Alternative method for introducing torque and angular momentum

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Abstract

An alternative method is presented for introducing the physical quantities that are represented by vector products, namely, torque and angular momentum. The basis of the definitions of these quantities becomes evident through this approach. The reason that the position vector of a particle appears in these definitions is also clarified. The present approach can be smoothly integrated into the concepts of generalized forces and coordinates in analytical mechanics.

1 Introduction

Torque is an important physical quantity that arises in regard to the circular motion of a particle and the motion of rigid bodies. Even today, the derivation of the rotational form of Newton's second law from the translational form is discussed in the field of physics education [1]. In standard textbooks on classical mechanics [2, 3], however, a convenient vector expression for torque is introduced without explaining the basis of the definition, although the idea of torque is familiar at the undergraduate level. There are two questions that should be addressed from a pedagogical standpoint. Why is the torque vector defined using a position vector that has its origin at the fulcrum and that points towards the location where the force is applied? How do we understand the circumstances that result in the torque vector being defined as a vector that is orthogonal to both the position and force vectors?

An example [4] of the way in which the concepts of torque and angular momentum are commonly introduced is as follows. First, the torque and angular momentum vectors are defined as the vector products of the position, force, and linear momentum vectors. Suppose a force $\boldsymbol{F}$ acts on a particle whose position with respect to the origin is the position vector $\boldsymbol{r}$. Then, the torque acting on the particle with respect to the origin located at the fulcrum is defined as

$$\boldsymbol{T} = \boldsymbol{r} \times \boldsymbol{F}. \tag{1}$$

Suppose the particle has a linear momentum $\boldsymbol{p}$ relative to the origin. Then, the angular momentum of the particle is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}. \tag{2}$$

Using the definitions of torque and angular momentum, the relationship between them can be derived. Starting from the equation of motion

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}, \tag{3}$$

the torque is

$$\boldsymbol{T} = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt}. \tag{4}$$

By slightly rearranging this expression, the torque can be expressed as

$$\boldsymbol{T} = \frac{d\boldsymbol{L}}{dt}. \tag{5}$$

This method of explaining the relationship between the torque and angular momentum is clear, but no basis is given for why we consider the vector products of the position, force, and linear momentum.

In the present article, we propose an alternative method for introducing torque by letting it develop smoothly from the relationship between the work done by a force and the kinetic energy; this is done by clarifying the basis of the definition of the angular velocity vector. The rate of change of the kinetic energy is familiar to beginning students, although the mathematical treatment of the transformation of the equations is not necessarily simple. The mathematical notes necessary to derive the expressions for torque and angular momentum are good examples for introducing students in advanced classes to vector analysis. Misconceptions about the dynamics of rigid bodies are also discussed using the definition of torque expressed with the position vector.

2 Introducing torque and angular momentum using the relationship between work and kinetic energy

2.1. Basis of vector representation of angular velocity

Suppose the time rate of change of the position vector of a particle is restricted to pure rotation. The position of the particle is

$$\boldsymbol{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$$

in the frame in which the particle is at rest, where x, y, and z are the coordinates of the particle and $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the three corresponding orthogonal unit vectors that define the Cartesian coordinate system. These coordinates do not change with time; that is, $dx/dt = 0$, $dy/dt = 0$, and $dz/dt = 0$, and thus

$$\boldsymbol{v} \equiv \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{i}}{dt}x + \frac{d\boldsymbol{j}}{dt}y + \frac{d\boldsymbol{k}}{dt}z.$$

Following a textbook on mechanics [5], $d\boldsymbol{i}/dt$, $d\boldsymbol{j}/dt$, and $d\boldsymbol{k}/dt$ are expressed with the given Cartesian unit vectors. We recall that $\boldsymbol{i}\cdot\boldsymbol{i} = 1$, $\boldsymbol{j}\cdot\boldsymbol{j} = 1$, and $\boldsymbol{k}\cdot\boldsymbol{k} = 1$, and thus

$$\begin{aligned} \boldsymbol{i}\cdot\frac{d\boldsymbol{i}}{dt} &= 0, \\ \boldsymbol{j}\cdot\frac{d\boldsymbol{j}}{dt} &= 0, \\ \boldsymbol{k}\cdot\frac{d\boldsymbol{k}}{dt} &= 0. \end{aligned} \tag{6}$$

