Explanation of reflection in Paraxial ray optics by matrix method

Sk. Kalimuddin(1) and A.Ghorai(2)

Department of Physics; Maulana Azad College
8, Rafi Ahmed Kidwai Road, Kolkata – 700013, India
e-mail :(1) premkhan444@rediffmail.com, (2) amitavaghorai61@gmail.com

(*Submitted*: 21-03-2016)

Abstract

Matrix method in paraxial ray optics is essential for methodical understanding in graduate physics students. However, there is almost no text book which gives a complete description about the matrix method of paraxial ray optics, in particular, the mechanism of reflection. So in this article the matrix method of reflection from plane and curved surfaces will be dealt with in details and for completeness refraction in optical systems will also be explained.

1. Introduction

Paraxial ray optics is essential for understanding of geometrical optics especially for an optical system consisting of several reflecting and refracting surfaces. Also it is an essential part of undergraduate physics syllabus. However there is almost no standard text book [1–5] which gives a complete description about the matrix method of paraxial ray optics, in particular, the mechanism of reflection although in two recent books on optics [6–7] an introductory outline have been made. So in this article the matrix method of reflection from plane and curved surfaces will be explained in detail and for completeness refraction at plane surface will also be considered.

2. Basic ideas

A combination of reflecting or refracting surfaces or a combination of thin lenses or a thick lens produces an optical system and for lenses it is called lens system. We take x axis as the axis of the optical system and configure this system with small aperture such that only paraxial rays enter the system. These paraxial rays are very close to the principal axis (x axis) and make very small angle with it. For this we take cylindrical symmetry such that a set of paraxial rays lie in a plane making same small angle of inclination.

3. Coordinate of a ray

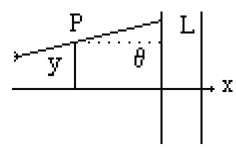

Figure 1

We consider an optical system L and a paraxial ray PL as shown in figure 1. The coordinates of point P can be specified by the distance from the x axis and the inclination of the ray PL with principal axis (x axis) measured in counter clockwise direction (for measurement in the clockwise direction the angle will be negative). The slope of PL is $\tan\theta \to \theta$ and it will vary with medium. So we choose a new coordinate $\lambda = \mu \tan\theta \to \mu\theta$. Thus the coordinates of the paraxial ray at P will be (λ, y), where y is the distance of point P from x axis or principal axis and the refractive index of the medium is μ. Of course, the coordinate at L will differ inside and outside the optical system because of refractive index.

4. Translation matrix

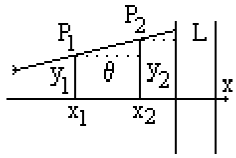

Figure 2

We take a paraxial ray P_1P_2 specified by inclination and distance from principal axis (x axis). The coordinates of P_1 and P_2 will be respectively (λ_1, y_1) and (λ_2, y_2). From geometry shown in figure 2 the slope of paraxial ray P_1P_2 is same and so for $x_2 - x_1 = x$ and for small angles for paraxial rays we get

$$\tan\theta_2 = \tan\theta_1 \Rightarrow \mu\theta_2 = \mu\theta_1 \Rightarrow \lambda_2 = \lambda_1$$

Also

$$y_2 = y_1 + (x_2 - x_1)\tan\theta \to y_1 + x\theta_1 = y_1 + x\lambda_1 / \mu$$

Hence in matrix notation

$$\begin{bmatrix} \lambda_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x/\mu & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix} = T\begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$$

Here $\begin{bmatrix} \lambda_2 \\ y_2 \end{bmatrix}$ and $\begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$ are two column matrices defined as the coordinate of the paraxial ray P_1P_2 at points P_1 and P_2 respectively. The 2×2 square matrix $T = \begin{bmatrix} 1 & 0 \\ x/\mu & 1 \end{bmatrix}$ is called **translation matrix**. Here x is the translation along x axis of point P_1 to point P_2 and the refractive index of the medium is μ. It should be noted that determinant of translation matrix is unity, i.e. $\det T = |T| = 1$. We can also write the matrix equation as $[2] = T[1]$.

