Travel near the speed of light: an instructive exercise in special relativity

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Abstract

An acceleration of *g* can bring a space vehicle to nearly the speed of light in less than a year. In this paper we study one possibility for such relativistic travel, as measured by both travelers and earthbound observers. Udergraduates who encounter special relativity toward the end of their introductory, calculus based physics course should find the paper's mathematics accessible, and be stimulated by its extension of the usual topics. The sections can be either studied as presented, or offered as problems for investigation and solution by the students.

1. Introduction

The fact that the mass of an object increases without limit as its speed approaches c does not of itself prevent the object from attaining a speed arbitrarily close to c. In the following examples we investigate changes in time, mass and distance when traveling at relativistic speeds, acknowledging the technical challenges to actually doing so. We will be particularly interested in comparing earthbound measurements with those of the travelers launched into space. With some guidance, students introduced to special relativity in a calculus based physics course should readily understand this study, intended to challenge and stimulate them [1].

The familiar equation $t = v/g$ reveals—perhaps surprisingly—that a constant acceleration of g (9.80665 m/s^2) takes an object from a speed of 0 to c (299,792,458 m/s) in just 30,570,323 s or 353.8231829 days. The number of days it takes with an acceleration of g to reach various fractions of c, with no relativistic considerations, is shown in column 1 of table 1.

Although we might adopt any acceleration for this non-relativistic example, we use g because it is familiar to us both intellectually and experientially. Moreover, it requires a force not unreasonable to expect a future space vehicle to maintain, given that even today we can generate a much greater acceleration for several minutes after launch. Thus we will assume that g is not only the constant acceleration for the non-relativistic version of column 1, but also the *initial* acceleration for the two relativistic versions tabulated in columns 2 and 3.

Note that we are *not* assuming that a constant acceleration is maintained. Rather we make a different assumption, one that seems to place a more sensible demand upon the mechanism of a space vehicle than that of constant acceleration. We assume a constant delivery of force—as measured by earthbound observers—one which launches the vehicle with an acceleration of g but which later produces a different acceleration as mass or time are relativistically modified. This force continues until a prescribed speed is attained, after which the vehicle can coast. Of course a model with constant force is only one possibility, although it is surely more practical than a model with constant acceleration, which could demand impossible outputs as mass increases. Students are encouraged to invent their own models with different assumptions.

2. Measurements from the earth, assuming external propulsion and relativistic mass increase

The achievement of relativistic speeds seems more likely to come about through a technology which utilizes an external source of propulsion rather than one which uses onboard fuel [2,3]. Even a cargo of matter-antimatter—the most promising current prospect—would be prohibitively heavy. The student might like to research the possibility of solar sails, and calculate the dimensions needed for feasible travel [4]. The prospects for utilizing “dark energy” seem more remote at this time, but may eventually materialize. Whatever the mechanism, since our study assumes an external source of propulsion, we will encounter no decrease in mass due to consumption of fuel. (We note that the assumption of a diminishing supply of onboard fuel leads to very different scenarios and equations from those studied in this paper.)

Time: Assume that the space-time coordinate system used by observers on the earth is an inertial frame of reference. That is, the accelerations due to the earth’s rotation and all other circular motions are negligible and can be disregarded.

In our model, if m_0 is the rest mass of the vehicle, a force of m_0g is continuously applied until acceleration is no longer needed. As the speed v increases, the mass increases by a factor of $1\big/\sqrt{1-v^2/c^2}$, as measured by the earthbound observers [5]. Since force is the derivative of the relativistic momentum, we have:

$$\frac{d(m_0 v / \sqrt{1-v^2/c^2})}{dt} = m_0 g, \qquad (1)$$

and after integration we obtain

$$\frac{v}{\sqrt{1-v^2/c^2}} = gt \qquad \text{or}$$

$$t = \frac{v}{g\sqrt{1-v^2/c^2}}. \qquad (2)$$

If we let $u = v/c$ then, with $0 \le u \le 1$,

$$t = \left(\frac{c}{g}\right)\frac{u}{\sqrt{1-u^2}} \qquad (3)$$

The number of days needed to reach various values of u is shown in column 2 of table 1. Compare these with those in column 1, where $t = v/g$. Since

$\dfrac{u}{\sqrt{1-u^2}} > u$ for $0 \le u \le 1$, the values in column 2 are always greater than those in column 1, with a ratio approaching ∞ as u approaches 1. Thus the vehicle can never achieve the speed of light, but can get arbitrarily close to it. As its mass increases without limit, it takes longer and longer to increase its speed by the slightest increment.

Let us remind ourselves that equation (3) and the values in column 2 specify how observers on earth measure the vehicle's motion. They are not the measurements of the travelers themselves, which we will consider below.