These inner products imply

$$\boldsymbol{i}\perp\frac{d\boldsymbol{i}}{dt},\ \ \boldsymbol{j}\perp\frac{d\boldsymbol{j}}{dt},\ \ \boldsymbol{k}\perp\frac{d\boldsymbol{k}}{dt}.$$

Thus, $d\boldsymbol{i}/dt$, $d\boldsymbol{j}/dt$, and $d\boldsymbol{k}/dt$ are in the yz, zx, and xy planes, respectively. We also recall that $\boldsymbol{i}\cdot\boldsymbol{j} = 0$, $\boldsymbol{j}\cdot\boldsymbol{k} = 0$, and $\boldsymbol{k}\cdot\boldsymbol{i} = 0$, and thus

$$\begin{aligned} \frac{d\boldsymbol{i}}{dt}\cdot\boldsymbol{j} + \boldsymbol{i}\cdot\frac{d\boldsymbol{j}}{dt} &= 0, \\ \frac{d\boldsymbol{j}}{dt}\cdot\boldsymbol{k} + \boldsymbol{j}\cdot\frac{d\boldsymbol{k}}{dt} &= 0, \\ \frac{d\boldsymbol{k}}{dt}\cdot\boldsymbol{i} + \boldsymbol{k}\cdot\frac{d\boldsymbol{i}}{dt} &= 0. \end{aligned} \tag{7}$$

Let $d\boldsymbol{i}/dt$, $d\boldsymbol{j}/dt$, and $d\boldsymbol{k}/dt$ be expressed as $\boldsymbol{j}c_1+\boldsymbol{k}c_2$, $\boldsymbol{k}c_3+\boldsymbol{i}c_4$, and $\boldsymbol{i}c_5+\boldsymbol{j}c_6$, respectively, where c_i $(i = 1, 2, \ldots, 6)$ are undetermined coefficients. Note that these six coefficients are not independent. From

$$\frac{d\boldsymbol{i}}{dt}\cdot\boldsymbol{j} = c_1$$

and

$$\boldsymbol{i}\cdot\frac{d\boldsymbol{j}}{dt} = c_4,$$

we obtain

$$c_4 = -c_1,$$

because we have

$$\frac{d\boldsymbol{i}}{dt}\cdot\boldsymbol{j} = -\boldsymbol{i}\cdot\frac{d\boldsymbol{j}}{dt}$$

from Eqs. (7). Similarly,

$$c_5 = -c_2, \quad c_6 = -c_3.$$

We represent c_1, c_2, and c_3 as ω_z, $-\omega_y$, and ω_x, respectively, giving

$$\begin{aligned} \frac{d\boldsymbol{i}}{dt} &= \boldsymbol{j}\omega_z - \boldsymbol{k}\omega_y, \\ \frac{d\boldsymbol{j}}{dt} &= \boldsymbol{k}\omega_x - \boldsymbol{i}\omega_z, \\ \frac{d\boldsymbol{k}}{dt} &= \boldsymbol{i}\omega_y - \boldsymbol{j}\omega_x, \end{aligned} \tag{8}$$

and thus we can represent the velocity vector in the form of the vector product

$$\begin{aligned} \boldsymbol{v} \ = \ & \boldsymbol{i}(\omega_y z - \omega_z y) + \boldsymbol{j}(\omega_z x - \omega_x z) \\ & + \boldsymbol{k}(\omega_x y - \omega_y x). \end{aligned} \tag{9}$$

Here, we define a vector $\boldsymbol{\omega}$ whose three components are ω_x, ω_y, and ω_z and call it the angular velocity vector from the standpoint of its physical meaning. This idea is the basis of the vector product $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, as follows. The magnitude of the velocity vector $|\boldsymbol{v}|$ is the product of $|\boldsymbol{\omega}|$ and $|\boldsymbol{r}|\sin\theta$, where θ is the angle between $\boldsymbol{\omega}$ and $\boldsymbol{r}$. The direction of the vector $\boldsymbol{\omega}$ is along the instantaneous axis of rotation, and $|\boldsymbol{r}|\sin\theta$ is the radius of a circle in the plane perpendicular to $\boldsymbol{\omega}$ (see Appendix).

