Scaling Analysis: Size of Lilliputs, Giants and Normal Humans

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Abstract

In this paper, we would like to illustrate the power of scaling analysis to estimate the order of magnitude of the size of Lilliputs, Giants and Normal humans. We further extend this scaling approach to estimate the typical life span of human being in terms of natural fundamental constants of physics.

1. Introduction

Scaling laws [1-9] are observed in all over science. The simplest is the Kepler's third law seen in the planetary motion governed by Newton's law of gravitation. The square of the time period of revolution of the planets is proportional to the cube of the distance between the planets and the Sun. In fact, there is indeed a deep symmetry between space and time for this particular power law. Considering Newton's inverse square law, we can write down the force equation

$$m\frac{d^2r}{dt^2} = -\frac{GMm}{r^2} \qquad (1)$$

A quick look between the power of space and time ensures that $t^2 \propto r^3$. However, instead of inverse square, for inverse cube law (arises when general theory of relativity is taken into account) gives us $t \propto r^2$.

Assuming earth to be homogeneous sphere of radius R and density ρ, it is easy to notice that the acceleration due to gravity g varies with the variables as

$$g \propto \rho R \qquad (2)$$

The important point of the above scaling relation is that it is free from the universal gravitational constant G and can be used in other astrophysical object such as moon. In fact, if we assume further that both earth and moon have the same density, then the ratio of acceleration due to earth's gravity is 3 times larger that due to moon. However, a correct factor of 6 can be visualized if one takes into the correct numerical value of the densities of moon and earth.

Simple spring also satisfies a scaling relation between the time-period T and the mass of the spring m as

$$T \propto \sqrt{\frac{m}{k}} \qquad (3)$$

where k is the stiffness constant, The famous Richter scale (1-10) or Marshili scale (1-12) used in the earthquake [9] follows the power law. The probability p for quake releasing energy E due to generation of shock wave is seen to proportional to $E^{-\beta}$, where β is the relevant exponent. It is evident that the above two scales are based on the logarithm of the energy. Even the recession of speed v of the galaxy at a distance R away from the observer was given by Hubble as $v \propto R^{-1}$.

Mathematically, all these power laws [4,5] can be expressed as

$$Y = f(x) = Cx^{\alpha} \qquad (4)$$

If we scale $x \to bx$, we get from the above equation that

$$Y(bx) = f(bx) = Cb^{\alpha}x^{\alpha} = b^{\alpha}f(x) \quad (5)$$

The above equation indicates that if we measure x in meters or inches, the form of Y-x relationship remains unchanged. This invariance of scale change justifies the name of scaling relation. In fact, taking the logarithm of both sides of equation (4), we can easily obtain the exponent α from the slope the straight line with log-log axis. Moreover, if the two functions $f(x)$ and $f(y)$ satisfy the following relation

$$f(x)f(y) = f(xy) \qquad (6)$$

one can show that the function must be $f(x) \sim x^{\lambda}$. It is easy to note from equation (6) that

$$y\frac{f'(y)}{f(y)} = x\frac{f'(x)}{f(x)} = \lambda \qquad (7)$$

Here, λ is a constant because the LHS is a function of y while the RHS is a function of x only. Naturally, the solution of equation (7) can be easily guessed as $f(x) \sim x^{\lambda}$. In non-equilibrium as well as equilibrium statistical mechanics, the scaling relation [6-8] can be of the form

$$Y(x,w) = x^{\alpha}g\left(\frac{w}{x^{\beta}}\right) \qquad (8)$$

Here, the observables Y depends on two parameters x and w. In such a case, one understands the scaling regime $Y \sim x^{\alpha}$ in the range w is sufficiently small so that $g\left(\frac{w}{x^{\beta}}\right) \sim g(0)$, a constant. In critical phenomena, the physical quantity at the critical point scales with a power law with anomalous rational exponent [7].

Sometimes, pure dimensional analysis can help one to deduce the scaling relation. For example, the typical phase velocity v of water waves in shallow water depends on the acceleration due to gravity g, and water depth h. Here, the surface tension and the viscous effects are neglected. A quick straightforward calculation reveals that

$$v = \sqrt{gh}\,f\left(\frac{h}{\lambda}\right) \qquad (9)$$

where λ is the wavelength of the water wave. In the limit, $h \ll \lambda$, $f\left(\frac{h}{\lambda}\right) \to f(0)$, we find in a place that the velocity $v \propto \sqrt{h}$, the well-known scaling relation in fluid dynamics.