Distance: While the time to attain any particular speed is understandably greater when mass is subject to relativistic increase, is the same true for the distance traveled? If we call the target speed v_t, then the non-relativistic distance traveled from launch until v_t is attained is the familiar

$$s_1 = \int_0^{v_t/g} gt\,dt = v_t^2/2g, \text{ or, if } u_t = v_t/c,$$

$$s_1 = \frac{c^2 u_t^2}{2g} \qquad (4)$$

The distance traveled with relativistic mass increase is

$$s_2 = \int_0^{v_t} v\,dt = gc\int_0^{t_t} \frac{t}{\sqrt{c^2+(gt)^2}}\,dt\,.$$

The upper limit of the integral, t_t, signifies $\dfrac{v_t}{g\sqrt{1-v_t^2/c^2}}$, obtained directly from (2), while the equivalence $v = \dfrac{gtc}{\sqrt{c^2+(gt)^2}}$ is an algebraic consequence of (2). After integration

$$s_2 = \frac{c^2}{g}\left(\frac{c}{\sqrt{c^2-v_t^2}} - 1\right),$$

and with $u_t = v_t/c$ this becomes

$$s_2 = \frac{c^2}{g}\left(\frac{1}{\sqrt{1-u_t^2}} - 1\right). \qquad (5)$$

It is easily shown that $s_2 > s_1$ for all $t > 0$. Inspection of (5) reveals that as $u_t \to 1$, $s_2 \to \infty$. This is to be expected, since for the earthbound observers the vehicle will be traveling at a speed near c for an amount of time that according to (3) increases without limit.

3. Measurements by the travelers

Time: How will the situation change if we take into account time dilation, whose effects the travelers' measurements will reflect?

Let T be the elapsed time since launch according to the travelers' onboard clock. T is a function of t, the time as measured by the observers on earth. This function depends upon the vehicle's entire history of motion, not just upon what is occurring at the moment. By contrast, history will prove irrelevant to the travelers' measurement of instantaneous speed, V. That is, we will see that regardless of the vehicle's history, the speed measured by the observers and the speed measured by the travelers are always equal. This of course is an ambiguous statement, but before we clarify what it means, let us show the simple derivation. Let s represent the distance traveled since launch as measured by the observers, and S the distance as measured by the travelers. For any brief interval, time dilation yields:

$$dT = \sqrt{1 - v^2/c^2}\,dt$$

But since the travelers measure the space outside them to be moving past their window at a velocity of v, length contraction yields [6]:

$dS = \sqrt{1 - v^2/c^2}\,ds$, and thus the instantaneous speed measured by the travelers is

$$V = \frac{dS}{dT} = \frac{\sqrt{1 - v^2/c^2}}{\sqrt{1 - v^2/c^2}}\frac{ds}{dt} = \frac{ds}{dt} = v \qquad (6)$$

Time dilation apparently compensates for length contraction, but again, what exactly does (6) assert? We can interpret it with a simple example. Suppose that both the observers and the travelers, with much more knowledge of the solar system and galaxy than we have, specify event *A* to be the vehicle's passage past a particular comet in the Oort cloud (perhaps even specifying that the vehicle's vector of motion is perpendicular to a line from the vehicle to the center of the comet.) Then the speed v which the observers measure at event *A* will equal the speed V which the travelers seeon their speedometer at their own event *A*. That is, the observers and the travelers will agree on the vehicle's speed at the moment event A occurs. There is no confusion as to whether these two equal speeds are "happening at the same time," which is not only confusing but also meaningless. We *can*, however, say something definite about time: as we will show below, if the travelers measure event *A* happening at time *T,* and the observers measure it happening at time t, then $T<t$. That is, the travelers measure its occurrence at an earlier time on their clock than the observers do on theirs.

We will use equation (6) to obtain T as a function of t for the specific motion portrayed in (2), but now as measured by the travelers. Without noticing anything strange, since they are not comparing their measurements to those of the observers, the travelers are subject to two relativistic effects: a dilation of time, and a contraction of the space outside as it flies by their vehicle.