2.2. Basis of definitions of torque and angular momentum

Based on the above preliminaries, we can consider the rate of change of the kinetic energy for a particle. Differentiating the kinetic energy gives

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}m|\boldsymbol{v}|^2\right) &= \frac{d}{dt}\left(\frac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v}\right)\\ &= \boldsymbol{v}\cdot m\frac{d\boldsymbol{v}}{dt},\end{aligned}$$

where m is the mass of the particle. Suppose a net external force $\boldsymbol{F}$ acts on the particle. From Newton's second law, the equation of motion is

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F},$$

where $\boldsymbol{F} = \boldsymbol{i}F_x + \boldsymbol{j}F_y + \boldsymbol{k}F_z$. From Eqs. (8), we have

$$\frac{d\boldsymbol{i}}{dt}\cdot\boldsymbol{j} = \omega_z,\quad \frac{d\boldsymbol{j}}{dt}\cdot\boldsymbol{k} = \omega_x,\quad \frac{d\boldsymbol{k}}{dt}\cdot\boldsymbol{i} = \omega_y,$$

and thus we can write

$$\begin{aligned}&\boldsymbol{v}\cdot m\frac{d\boldsymbol{v}}{dt}\\
&= \left(\frac{d\boldsymbol{i}}{dt}x + \frac{d\boldsymbol{j}}{dt}y + \frac{d\boldsymbol{k}}{dt}z\right)\cdot(\boldsymbol{i}F_x + \boldsymbol{j}F_y + \boldsymbol{k}F_z)\\
&= \frac{d\boldsymbol{i}}{dt}\cdot\boldsymbol{i}\ xF_x + \frac{d\boldsymbol{i}}{dt}\cdot\boldsymbol{j}\ xF_y + \frac{d\boldsymbol{i}}{dt}\cdot\boldsymbol{k}\ xF_z\\
&\quad + \frac{d\boldsymbol{j}}{dt}\cdot\boldsymbol{i}\ yF_x + \frac{d\boldsymbol{j}}{dt}\cdot\boldsymbol{j}\ yF_y + \frac{d\boldsymbol{j}}{dt}\cdot\boldsymbol{k}\ yF_z\\
&\quad + \frac{d\boldsymbol{k}}{dt}\cdot\boldsymbol{i}\ zF_x + \frac{d\boldsymbol{k}}{dt}\cdot\boldsymbol{j}\ zF_y + \frac{d\boldsymbol{k}}{dt}\cdot\boldsymbol{k}\ zF_z\\
&= \frac{d\boldsymbol{i}}{dt}\cdot\boldsymbol{j}\ (xF_y - yF_x) + \frac{d\boldsymbol{j}}{dt}\cdot\boldsymbol{k}\ (yF_z - zF_y)\\
&\quad + \frac{d\boldsymbol{k}}{dt}\cdot\boldsymbol{i}\ (zF_x - xF_z)\\
&= (yF_z - zF_y)\ \omega_x + (zF_x - xF_z)\ \omega_y\\
&\quad + (xF_y - yF_x)\ \omega_z\\
&= (\boldsymbol{r}\times\boldsymbol{F})\cdot\boldsymbol{\omega}. \qquad (10)\end{aligned}$$

This result means that the change in kinetic energy, $\frac{d}{dt}\left(\frac{1}{2}m|\boldsymbol{v}|^2\right)$, is caused by $(\boldsymbol{r}\times\boldsymbol{F})\cdot\boldsymbol{\omega}$, and thus this inner product indicates the work done on the particle. The vector product $\boldsymbol{r}\times\boldsymbol{F}$ is a measure of how much a net external force acting on a particle causes that particle to rotate, so we call this vector product the torque.