Even when x is independent of Y, we can regard that as a power law of exponent being zero. In fact from biology, it has been seen that the maximum height of animals scales with size (L) as L^0. This surprising result [10-13] can be understood physically as follows. The maximum height that an animal can achieve must be proportional to the achievable potential energy divided by its weight. The maximum applied force or the strength of the muscle of animal scale with the characteristic length/ size L of an animal body as L^2. Therefore, the achievable potential energy will scale as L^3 (Force$\times$distance). However, the weight of the animal (assuming the density of body remains constant) is proportional to its volume, L^3. Hence, the height to which an animal can jump turns out to be independent of the size L.

The above scaling behavior can be used in another way. A bone's strength increases as its cross-sectional area while an animal's weight is proportional to its volume, so that to support its own weight an elephant's legs need to be relatively stouter than a dog's. The scaling law is that $w \propto L^{3/2}$, where w is the leg width and L its length.

It is known from biological scaling [13] analysis that the strength of a muscle is proportional to L^2 while the weight of the muscle varies as L^3. Therefore, the ratio of the strength to weight is proportional to $\frac{1}{L}$. This result has a remarkable consequence as follows. For example, an elephant bears the ratio as 1 m^{-1} while a tiny flee 1000 m^{-1}. This indicates the impossibility of jumping of an elephant over a flee. Moreover, a flee can support almost 100 times of its body weight while an elephant can something of $\frac{1}{10}$ of its body weight. This is in accord with the fact that the characteristic dimension of an elephant is 1000 times that of a flee.

Scaling analysis along with dimensional analysis can be used to estimate the maximum height [14] of a tree. Each tree is characterized by two different length scales namely the radius and its length or height. In this case one has to compare the two energy scales – namely gravity energy and the elastic one. We know that the typical potential energy of a tree of mass *m* and height *h is* simply $E_{grav} \sim mgh = \rho g r^2 h^2$ with *r* being the radius of the tree. But the elastic energy E_{elast} stored in a tree can be written in terms of elastic modulus as $\frac{Yr^4}{h}$, where *Y* is the Young's modulus of the material of the tree. Now, for the stability of a tree i.e. no buckling, the condition is $E_{grav} \leq E_{elast}$. Thus, for a given region with the same type of tree, we must have scaling relation with the radius *r* and the height *h* as $r^2 \propto h^3$. Thus, for a given radius *r*, the maximum height of a tree can be estimated as

$$h_{\max} \leq \left(\frac{Y}{\rho g}\right) r^{\frac{2}{3}} \qquad (10)$$

In case of a typical length *L* of an animal, the relation is modified as $r^2 \propto L^3$. Moreover, the mass of an animal scales $M \sim \rho L r^2 = \frac{\rho^2 g}{Y} L^4 \sim L^4$. This eventually gives an important scaling between the mass and length of an animal as $L \sim M^{\frac{1}{4}}$, which is known as Kleiber's law [13]. Hence, the radius scales simply $r \propto L^{\frac{3}{2}} \propto M^{\frac{3}{8}}$. It is interesting to note that this scaling is quite different from the isometric one where $L \sim M^{\frac{1}{3}}, r \sim M^{\frac{1}{3}}$.This scaling can also be applied to the situation of exhausting to animals for climbing a hill. This is related to the metabolic rate of a typical animal. However, the metabolic rate essentially depends on the flow rate of oxygen. Note that the flow rate of O_2 scales as the surface area of the lungs $\sim M^{\frac{3}{4}}$. Now the comparison of the two energy scales indicates that $Mgh \sim M^{\frac{3}{4}}$ or $h \sim M^{-\frac{1}{4}}$. Estimation reveals that it is almost 13 times exhausting for a 1 ton horse than for 30 gm mouse to climb a hill.

The paper is organized as follows. With the brief introduction, we would like to use the scaling analysis for the existence of Lilliput and Giants as depicted in Gullivers' travel book. Finally, in section 3, we point out the order of magnitude of height and life-span of common man in terms of fundamental constants of physics. In section 4, we give our conclusions.