Solving for v in (2) gives $v = \dfrac{tgc}{\sqrt{c^2 + (gt)^2}}$, and substituting this into

$dT = \sqrt{1 - v^2/c^2}\,dt$ yields

$$dT = \sqrt{1 - \frac{t^2 g^2 c^2}{c^2(c^2 + (gt)^2)}}\,dt = \sqrt{\frac{c^2 + t^2 g^2 - t^2 g^2}{c^2 + (gt)^2}}\,dt = \frac{c}{\sqrt{c^2 + (gt)^2}}\,dt.$$

Integrating and solving for the constant of integration gives:

$$T = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right), \text{ or} \qquad (7)$$

$$t = \frac{c}{g}\sinh\left(\frac{gT}{c}\right) \qquad (8)$$

Replacing t in (2) with t from (8) produces the equation

$\dfrac{c}{g}\sinh\left(\dfrac{gT}{c}\right) = \dfrac{v}{g\sqrt{1 - v^2/c^2}}$, which yields

$$T = \frac{c}{g}\ln\left(\frac{v+c}{\sqrt{c^2 - v^2}}\right) \qquad (9)$$

Letting $u = v/c$,

$$T = \frac{c}{g}\ln\left(\frac{u+1}{\sqrt{1-u^2}}\right). \qquad (10)$$

This is still less than satisfactory, for T on the left side of (10) represents the travelers' measurement of time, while u on the right side is the fraction of c as measured by the observers. However, since for any event A the speed V equals the speed v, u could just as well be equated with V/c, and (10) would be exclusively in terms of the travelers' measurements. The value of T for various speeds u is given in column *3*, table 1. The entire column is in italics to indicate that it represents the measurements of the travelers. $T<t$ for all u, an inequality which grows with increasing u.

A note on measurement: Before calculating the distance as measured by the travelers, let us briefly consider the term *measurement* itself. This is a term used repetitively, not only in this paper but in all physical discussions. While the term conjures up—and indeed almost always signifies—an observation using some calibrated instrument or apparatus, it might be useful to explore the concept somewhat. Consider this line of reasoning. Between the above event A and a later event B, such as the encounter with a second comet, there is an interval of time and distance measured differently by the observers and the travelers. (To be clear, here the word "interval" is *not* being used to designate the invariant quantity $[(c\Delta t)^2 - (\Delta x)^2]^{1/2}$.) Let us say that during this interval the traveler (we will assume female), using her onboard clock and her right index finger placed on her left wrist, calculates that during this interval her pulse has held steady at 60 beats per minute. An observer on earth, with electromagnetic access to the traveler's pulse but using his own clock, will calculate that the traveler's pulse began the interval at, say, 40 beats per minute, and after the vehicle's acceleration through the interval, finished at 35 beats per minute. Between events A and B the total number of beats had to be the same for the traveler and the observer, but the onboard clock measured fewer minutes than the clock on earth. To the observer, the traveler's heart seemed to beat increasingly slowly, while the traveler herself felt quite normal.

Any onboard activity would give a similar result. For example, the observer would measure a male traveler's facial hair to be growing slowly, although by the onboard clock the traveler himself would shave according to his usual schedule. To the observer the traveler's speech would get slower and slower as the vehicle accelerated, as would the appearance of emotionson his face, or even—if the capacity to register this had been discovered—the succession of his thoughts. None of these activities would appear or feel unusual to the traveler. To him the onboard clock itself would advance normally, although the observer would perceive it as slowing down compared with his own clock.

The statement that *time* itself slows down can be misleading, although it does give a sense that something more substantial than a mere appearance is actually happening. One must remember that in this study it ultimately refers to a measurement by a clock in an inertial frame, of activities that are accelerating relative to that inertial frame. (A more complete understanding is beyond the scope of this paper, but a comprehensive and comprehensible reference is [7].)

Distance: Taking a closer look at columns 2 and *3*, is there some contradiction here? If the travelers take only 936.45 days to reach $u = .99$ (column *3*), and the observers measure it as 2483.10 days (column 2), how could the speeds always be the same at every event along the route, such as event *A*? Wouldn't the travelers measure an average speed much faster than that measured by the observers? We can resolve this difficulty by considering the contraction of exterior distance as measured by the travelers.

How far do the travelers measure that they have gone since the moment of launch until the moment they attain a speed of v_t? Call this distance s_3.

$s_3 = \int_0^{t} V\,dT$. Using (9), integrating, and replacing v_t/c by u_t yields

$$s_3 = \left(\frac{c^2}{2g}\right)\ln\left(\frac{1}{1-u_t^2}\right). \qquad (11)$$

Substitution of various values of u into (5) and (11) indicates that for $0 < u < 1$, the travelers' distance will be shorter than the distance observed from earth. However, we can be more convincing than this. Consider two specific events, B followed closely by C. The observers and travelers see the same speed at B and the same speed at C, and thus the same difference du_t. The observers measure the distance between these events as ds_2 while the travelers measure it as ds_3. Consider the ratio

$$\frac{ds_3}{ds_2} = \frac{ds_3/du_t}{ds_2/du_t} = \left(\frac{d\left(\frac{c^2}{2g}\ln\left(\frac{1}{1-u_t^2}\right)\right)\Big/du_t}{d\left(\frac{c^2}{g}\left(\frac{1}{\sqrt{1-u_t^2}}-1\right)\right)\Big/du_t}\right) = \frac{u_t/\left(1-u_t^2\right)}{u_t/\left(1-u_t^2\right)^{3/2}} = \sqrt{1-u_t^2}\,. \qquad (12)$$