The same result can also be confirmed in the following way. For simplicity, we assume that $\boldsymbol{\omega}$ is constant. Differentiating the kinetic energy yields

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}m|\boldsymbol{v}|^2\right) = \frac{d}{dt}\left(\frac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v}\right)\\
&= \frac{d}{dt}\left[\frac{1}{2}m(\boldsymbol{\omega}\times\boldsymbol{r})\cdot\boldsymbol{v}\right]\\
&= \frac{d}{dt}\frac{1}{2}m\left[(\omega_y z - \omega_z y)\ v_x + (\omega_z x - \omega_x z)\ v_y\right.\\
&\quad \left. + (\omega_x y - \omega_y x)\ v_z\right]\\
&= (\omega_y v_z - \omega_z v_y)\ mv_x + (\omega_z v_x - \omega_x v_z)\ mv_y\end{aligned}$$

$$
\begin{aligned}
&+ (\omega_x v_y - \omega_y v_x)\ m v_z \\
&+(\omega_y z - \omega_z y)\ m\frac{dv_x}{dt} \\
&+(\omega_z x - \omega_x z)\ m\frac{dv_y}{dt} \\
&+ (\omega_x y - \omega_y x)\ m\frac{dv_z}{dt} \\
= &\ (\omega_y z - \omega_z y)\ F_x + (\omega_z x - \omega_x z)\ F_y \\
&+ (\omega_x y - \omega_y x)\ F_z, \\
= &\ (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot \boldsymbol{F} \\
= &\ \boldsymbol{v} \cdot \boldsymbol{F},
\end{aligned}
\tag{11}
$$

which indicates the rate at which work is done. To express a measure of the amount of rotation and the tendency of a force to rotate a particle about an axis, the power expressed by Eq. (11) can be transformed into

$$
\begin{aligned}
&(yF_z - zF_y)\ \omega_x + (zF_x - xF_z)\ \omega_y \\
&+ (xF_y - yF_x)\ \omega_z \\
= &\ (\boldsymbol{r} \times \boldsymbol{F}) \cdot \boldsymbol{\omega}.
\end{aligned}
$$

We recall that $p_x = mv_x$, $p_y = mv_y$, and $p_z = mv_z$, so the rate of change of kinetic energy can be expressed as

$$
\begin{aligned}
&(\omega_y z - \omega_z y)\ m\frac{dv_x}{dt} \\
&+ (\omega_z x - \omega_x z)\ m\frac{dv_y}{dt} \\
&+ (\omega_x y - \omega_y x)\ m\frac{dv_z}{dt} \\
= &\ \left(y\ \frac{dp_z}{dt} - z\ \frac{dp_y}{dt}\right)\omega_x \\
&+ \left(z\ \frac{dp_x}{dt} - x\ \frac{dp_z}{dt}\right)\omega_y \\
&+ \left(x\ \frac{dp_y}{dt} - y\ \frac{dp_x}{dt}\right)\omega_z \\
= &\ \left[\frac{d}{dt}\left(yp_z - zp_y\right)\right]\omega_x \\
&+ \left[\frac{d}{dt}\left(zp_x - xp_z\right)\right]\omega_y \\
&+ \left[\frac{d}{dt}\left(xp_y - yp_x\right)\right]\omega_z \\
= &\ \left[\frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p})\right] \cdot \boldsymbol{\omega}.
\end{aligned}
\tag{12}
$$

By comparing $[d(\boldsymbol{r} \times \boldsymbol{p})/dt] \cdot \boldsymbol{\omega}$ in Eq. (12) with $(\boldsymbol{r} \times \boldsymbol{F}) \cdot \boldsymbol{\omega}$ transformed from Eq. (11), we obtain

$$
\frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = \boldsymbol{r} \times \boldsymbol{F},
$$

which is the same as Eq. (5). The physical quantity $\boldsymbol{r} \times \boldsymbol{p}$ is a measure of the amount of rotation, so we call this vector product the angular momentum. This relationship indicates that the rate of change of the angular momentum is equal to the torque.