5. Reflection at a plane surface

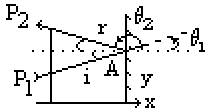

Figure 3

We take an incident paraxial ray P_1A which after reflection at A by a plane mirror produces a reflected ray AP_2 as shown in figure 3. From laws of reflection we know the angle of incidence is equal to the angle of reflection. Also at the point of incidence $P_1 \to A$ and $P_2 \to A$. Thus the angles subtended with dotted line parallel to x axis by incident and reflected ray will be $\theta_1 = i$ and $\theta_2 = \pi - r$ (or $\theta_2 = -r$). The negative sign is due to the fact that angle of incidence and angle of reflection are measured in the opposite direction from the x axis or principal axis (angle in counter clockwise direction measured from the dotted line is positive while in clockwise direction is negative). Thus $i = r$ gives

$$\mu\tan i = \mu\tan r \Rightarrow \mu\tan\theta_1 = \mu\tan(\pi - \theta_2)$$

Or, $\lambda_1 = -\lambda_2$

Also $y_2 = y_1 = y$

Hence in matrix notation

$$\begin{bmatrix} \lambda_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix} = R_F\begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$$

Here $\begin{bmatrix} \lambda_2 \\ y_2 \end{bmatrix}$ and $\begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$ are two column matrices defined as the coordinate of the paraxial ray P_1P_2 at points P_1 and P_2 respectively which are very close to point A on the reflecting plane surface. The 2×2 square matrix $R_F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **reflection matrix** for plane surface. It should be noted that determinant of reflection matrix is negative unity, i.e. $\det R_F = |R_F| = -1$. We can also write the matrix equation as $[2] = R_F[1]$. This is the essential difference between reflection and refraction.

6.Reflection at a concave surface

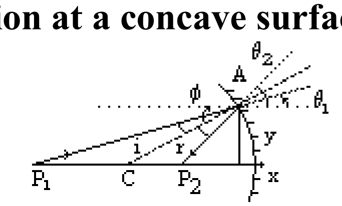

Figure 4

In case of reflection at a concave spherical surface we take an incident paraxial ray P_1A which after reflection at A by a spherical mirror produces a reflected ray AP_2 (figure 4). Also at the point of incidence $P_1 \rightarrow A$ and $P_2 \rightarrow A$. In this case we have to measure angles from principal axis for which a horizontal line is drawn at A. Thus the angles subtended with dotted line parallel to x axis by incident and reflected rays will be $i = \phi - \theta_1$ and $r = \theta_2 - \phi$ respectively.Here ϕ is the angle subtended by AC, the radius of curvature with positive direction of x axis or principal axis. From laws of reflection we know angle of incidence is equal to angle of reflection or

$$\tan i = \tan r \Rightarrow i = r \Rightarrow \mu(\phi - \theta_1) = \mu(\theta_2 - \phi)$$

Or, $\mu\theta_2 = 2\mu\phi - \mu\theta_1$

Or, $\lambda_2 = -\lambda_1 + 2\mu y_1 / R$

Here the radius of curvature of the spherical mirror is R and so $\phi = y_1 / R$.

Also $y_2 = y_1 = y$

Hence in matrix notation

$$\begin{bmatrix} \lambda_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 2\mu/R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix} = R_F \begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$$

The 2×2 square matrix $R_F = \begin{bmatrix} -1 & 2\mu/R \\ 0 & 1 \end{bmatrix}$ is called **reflection matrix** for concave spherical surface. It should be noted that determinant of reflection matrix is negative unity i.e. $\det R_F = |R_F| = -1$. For $R \rightarrow \infty$ we have plane reflecting surface and $R_F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. We can also write the matrix equation as $[2] = R_F[1]$.

7. Reflection at a convex surface

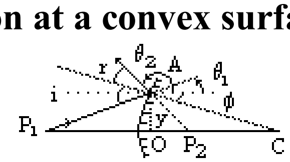

Figure 5

In case of reflection at a convex spherical surface we take an incident paraxial ray P_1A which after reflection at A by the mirror produces a reflected ray AP_2. Also at the point of incidence $P_1 \rightarrow A$ and $P_2 \rightarrow A$. Here we have to measure angles from principal axis for which a horizontal line is drawn at A. Thus from figure 5 the angles subtended with dotted line parallel to x axis by incident and reflected ray will be respectively $i = \theta_1 + (-\phi)$ (ϕ is measured from the dotted line in clockwise direction and so negative) or $i = \theta_1 + (\pi - \phi)$ and $r = (-\theta_2) - (-\phi)$ (θ_2 is negative because it is measured in the clockwise direction)or $r = (\pi - \theta_2) - (\pi - \phi)$. Hence from laws of reflection we know angle of incidence is equal to angle of reflection or

$$\tan i = \tan r \Rightarrow \tan(\theta_1 - \phi) = \tan(-\theta_2 + \phi)$$

[or, $\tan i = \tan r \Rightarrow \tan(\pi + \theta_1 - \phi) = \tan(-\theta_2 + \phi)$]

Or, $\theta_1 - \phi = -\theta_2 + \phi$

Or, $\mu\theta_2 = -\mu\theta_1 + 2\mu\phi \Rightarrow \lambda_2 = -\lambda_1 + 2\mu y_1 / R$

If the radius of curvature of the spherical mirror be R then $\phi = y_1 / R$.