2. Scaling Analysis of Lilliput

To introduce the topic of scaling analysis, let us look at a classical example of the romantic literature, in which Dean Swift, in "The Adventures of Gulliver" describes the imaginary voyages of Lemuel Gulliver to the kingdoms of Lilliput and Brobdingnag. In these two places life was identical to that of normal humans; their geometric dimensions were, however, different. In Liliput, man, houses, dogs, trees were twelve times smaller than in the country of Gulliver, and in Brobdingnag, everything was twelve times taller. The man of Lilliput was a geometric model of Gulliver in a scale 1:12, and that of Brobdingnag a model in a scale of 12:1.

One can come to interesting observations of these two kingdoms through dimensional analysis. Much time before Dean Swift, Galileus already found out that amplified or reduced models of man could not be like we are. The human body is built of columns, stretchers, bones and muscles. The weight of the body that the structure has to support is proportional to its volume, that is, L^3, and the resistance of a bone to compression or of a muscle for fraction, is proportional to L^2.

Let's compare Gulliver with the giant of Brobdingnag, which has all of his linear dimensions twelve times larger.

It is known that a person's food intake capacity is related to his mass (volume). Gulliver was 12 times taller than Lilliputians. Let us assume that the linear dimension of Gulliver be L_G with the volume V_G. Similarly for the Lilliputian, the volume is V_L with L_L be the linear dimension. Then, we can write simply $V_G/V_L=(L_G/L_L)^3=12^3=1728$. Therefore, Gulliver needs the food of 1728 times the amount of food of each as the Lilliputians. This simple problem has a quite good impact in modern days in drug dosage in humans.

The resistance of his legs would be 144 times larger than that of Gulliver, and his weight 1728 times larger. The ratio resistance/weight of the giant would be 12 times less than ours. In order to sustain its own weight, he would have to make an equivalent effort to that we would have to make to carry eleven other men.

Galileus treated this subject very clearly, using arguments that deny the possibility of the existence [6] of giants of normal aspect. If we wanted to have a giant with the same leg/arm proportions of a normal human, we would have to use a stronger and harder material to make the bones, or we would have to admit a lower resistance in comparison to a man of normal stature. On the other hand, if the size of the body would be diminished, the resistance would not diminish in the same proportion. The smaller the body, the greater is its relative resistance. In this way, a very small dog could, probably, carry two or three other small dogs of his size on his back; on the other hand, an elephant could not carry even another elephant of his own size.

In figure 1, we show the schematic picture from Gullivers' book to compare the typical length scales of a normal human being with Lilliput.

Figure 1: A picture from Gullivers' book

Let's analyze the problem of the Lilliputians from the idea of heat loss. The heat that a body loses to the environment goes through the skin, being proportional to the area covered by the skin, that is, L^2, maintaining constant the body temperature and skin characteristics. The food taken in must supply this amount of heat. Therefore the minimum food needs should scale as L^2.

If Gulliver would be happy with a broiler, a bread and a fruit per day, a Lilliputian would need a $(1/12)^2$ smaller food volume. But a broiler, a bread, a fruit when reduced to the scale of his world, would have volumes $(1/12)^3$ smaller. He would, therefore, need twelve broilers, twelve breads and twelve fruits to be as happy as Gulliver. Besides Lilliputians must be hungry enough, famine, restless, active and can become easily water-logged. It is easy to recognize that these properties with many small mammals such as mouse for example. It is interesting to note that there are not many warm-blooded animals smaller than mouse, probably in light of the scale laws discussed above. Notice that in nature, fish, frogs and insects can have much smaller size because of the fact that this body temperature is not higher than their surroundings. As a consequence, large animals by the above scaling law require relatively a good deal of food in compared to smaller creatures to maintain a relatively higher body temperature. In fact, it is indeed very difficult for small animals to gather such an enormous amount of food. Not only that if the foods were collected, that could not be digested over a feasible time. Thus, it is clear that

the agriculture of the Lilliputians could not have

supported a kingdom like described by Gulliver.

We also illustrate the scale factor from the famous

book in Fig.2.

Therefore, in our earth it is thus impossible for Lilliput to survive with giants as compared to them.