If students are familiar with the scalar triple product, the above results can be derived from $(\boldsymbol{r} \times \boldsymbol{F}) \cdot \boldsymbol{\omega} = (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot \boldsymbol{F}$ and $(\boldsymbol{r} \times \boldsymbol{p}) \cdot \boldsymbol{\omega} = (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot \boldsymbol{p}$, which means that the dot and cross products in the scalar triple product may be interchanged without altering the value of the product. Pedagogically, it is important for beginning students to rearrange the equation without relying on the formulae.

From the point of view of analytical mechanics, we can understand the introductory remarks by Feynman [7], which are based on an analogy between linear and angular quantities. The definition of work as the force times the displacement is thus readily converted to the torque times the rotational angle. Suppose that a force is applied to a particle at a certain point (x, y) in the xy plane, and the particle rotates by a very small angle $\Delta\theta$ in this plane. Slightly rearranging the expression of the work $F_x\Delta x + F_y\Delta y$ yields $(xF_y - yF_x)\Delta\theta$, where Δx and Δy are the change in x and y, respectively, because $\Delta x = -y\Delta\theta$ and $\Delta y = +x\Delta\theta$ [7] by reference to the kinematics of two-dimensional rotation. The generalized force is the coefficient of the variation of a generalized coordinate in the formulation of virtual work. By replacing the changes Δx, Δy, and $\Delta\theta$ by the virtual displacements, δx, δy, and $\delta\theta$, $xF_y - yF_x$ and θ can be regarded as the generalized force and the generalized coordinate, respectively. The generalized force in this motion is a kind of rotational force called torque.

The main results of the process described above are summarized as follows. A pedagogical framework of elementary mechanics can be developed from temporal and spatial viewpoints [6].

1. From a spatial viewpoint, the change in the kinetic energy of a rotating particle is caused by the total work done on that particle by all the torques that act on it during the process of rotation. In the simple case of rotation in the xy plane, this theorem can be expressed as

$$d\left(\frac{1}{2}m|\boldsymbol{v}|^2\right) = \boldsymbol{T}\cdot\boldsymbol{k}d\theta,$$

where $\boldsymbol{\omega}$ is along the z-axis and can be represented as $(d\theta/dt)\boldsymbol{k}$.

2. From a temporal viewpoint, the change in the angular momentum of a rotating particle is caused by the total impulse of all the torques that act on it during the process of rotation:

$$d(\boldsymbol{r}\times\boldsymbol{p}) = \boldsymbol{T}dt,$$

where $\boldsymbol{T}dt$ is the vector product of $\boldsymbol{r}$ and $\boldsymbol{F}dt$.

We can develop a framework of elementary mechanics from these temporal and spatial viewpoints [6]. There are three alternatives to the equation of motion that can act as starting points for elucidating mechanical phenomena: (1) The change in the linear momentum of a particle is caused by an impulse. (2) The change in the kinetic energy of a particle is caused by the work done on that particle by an applied force. (3) The change in the angular momentum of a particle is caused by torque. Of these three propositions, (2) is a theorem common to both translational and rotational motion. We can consider that these three propositions result in the equation of motion. From this standpoint, we have introduced the concept of torque to describe rotational motion by (2) in the present article. Temporal and spatial viewpoints are two perspectives on the same physical phenomenon. Thus, we can translate between (1) and (3) for translational motion and between (2) and (3) for rotational motion. In this sense, we can say that (3) has been translated from (2) in the present article.

3 Misconceptions about the dynamics of rigid bodies

Unexpectedly, in the field of mechanical engineering, there are misconceptions about the definitions of torque and angular momentum. From this fact, we expect similar misconceptions are found in college physics, and thus we consider the following example. By gaining an understanding of this issue, beginning students can improve their ability to find misconceptions and to correct them.

