1 \in \mathbb{N}$, in $n!L_n(x)$, is an integer divisible by ns. Let us not forget that our binomial coefficients and factorials are positive integers (see Eqs. (24), (25), (27), (28)). Remember that $n!L_n(x)$ is an integer polynomial [6].

Since the power series expansion of the Hermite polynomials is given by [1, 2, 3]

$$\begin{aligned} H_n(x) &= \sum_{s=0}^{[n/2]} \frac{(-1)^s\, n!\, (2x)^{n-2s}}{s!\,(n-2s)!} \\ &= \sum_{s=0}^{[n/2]} (-1)^s\, 2^{n-2s}\, nn-2s2ss\;\; s!\, x^{n-2s}, \end{aligned} \tag{25}$$

it is clear that the coefficient of $x^{n-2s}, 0 \le s \le [n/2]$, $n+1 \in \mathbb{N}$, in $H_n(x)$, is an integer divisible by the binomial coefficient $nn-2s$. All the Hermite coefficients are *even* when $n \ge 1$.

However, Pascal cannot predict the parity of Laguerre (coefficient multiplied by $n!$) and Hermite (coefficient) perfectly!

By playing with the *modified* Hermite Polynomials [1, p. 189], [10], we have

$$He_n(x) \triangleq 2^{-n/2} H_n(x/\sqrt{2}). \qquad (26)$$

Using Eqs. (25) and (26), we find that

$$He_n(x) = \sum_{s=0}^{[n/2]} (-1)^s \times nn-2s \times \Psi_s \times x^{n-2s}, \qquad (27)$$

where

$$\Psi_s \triangleq (2s)!/(2^s\, s!) = \begin{cases} 1 = odd\ \#, & s = 0, 1, \\ 1 \times 3 \times 5 \times \cdots \times (2s-1) = odd\ \#, & s - 1 \in \mathbb{N}. \end{cases} \qquad (28)$$

As a simple check on Eq. (28) (for $s \geq 2$), $\nu_2(\Psi_s) = 0 \Rightarrow \Psi_s$ *is odd*. See Eqs. (5) and (6). It follows from Eqs. (27) and (28) that (a) the *modified* Hermite polynomials are integer polynomials [6], (b) the coefficient of x^{n-2s} in $He_n(x)$, an integer, is divisible by the binomial coefficient $nn-2s$, (c) both of them (*modified* Hermite and Pascal) have the same parity, (d) their ratio is always odd; for $s \geq 2$, this ratio is a product of s consecutive odd positive integers, starting from 1 (multiplied by a phase factor, ± 1), and (e) the nature of the *modified* Hermite is predicted by Pascal as in the case of Legendre!! Hence Legendre too can predict the nature of *modified* Hermite (*1822, †1901)!! Pascal is equally friendly with Legendre and *modified* Hermite!! Let us not fail to note the *French* Connection: French by birth, Pascal, Legendre, Hermite, and Laguerre are world citizens / world–class mathematicians!! Know Pascal (Legendre), know Legendre (*modified* Hermite)!! What Pascal can do for Legendre, Legendre can do for *modified* Hermite!! Even though Pa (pascal) is a unit of pressure in Physics, Pascal is unable to put pressure on Laguerre and Hermite mathematically!!

7 Completeness can come with a Table

For the sake of completeness, we present a Table of $K_n(x)$, the numerator of the Legendre Polynomial $P_n(x)$, for $2 \leq n \leq 11$ (see Eq. (2); Table 2). Let us note the following: (a) $K_0(x) = 1, K_1(x) = x$, (b) the adjacent coefficients of $P_n(x), n \geq 2$, alternate in sign [15] (Pearl # 11), (c) there are no missing powers [15], consistent with the degree and definite parity of $P_n(x)$ [1, 2, 3] (Pearl # 12). Here is a nice chance for you to enjoy the beauty of our 12 Pearls and check the correctness of our results [11]!! Since the nature of Legendre is predicted by Pascal, we can confidently conclude, from $K_{11}(x)$, that 119 is odd, but not divisible by 3, nor by 5^2, nor by 7; 117 is even, but not divisible by 4. *Mnemonic: Legendre by Pascal is always odd!!* The nature (odd or even) of the coefficients of $K_{10}(x)$ and $K_{11}(x)$ is *palindromic* (Pearl # 4): (*Odd*, Odd, Even, Even, Odd, *Odd*). ✓*cfc*. $P_8(x)$ and $P_9(x)$ have the *same* denominator (Pearl # 1, Pearl # 6; see Eq. (2)). ✓*cfc*. The sum of the coefficients of $K_8(x)$ is exactly the denominator of $P_8(x)$ (Pearl # 1, Pearl # 7; see Eq. (2)). ✓*cfc*. The odd–valued coefficients of $K_6(x)$, $K_7(x)$, $K_{10}(x)$, and $K_{11}(x)$ are even (= 4) in number (Pearl # 8). ✓*cfc*. But for the last and the leading coefficients, all the coefficients of $K_2(x)$, $K_4(x)$, and $K_8(x)$ are even (Pearl # 9), thanks to Glaisher (1899) [13, p. 156]! ✓*cfc*. All the coefficients of $K_2(x), K_3(x), K_6(x)$, and $K_7(x)$ are odd (Pearl # 10); thank you Glaisher!! ✓*cfc*. The adjacent coefficients of $P_{10}(x)$ alternate in sign (Pearl # 11) [15]. ✓*cfc*. All the odd powers of x, right from 1 up to 11, are present in $P_{11}(x)$ (Pearl # 12) [15]. ✓*cfc*.