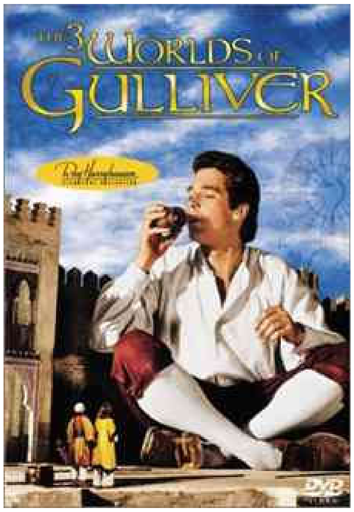

Fig 2. A Picture from the famous book

It is also evident from the above scaling arguments that the heat loss/mass is proportional to L^{-1}. In other words, small animals will lose more heat compared to bigger one and naturally, they will not survive in the polar region at that cold atmosphere. Let us give some numerical estimation on mouse and polar bear. Considering a small mouse of length 5 *cm*, heat loss is of the order of 20 m^{-1}.

However, polar bear of length 2 *m*, the heat loss is 0.5 m^{-1}. Therefore, the ratio of heat loss of mouse to polar bear is 20:0.5=40:1.

From all the above observations, we come to the following conclusions that, although being geometric models of our world, Brobdingnag and Lilliput could never exist in our physical models, since they would not have the necessary physical similarity which is found in natural phenomena. In the case of Brobdingnag, for example, the giant would be able to support his own weight having the stature of humans, only if he would be living on a planet having a gravitational force of (1/12)g.

2.1 Absence of giants of normal size

To argue this, we know that the human body is built of columns, stretchers, bones and muscles. Naturally, the weight [1, 10-12] of the body that the structure has to support is proportional to its volume, L^3. However, the resistance of a bone to compression or of a muscle for fraction is proportional to L^2. Therefore, if we wanted to have a giant with the same leg/arm proportions of a normal human being, we would have two options. Either, we must have to use a stronger and harder material to make the bones, or we would have to admit a lower resistance in comparison to a man of normal stature. If his/her height is proportionally increased, naturally he/she will fall and be crushed completely under his/her own weight. In fact, smaller is the body; greater is its relative strength. On the other hand, if the size of the body is diminished, then, the resistance will not diminish in the same proportion. Therefore, the smaller is the body, the greater its relative resistance. In this way, a very small dog could, probably, carry [2, 10-12] two or three other small dogs of his size on his back. But, an elephant will never be able to carry even another elephant of his own size.

3. Size and life span of a normal man

It is known [10-13] that smaller animals have quick pulse rate and short lives while larger animals have slow pulse rates and long lives. The biological scaling analysis suggests that

$$\begin{aligned} LS &\propto m^{1/4} \propto L^{3/4} \\ HR &\propto m^{-1/4} \propto L^{-3/4} \\ BMR &\propto m^{3/4} \propto L^{9/4} \end{aligned} \qquad (11)$$

where LS, HR and BMR refer to life span, heart rate and Basal metabolic rate respectively.

Below we follow an order of magnitude estimation of the size and life span of normal human being in accord with the model developed by William H. Press [15]. In brief, we attempt here to express the characteristic size L_H and life span t_H in terms of natural fundamental constants such as $e, \hbar, c, G$ etc.

It is important to note that this simple estimation however, cannot distinguish between male and female; moreover, since it is based on the scaling arguments, the accuracy level is unable to distinguish between the size of elephant and human being. More, sophisticated model calculations are welcome to match with the experimental data. The model computation however, is based on three fundamental assumptions [15].

(i) Human being is composed of very complicated molecules.

(ii) For the survival of human race, it is desirable that the atmosphere should not be primordial or cosmological in nature. This excludes eventually the presence of hydrogen and helium in the atmosphere.

(iii) Lastly, it is supposed that the height will be sufficiently large to carry its huge (heavy) brain. The person, however, can stumble or fall, but should not break at all by doing so.

With these three assumptions, let us look into the characteristic sizes of atoms, density and the binding energy. We choose the characteristic size of the atom as Bohr's length $a_0 = \frac{\hbar^2}{m_e e^2} = 0.52 A^\circ$. This can be justified from simple dimensional analysis and uncertainty principle [14, 16]. If we assume that one proton in a cube of 1 Bohr diameter, the relevant density scale turns out to be $\rho_0 = \frac{m_p}{(2a_0)^3} = 1.44 g/cc$. Now, the scale of all molecular binding energy can be measured in terms of the hydrogen binding energy fixed at $E_B = \frac{e^2}{2a_0} = 13.6$ eV.