According to commentary [8] on the free rotation of rigid bodies, the definitions of torque and angular momentum as the vector products of the position vector of the particle relative to the origin, the net external force on that particle, and the linear momentum vector of the particle are not correct, because the origin is considered, but the axis of rotation is not. This commentary claims that the definitions of torque and angular momentum are the distance from the axis of rotation times the transverse components of force and linear momentum, respectively.

The above interpretation contains a misunderstanding. The commentary [8] confuses the definitions of torque and moment of inertia. Beginning students may also share this misunderstanding. We can explain the circumstances through a simple example. Suppose that a thin slab lies in the xy plane and rotates about the z-axis with an angular velocity vector $\boldsymbol{\omega}$ in the z-direction (Figure 1). For the present purpose, it is convenient to use a cylindrical coordinate system (Figure 2). The position vector $\boldsymbol{r}_i$ for the mass element m_i can be expressed as

$$\boldsymbol{r}_i = \hat{\boldsymbol{\rho}}_i \rho_i + \boldsymbol{k} z_i, \qquad (13)$$

where $\hat{\boldsymbol{\rho}}_i$ is a unit vector perpendicular to the z-axis and defined by coordinates ρ_i and φ_i, and the relationships between the cylindrical coordinates ρ_i, φ_i, and z_i, and the Cartesian coordinates are given by $x_i = \rho_i \cos\varphi_i$, $y_i = \rho_i \sin\varphi_i$, and $z_i = z_i$.

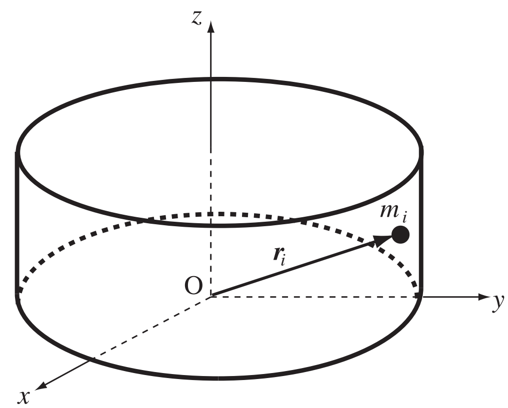

Figure 1. Mass element of a slab.

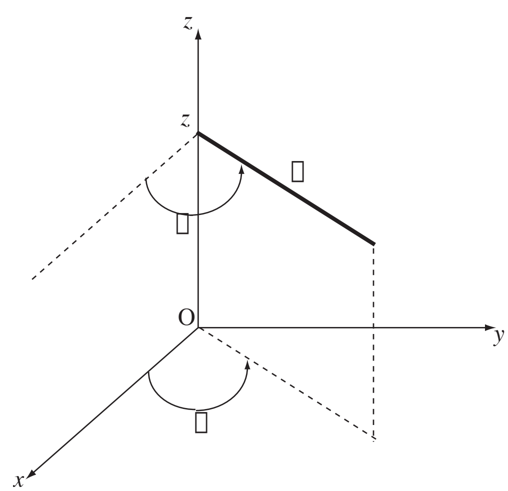

Figure 2. Cylindrical coordinate system.

For the relationships between the Cartesian and cylindrical unit vectors, we have

$$\begin{aligned} \hat{\boldsymbol{\rho}}_i &= \boldsymbol{i}\cos\varphi_i + \boldsymbol{j}\sin\varphi_i, & (14)\\ \hat{\boldsymbol{\varphi}}_i &= \boldsymbol{i}\cos\left(\varphi_i + \frac{\pi}{2}\right) + \boldsymbol{j}\sin\left(\varphi_i + \frac{\pi}{2}\right) \\ &= -\boldsymbol{i}\sin\varphi_i + \boldsymbol{j}\cos\varphi_i, & (15) \end{aligned}$$

and thus

$$\begin{aligned} \frac{d\hat{\boldsymbol{\rho}}_i}{dt} &= (-\boldsymbol{i}\sin\varphi_i + \boldsymbol{j}\cos\varphi_i)\ \frac{d\varphi_i}{dt} \\ &= \hat{\boldsymbol{\varphi}}_i\ \frac{d\varphi_i}{dt}. & (16) \end{aligned}$$