Table 2. Table of the Polynomials $K_n(x) = 2^{\nu_2(n!)}P_n(x)$, $2 \le n \le 11$. $K_n(x)$ is the numerator of $P_n(x)$. Column 2 is the ratio $K_n(x)/P_n(x)$, the denominator of $P_n(x)$ (see Eq. (2)).

n	$2^{\nu_2(n!)}$	Integer Polynomial $K_n(x)$, the numerator of $P_n(x)$
2	2	$3\,x^2 - 1$
3	2	$5\,x^3 - 3\,x$
4	8	$35\,x^4 - 30\,x^2 + 3$
5	8	$63\,x^5 - 70\,x^3 + 15\,x$
6	16	$231\,x^6 - 315\,x^4 + 105\,x^2 - 5$
7	16	$429\,x^7 - 693\,x^5 + 315\,x^3 - 35\,x$
8	128	$6435\,x^8 - 12012\,x^6 + 6930\,x^4 - 1260\,x^2 + 35$
9	128	$12155\,x^9 - 25740\,x^7 + 18018\,x^5 - 4620\,x^3 + 315\,x$
10	256	$46189\,x^{10} - 109395\,x^8 + 90090\,x^6 - 30030\,x^4 + 3465\,x^2 - 63$
11	256	$88179\,x^{11} - 230945\,x^9 + 218790\,x^7 - 90090\,x^5 + 15015\,x^3 - 693\,x$

You can also nicely play similarly with a Table of the *modified* Hermite Polynomials [10]. You can definitely enjoy how *modified* Hermite follows *Legendre* faithfully!! Thus, the nature (odd or even) of the coefficients of $He_{10}(x)$ and $He_{11}(x)$ is *palindromic* (Pearl # 4): (*Odd*, Odd, Even, Even, Odd, *Odd*). ✓ *cfc*. The odd–valued coefficients of $He_6(x)$, $He_7(x)$, $He_{10}(x)$, and $He_{11}(x)$ are even (= 4) in number (Pearl # 8). As a simple corollary, $He_n(1), n \ge 2$, is *even*!! *Mathematica* says:

$$He_{10}(x) = x^{10} - 45\,x^8 + 630\,x^6 - 3150\,x^4 + 4725\,x^2 - 945. \quad (29)$$

Example # 7: $He_{10}(1) = 1216 =$

even. ✓ cfc. With $n = 10$ and $n - 2s = 2, coefficientof\ x^2/102 = 4725/45 = 105 = 1 \times 3 \times 5 \times 7$, a product of $s\ (= 4)$ consecutive odd integers. ✓ cfc. See Eqs. (28) and (29). *Modified Hermite by Pascal is always odd!!*

Using the Differential Recurrence Relation, $DHe_m(x) = m\ He_{m-1}(x),\ D \triangleq d/dx,\ m \in \mathbb{N}$, you can generate, from Eq. (29), a Table of $He_n(x), 0 \leq n \leq 9$, and enjoy the beauty of our Pearls. Thus, all the even powers of x, right from 0 up to 8, are present in $He_8(x)$ (Pearl # 12). All the coefficients of $He_7(x)$ are odd (Pearl # 10), thanks to Glaisher [13].