Now, according to the first assumption, since the involved chemistry of the molecules in human being is complex in nature, we can take the binding energy of these molecules to be a small fraction ($\varepsilon \approx 0.003$) of E_B. This eventually gives an order of magnitude of the environment around the human being as

$$T_{env} = \frac{0.003 \times 13.6}{k_B} eV \qquad (12)$$

This temperature will naturally provide the perfect environment for continuing the internal chemical processes and hence the survival.

Considering the second assumption, we know that the atmosphere of the human race is neither hydrogen or helium nor vacuum. This is possible if the escape velocity from the surface of the earth is greater than the thermal velocity of hydrogen at T_{env}. Therefore, we must have

$$\frac{GM_E}{R_E} \approx \frac{0.003 \times 13.6}{m_p} \quad (13)$$

and again from density consideration,

$$\frac{M_E}{R_E^3} \approx \frac{m_p}{(2a_0)^3} \quad (14)$$

These two simple equations (13) and (14) can be used to estimate the radius R_E and mass M_E of the earth as

$$R_E = \sqrt{\varepsilon}(2a_0)\sqrt{\frac{e^2}{Gm_p^2}}$$
$$M_E = \varepsilon^{3/2} m_p \left(\frac{e^2}{Gm_p^2}\right)^{3/2} \quad (15)$$

The equation (15) also points out the emergence of a dimensionless constant $\left(\frac{e^2}{Gm_p^2}\right)$ formed from the natural fundamental constants.

Now, if the characteristic size of human being is L_H, then we can estimate its order of magnitude from the mass M_H as

$$M_H \approx \rho_0 L_H^3 \quad (16)$$

The characteristic energy scale of the potential energy of the fall of human being is fixed as $M_H L_H g = M_H L_H \left(\frac{GM_E}{R_E^2}\right)$.The typical number of atoms N_H in a human being is of the order $\left(\frac{M_H}{m_p}\right)$.The breaking energy can be fixed at the scale $\varepsilon\left(\frac{e^2}{2a_0}\right)\left(\frac{M_H}{m_p}\right)^{2/3}$, the last factor originates from the two dimensional surface. Now, considering the third and final assumption, we can write an equation

$$M_H L_H \left(\frac{GM_E}{R_E^2}\right) \approx \varepsilon\left(\frac{e^2}{2a_0}\right)\left(\frac{M_H}{m_p}\right)^{2/3} \quad (17)$$

which after simplification, we obtain the size of the human being as

$$L_H \approx \varepsilon^{1/4}(2a_o)\left(\frac{e^2}{Gm_p^2}\right)^{1/4} \quad (18)$$

The estimated height (2.6 cm) of the human being turns out to be 10^2 smaller than the actual data (180-212 cm) because of the assumption (iii). If one equates the excess breaking energy (which is 10^4-10^5 times that used above) to the number of atoms in a protein, one gets a reasonable estimation of the length of the human being.

The life span t_H of a human being can be estimated [15] from the solar radiation in an environment temperature T_{env} in terms of Stefan-Boltzmann constant $\sigma = \frac{\pi^2 k_B^2}{60\hbar^2 c^3}$. The life-span or shelter-seeking time can be set if one equates the total energy of chemical bond in human being to the solar flux radiation on human's surface area L_H^2 multiplied by t_H. Thus, we get the desired equation as

$$\varepsilon\left(\frac{e^2}{2a_0}\right) = \sigma\left(\frac{\varepsilon \times e^2}{2a_0 k_B}\right)^4 L_H^2 \times t_H$$
$$t_H = \left(\frac{k_B^4}{\sigma}\right)\left(\frac{\rho_0}{m_p}\right)\frac{L_H}{\left(\varepsilon\frac{e^2}{2a_0}\right)^3} \quad (19)$$

The estimated value (5×10^4 sec) however does not match with the observed data (2.2×10^9 sec).

4. Conclusions

Even within the cutting edge support from the technology, the scaling arguments presented above are quite important. Sometimes, we design a new large object on the basis of knowledge gathered from the small one, we are warned that the new effects may become a serious issue to consider. One cannot just scale up and down blindly, geometrically; but by scaling in the light of physical reasoning, one can predict some new things about the unknown system. Like any other order of magnitude estimation, they are extremely important and helpful to study any physical system. This can in fact serve as a best guide to a detailed analysis of the physical system.

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