The height of the slab is constant, because the slab lies in the xy plane, and thus $dz_i/dt = 0$ for mass element m_i. The radius of rotation ρ_i is also constant, because the slab is a rigid body in which deformation is neglected, and thus the distance between any two points on the slab remains constant in time regardless of the external forces exerted on the slab. The angular velocity is common for all mass elements, so $d\varphi_i/dt = \omega$, where ω is the magnitude of $\boldsymbol{\omega}$. From Eq. (16), the velocity vector of mass element m_i is

$$\begin{aligned} \boldsymbol{v}_i &\equiv \frac{d\boldsymbol{r}_i}{dt} \\ &= \rho_i \frac{d\hat{\boldsymbol{\rho}}_i}{dt} \\ &= \hat{\boldsymbol{\varphi}}_i\ \rho_i \frac{d\varphi_i}{dt} \\ &= \hat{\boldsymbol{\varphi}}_i\ \rho_i\omega, & (17) \end{aligned}$$

where $\hat{\boldsymbol{\varphi}}_i$ is a unit vector perpendicular to $\hat{\boldsymbol{\rho}}_i$. By reference to $\hat{\boldsymbol{\rho}}_i \times \hat{\boldsymbol{\varphi}}_i = \boldsymbol{k}$ and $\boldsymbol{k} \times \hat{\boldsymbol{\varphi}}_i = -\hat{\boldsymbol{\rho}}_i$ obtained from Eqs. (14) and (15), we can calculate the angular momentum $\boldsymbol{L}_i$ for mass element m_i. Thus, from Eqs. (13) and (17), we have

$$\begin{aligned} \boldsymbol{L}_i &\equiv \boldsymbol{r}_i \times m_i \boldsymbol{v}_i \\ &= (\hat{\boldsymbol{\rho}}_i \rho_i + \boldsymbol{k} z_i) \times \hat{\boldsymbol{\varphi}}_i m_i \rho_i \omega \\ &= \boldsymbol{k} m_i {\rho_i}^2 \omega - \hat{\boldsymbol{\varphi}}_i m_i \rho_i z_i \omega. \end{aligned}$$

We sum over all elements, and the angular momentum for the slab is then

$$\begin{aligned} \boldsymbol{L} &= \sum_i \boldsymbol{L}_i \\ &= \boldsymbol{k}\left(\sum_i m_i {\rho_i}^2\right)\ \omega \\ &\quad - \left(\sum_i \hat{\boldsymbol{\varphi}}_i m_i \rho_i z_i\right)\ \omega \\ &= \boldsymbol{k} I_z \omega, \end{aligned}$$

where, in the first term, we have defined the physical quantity I_z, called the moment of inertia, for the slab with respect to the z-axis as $I_z \equiv \sum_i m_i {\rho_i}^2$. The physical quantity of the second term is zero, because the slab is symmetric with respect to the xy plane, as shown in Figure 1. This result indicates that the angular momentum is expressed as a vector in the direction of the rotational axis. The moment of inertia is determined by ${\rho_i}^2$, the square of the distance from the z-axis to the mass element m_i, instead of by $|\boldsymbol{r}_i|^2$, the square of the distance from the origin to that element.

Therefore, the angular momentum is properly defined as the vector product of the position vector of a particle relative to the origin and the linear momentum vector of that particle. As a result, we can say that a net external torque is needed for the angular acceleration of the particle about the axis of rotation. The angular momentum with respect to the rotational axis described in the commentary [8] is exactly the z-component of $\boldsymbol{L}$ expressed as $\boldsymbol{L}\cdot\boldsymbol{k}$, which is $I_z\omega$. The moment of inertia I_z is determined by the distance from the axis of rotation. Similarly, the net external torque giving rise to the changing angular momentum vector is exactly the z-component of $\boldsymbol{T}$ expressed as $\boldsymbol{T}\cdot\boldsymbol{k}$, where $\boldsymbol{T}$ includes the sum of the interactions between particles [9].