8 Conclusion

The (non-zero) coefficient of x^s in $\Phi_n(x)$, an integer polynomial [6], is divisible by the associated binomial coefficient, ns, when $\Phi_n(x) = 2^{\nu_2(n!)}P_n(x), n!L_n(x), H_n(x), He_n(x)$. The Chebyshev Polynomials [2, 3], $T_n(x), n \geq 4$, do *not* satisfy this divisibility property in general. If you want to search for more pearls, you have to dive below the (common) denominator and the divisibility of (common denominator times) Legendre (coefficient) by Pascal (binomial coefficient)!! D for *Dive*, D for *Denominator*, D for *Divisibility*!! We have checked the correctness of our results [11] for $P_n(x)$, using *Mathematica* (for $2 \leq n \leq 150; 2 \leq p \leq 97$); in the case of $He_n(x)$, we have checked [11] the divisibility for $2 \leq n \leq 150$. We can relish Legendre (Polynomials) with Legendre (Eq. (5)), Pascal, Kummer, Holt [5], and Glaisher [13]!! *Blessed are those who are friendly with the Legendre Polynomials and the modified Hermite Polynomials!!*

Acknowledgment

BVJ would like to thank the UGC for the award of a Maulana Azad National Fellowship. This work has been classroom tested and we are extremely grateful to our nice M. Sc. (Physics) and M. Phil. (Computer Science) Students, University of Madras, for their encouraging response. We thank Prof. Dr Rita John, Dr D. Nedumaran, and Prof. Dr P. Thangavel, University of Madras, and Prof. Dr Ranabir Chakrabarti, Chennai Mathematical Institute, for their constant support. It is a pleasure for PRS and AJL to thank Ms. M. Sumathi for her kind help, support, and valuable suggestions. With all humility, we would like to dedicate this work to Dr J. W. L. Glaisher [13], J. B. Holt [5], Prof. Dr E. E. Kummer, A. M. Legendre, B. Pascal, and Prof. Dr Stephen Wolfram (*Mathematica*).

References

[1] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.

[2] W. W. Bell, *Special Functions for Scientists and Engineers*, Van Nostrand, London, 1968; Dover, New York, 2004.

[3] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, 5^{th} Edition, Harcourt (India), New Delhi, 2001.

[4] V. Devanathan, *Angular Momentum Techniques in Quantum Mechanics*

(Kluwer, Dordrecht, 1999); *Quantum Mechanics*, 2^{nd} Edition (Narosa, New Delhi, 2012); *Nuclear Physics*, 2^{nd} Edition (Narosa, New Delhi, 2013).

[5] J. B. Holt, On the irreducibility of Legendre polynomials I, *Proc. London. Math. Soc.* **11** ser. 2 (1912) 351–356. For Errata, see *ibid.* **12** ser. 2 (1912) 126–132.
`http://plms.oxfordjournals.org/content/s2-11/1/351.full.pdf`
`http://plms.oxfordjournals.org/content/s2-12/1/126.full.pdf`

[6] E. W. Weisstein, "*Integer Polynomial.*" From *MathWorld–A Wolfram Web Resource.*
`http://mathworld.wolfram.com/IntegerPolynomial.html`

[7] B. Sury, Revisiting Kummer's and Legendre's formulae, *Resonance* **10** (2005) 62–66.
`http://www.ias.ac.in/article/fulltext/reso/010/02/0062-0071`

[8] D. Miheţ, Legendre's and Kummer's theorems again, *Resonance* **15** (2010) 1111–1121.
`http://www.ias.ac.in/article/fulltext/reso/015/12/1111-1121`

[9] A. Granville, Binomial coefficients modulo prime powers, pp. 1–24.
`http://www.dms.umontreal.ca/~andrew/PDF/BinCoeff.pdf`

[10] E. W. Weisstein, "*Hermite Polynomial.*" From *MathWorld–A Wolfram Web Resource.*
`http://mathworld.wolfram.com/HermitePolynomial.html`

[11] P. R. Subramanian, Culture of checking the correctness of calculations, *Phys. Educ.* (*India*) **8** (1992) 387.

[12] See, for example, B. Kolman, R. C. Busby, and S. C. Ross, *Discrete Mathematical Structures*, 6^{th} Edition (Pearson Prentice–Hall, New Delhi, 2009).

[13] J. W. L. Glaisher, On the residue of a binomial–theorem coefficient with respect to a prime modulus, *Quart. J. Pure Appl. Math.* **30** (1899) 150–156. Thanks to *Niedersaechsische Staats- und Universitaetsbibliothek Digitalisierungszentrum, 37070 Goettingen, Germany. Email: gdz@www.sub.uni-goettingen.de*
`http://gdz.sub.uni-goettingen.de/dms/load/pdf/?PPN=PPN600494829_0030&DMDID=DMDLOG_0007`

[14] Chapter 4, Binomial Coefficients, October 6, 2009, pp. 1–61.
`http://www.cs.columbia.edu/~cs4205/files/CM4.pdf`

[15] P. R. Subramanian, On the adjacent coefficients of certain real orthogonal polynomials, *Eur. J. Phys.* **8** (1987) 149–150.
`http://www.iopscience.iop.org/0143-0807/8/2/116/pdf/0143-0807_8_2_116.pdf`