Also $y_2 = y_1 = y$

Hence in matrix notation

$$\begin{bmatrix} \lambda_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 2\mu/R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix} = R_F \begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$$

The 2×2 square matrix $R_F = \begin{bmatrix} -1 & 2\mu/R \\ 0 & 1 \end{bmatrix}$ is called **reflection matrix** for convex spherical surface. It should be noted that determinant of reflection matrix is negative unity i.e. $\det R_F = |R_F| = -1$. For $R \rightarrow \infty$ we have plane reflecting surface and $R_F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. We can also write the matrix equation as $[2] = R_F[1]$.

8. Object image relation for spherical mirror

It is now easy to derive the system matrix for an object at a distance u from the surface of the mirror and the corresponding image distance υ in case of reflection at the spherical curved surface as follows:

Object → Translation → Reflection → Translation → Image

$$[2] = T_u[1]$$
$$[3] = R_F[2]$$
$$[4] = T_\upsilon[3]$$

Or, $[4] = T_\upsilon R_F[2] = T_\upsilon R_F T_u[1] = S[1]$

Here $S = \begin{bmatrix} -1+2u/R & 2\mu/R \\ -\upsilon/\mu + 2u\upsilon/\mu R + u/\mu & 2\upsilon/R+1 \end{bmatrix}$

$$\begin{bmatrix} \lambda_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1+2u/R & 2\mu/R \\ -\upsilon/\mu + 2u\upsilon/\mu R + u/\mu & 2\upsilon/R+1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$$

Thus $\lambda_4 = (-1+\frac{2u}{R})\lambda_1 + \frac{2\mu}{R} y_1$

$$y_4 = (\frac{u}{\mu} + \frac{2u\upsilon}{\mu R} - \frac{\upsilon}{\mu})\lambda_1 + (1+\frac{2\upsilon}{R}) y_1$$

Equating the coefficient of λ_1 to zero (or for axial point object and image $y_4 = y_1 = 0$) we get

$$\frac{1}{\upsilon} - \frac{1}{u} = \frac{2}{r} = \frac{1}{f} = P$$

9. Refraction at a plane surface

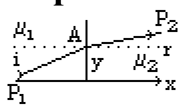

Figure 6

We take an incident paraxial ray P_1A which after refraction at A by aplane interface separating two medium 1 and 2 with refractive indices respectively μ_1 and μ_2, produces a refracted ray AP_2. From laws of refraction we know Snell's law is valid. Also at the point of incidence $P_1 \rightarrow A$ and $P_2 \rightarrow A$. Thus the angles subtended with dotted line parallel to x axis by incident and reflected ray will be $\theta_1 = i$ and $\theta_2 = r$. Thus

$$\mu_1 \sin i = \mu_2 \sin r \Rightarrow \mu_1 i = \mu_2 r \Rightarrow \lambda_2 = \lambda_1$$

Also $y_2 = y_1 = y$

Hence in matrix notation

$$\begin{bmatrix} \lambda_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$$

Here $\begin{bmatrix} \lambda_2 \\ y_2 \end{bmatrix}$ and $\begin{bmatrix} \lambda_1 \\ y_1 \end{bmatrix}$ are two column matrices defined as the coordinate of the paraxial ray P_1P_2 at points P_1 and P_2 respectively. The 2×2 square matrix $R_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **refraction matrix** for plane surface. It should be noted that determinant of refraction matrix is unity, i.e. $\det R_R = |R_R| = 1$. This gives the difference between reflection and refraction as stated earlier. We can also write the matrix equation as $[2] = R_R[1]$.

10. Conclusions

It is now a simple task to derive the refraction matrix for refraction at spherical surface, the object – image relation, the system matrix for lens and lens system, so on. All these are mentioned in the standard text books mentioned above.

Thisformalism clearly depicts the insight concept of matrix method of paraxial ray optics based on the undergraduate theoretical knowledge. Emphasis is given so that undergraduate students can think of new and novel ideas to increase their skill to create new and novel problems for proper understanding.

11. Acknowlegements

The authors gratefully acknowledge the DBT Star College fund for this theoretical project work.

12. References :

[1] Ajoy Ghatak, *Optics,* 4th edition,(Tata McGraw-Hill Pub. New Delhi, 2008).

[2] F. A. Jenkins and H. E. White, *Fundamentals of optics,* 4th ed. (McGraw Hill, Inc, 1976).

[3] R. S. Longhurst, *Geometrical and physical optics*, *(*Orient Longman, Hyderabad, 1973).

[4] Eugene Hecht, *Optics*, 4th ed. Addison Wesley, 2002.
[5] M. Born and E. Wolf, *Principles of Optics*, 5th ed. Pergamon Press, 1975.
[6] F. A. Pedrotti, L. M. Pedrotti and L. S. Pedrottii, *Introduction to optics*, 3rd ed. Pearson, 2014.
[7] S. Prakash, *Optics*, 4th ed. Pragati Prakashan, 2015.