4 Concluding remarks

A physical law is extended by defining new physical quantities such that the elementary principles can be maintained. An extension of a physical law to include a fundamental law is similar to the mathematical principle of the permanence of form and its transition [10].

For the present subject, two approaches can be used to introduce the concept of torque through the kinetic energy and work theorem. One is to maintain the analogy between linear and angular quantities through the corresponding arrangement of the expression of work [7]. The amount of work is expressed as the rotational angle multiplied by a combination of the force and the distance. The other is to arrange the expression of the kinetic energy of a rotating particle as shown in Section 2.2.

Understanding the relationships between physical quantities is essential to learning the meaning of physical laws. It is pedagogically important to explore the basis of the definitions of physical quantities.

Appendix: Direction of vector obtained by vector product

For each direction, a line segment has two senses, a positive and a negative sense. Similarly, a plane has two sides. A directed area element corresponds to the length of the directed line segment. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the area of the plane determined by these vectors is expressed as the exterior product $\boldsymbol{a}\wedge\boldsymbol{b}$. When the vector $\boldsymbol{a}$ in the first position in the expression is rotated by the smallest angle that will cause it to coincide with the direction of $\boldsymbol{b}$, the area of the plane is defined as positive.

The vector product $\boldsymbol{a}\times\boldsymbol{b}$ is defined by a vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ and has a magnitude of $|\boldsymbol{a}\wedge\boldsymbol{b}|$. The sense of the plane determined by $\boldsymbol{a}$ and $\boldsymbol{b}$ is that of the motion of a right-handed screw. Thus, $\boldsymbol{a}\times\boldsymbol{b} = (\boldsymbol{a}\wedge\boldsymbol{b})\,\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector with a positive sense.

Following this rule, the senses of $\boldsymbol{T}$, $\boldsymbol{L}$, and $\boldsymbol{v}$ are defined, and thus $\boldsymbol{T}$ is perpendicular to $\boldsymbol{r}$ and $\boldsymbol{F}$, and so on. When a particle rotates in a plane, the direction of $\boldsymbol{n}$ is that of the axis of rotation. The direction of $\boldsymbol{\omega}$ is defined as that of the axis of rotation, and thus the velocity of a particle $\boldsymbol{v}$ is perpendicular to the plane determined by $\boldsymbol{\omega}$ and $\boldsymbol{r}$.

References

[1] D. J. Cross (2015). Am. J. Phys. 83, 121-125

[2] C. Kittel, W. D. Knight, and M. A. Ruderman. Berkeley Physics Course Volume 1, Mechanics, (New York: McGraw-Hill, 1973) pp. 102- 115, p. 185

[3] H. Goldstein. Classical Mechanics, (Amsterdam: Addison-Wesley, 1978) pp. 24-25

[4] K. Morgan. Torque and angular momentum in circular motion, (Project PHYSNET, Physics Bldg, Michigan State University, 2002) p. 6

[5] K. Husimi and S. Nakai. Rikigaku, (Tokyo: Kyoritsu Shuppan, 1959) pp. 3-4 (in Japanese)

[6] Y. Kobayashi (2012). Phys. Educ. 28, 6

[7] R. P. Feynman, R. B. Leighton, and M. Sands. The Feynman Lectures on Physics, (Amsterdam: Addison-Wesley, 1970) 18-2

[8] Y. Yamada (2000). Gokiso 370 (in Japanese)

[9] T. Matsuda (1987). J. Phys. Educ. Soc. Jpn. 35, 262 (in Japanese)

[10] Y. Kobayashi (2014). Eur. J. Pure Appl. Math. 7, 405-411