Since $ds_3 < ds_2$ for all such intervals, integrating to find the distance between launch and any later event will always result in $s_3 < s_2$. Comparing (11) with (5), we find that at u = .5, s_3/s_2 = .93 and at u = .99, s_3/s_2 = .32. In fact, by applying L'Hôpital's rule to the ratio s_3/s_2 , we can show that $s_3/s_2 \to 0$ as $u \to 1$. To sum up, although the time measured is less for the travelers, so is the distance. Thus the equality of speeds shown in (6) is compatible with the briefer travel times of column *3*.

Although we will not consider it here, we might note that once the travelers are moving at a speed near *c* the passage of time is so slow by their measurement that within their lifetime they can reach distant locations in the galaxy or even beyond [8,9].

While the assumption of a constant delivery of force may be too simple and even unrealistic, it generates an interesting question: how does the vehicle's power output—the delivery of energy per unit of time—depend upon time as time itself dilates? That is, how do the travelers measure the delivery of force? We leave this for the motivated student to research.

4. Data

Table 1. Exterior propulsion. Time to reach *u=v/c*, or *V/c* to nearest hundredthof a day. In columns 1 and 2, *t* is time for observers; in column *3*, *T* is time for travelers. c = 299,792,458 m/s, g=9.80665 m/s^2

U=v/c	**1**	**2**	**3**
.9999	353.79	25017.19	1752.03
.99	350.28	2483.10	936.45
.98	346.75	1742.47	812.93
.95	336.13	1076.48	648.13

.90	318.44	730.55	520.91
.80	283.06	471.76	388.71
.70	247.68	346.82	306.87
.60	212.29	265.37	245.25
.50	176.91	204.28	194.36
.40	141.53	154.42	149.90
.30	105.15	111.27	109.52
.20	70.76	72.56	71.73
.10	35.38	35.56	35.50

1:No relativistic effects; $t = v/g$.

2:Relativistic mass increase, no time dilation; as observed from earth; $t = (c/g)\left(u/\sqrt{1-u^2}\right)$.

3: Relativistic mass increase, time dilation; as experienced by the travelers;

$$T = (c/g)\ln\left((u+1)/\sqrt{1-u^2}\right).$$

5. Conclusion: turning student questions into research

Let us look at three examples of questions which open the way to stimulating research. (Like many other instructors, I've finally learned to resist giving too complete an answer.)

A thoughtful student will inevitably reason like this: since the travelers see the earth accelerating away from them, why not reverse the calculations? Now the traveler measures the observer's clock to be going slow. What happens when the vehicle returns to earth 35.28 35.56 35.50and the two clocks are compared side by side? How can they both be slow? Isn't there some contradiction? No, in fact, there isn't. The earth is assumed to carry a fixed, inertial frame of reference, while the vehicle carries an accelerating frame, not only because of its acceleration of *g*, but also because of the further acceleration it requires to turn around and return to earth. Our calculations apply to inertial frames only, and these accelerations would affect the clock on the vehicle but not the clock on earth. The situation is not symmetric. Research the "twin paradox."

Another student might point out the imprecision in our calculations: the effects of general relativity, which we have not included in this study, would surely modify our results. True, but the effects of accelerating at *g* are indistinguishable from those of the earth's gravitational field, which we know from experiment has a negligible effect on mass and time [10]. Our calculations are accurate to several decimal places beyond the hundredths to which we've rounded off.

A third questioner, perhaps a down-to-earth engineering student, might dismiss relativistic travel as mere fantasy, theoretically impossible, beyond any feasible technology. This is the stuff of pop culture, why bother with it? Perhaps so, but an overly conservative attitude—however well informed—can be too restrictive. At the end of the nineteenth century many scientists believed that physics had reached its culmination. Probably the most celebrated comment to this effect was A.A.Michelson's of 1894 [11]: "The more important fundamental laws and facts of physical science have all been discovered, and these are so firmly established that the possibility of their ever being supplanted in consequence of new discoveries is exceedingly remote." Perhaps every era, satisfied with its accomplishments, believes it has made the ultimate discoveries [12]. This attitude is somewhat tempered today, for recent discoveries—the acceleration of the universe's expansion, or the finding of the Higgs boson in 2012—have clearly posed new fundamental questions. Without apology we can speculate sensibly about the future and make tentative assumptions such as those in this paper.

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