Teaching Special Relativity with Real Rotation Matrix

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Abstract

This brief note has pedagogical motivation and we want to emphasize that an introduction of Loedel representation of space-time, is very useful for high school students to understand the special relativity geometrically. We summarize a series of lectures given at the high school "Torquato Tasso" of Salerno by the author of this paper and by his tutor teacher Rachele Lanzillotti. The aim of this short note is to stimulate curiosity and it wants to be a starting point to explore this topic.

1. Introduction

It is well know that Minkowski introduced the concept of space-time continuum and a geometrical representation of Lorentz trasformations is usually studied using his diagrams. In this way it is graphically possible to understand relativistic phenomena even without using mathematical equations [1]. In this approach two inertial observers are represented by an orthogonal system and by a system with oblique axes. Moreover the Minkowski diagram uses different scales for the orthogonal and the non-orthogonal axes and therefore it is necessary to calculate a scale factor to convert the units of a frame into the units of the other one moving with a constant velocity. All this is often difficult for students. The main attraction of the Loedel Space time diagram is, instead, that it treats the reference frame and the first moving frame symmetrically and hence they have identical scales in geometric units[2],[3],[4]. Furthermore the students can use standard matrix of rotation without recourse to imaginary axes and angles. This approach became known and were studied in more detail by several authors [5],[6],[7],[8],[9] but it is completely ignored in most textbooks. These underused diagrams are fundamental to understanding the real relativistic physical consequences that should not be confused with the change of units that is only a product of the choice of the graphical representation. We will use a simple way to show the invariance of units of length and time for the two Loedel observers.

2. Galileian space-time geometry

In classical physics the link between the Cartesian coordinates of an object in two inertial reference frames in uniform rectilinear motion relative to one another with velocity v, is given by the Galileian transformations

$$\begin{cases} x' = x - vt \\ t' = t \end{cases} \qquad (1)$$

From a geometric point of view, they convert coordinates from Cartesian system to non-orthogonal axes. Indeed the x' axis coincides with the x axis but the t' axis is inclined, relative to the t axis, of $\alpha = arctgv$. Indeed the points of this axis are those that have null abscissa and therefore are those that are located on the straight line $x - vt = 0$. Furthermore one can easily observe that the point $A'(1{,}0)$ has still coordinates (1,0) on the (x, t) plane. Instead $B'(0{,}1)$ has coordinates $(v, 1)$ on the (x, t) plane, that is, in this reference, $OB' = \sqrt{1+v^2} > 1$. So the unit of measure of t' axis is larger than that of the t axis.

3. Relativistic space-time geometry

In special relativity, instead, the link between coordinates is governed by Lorentz transformations

$$\begin{cases} x' = \frac{x-\beta\tau}{\sqrt{1-\beta^2}} \\ \tau' = \frac{\tau-\beta x}{\sqrt{1-\beta^2}} \end{cases} \qquad (2)$$

and its inverse formulas

$$\begin{cases} x = \frac{x'+\beta\tau'}{\sqrt{1-\beta^2}} \\ \tau = \frac{\tau'+\beta x'}{\sqrt{1-\beta^2}} \end{cases} \qquad (3)$$

where $\tau = ct$ e $\beta = v/c$.

In this case the situation is further complicated because, with Lorentz transformations, both axes are inclined by the same angle α = arctgβ. Indeed the x' axis ($\tau' = 0$) is the straight line $\tau = \beta x$ and the τ' axis ($x' = 0$) is $\tau = x/\beta$. Moreover the points $A'(1{,}0)$ and $B'(0{,}1)$ have, on the (x, τ) plane, coordinates respectively $\left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}\right)$ and $\left(\frac{\beta}{\sqrt{1-\beta^2}}, \frac{1}{\sqrt{1-\beta^2}}\right)$. In both cases the unit vectors of the primed axes become long, in unprimed reference, $\sqrt{\frac{1+\beta^2}{1-\beta^2}} > 1$. It is necessary to note that this representation may confuse students because there seems to be a privileged observer, assigning to him a system of orthogonal axes. To eliminate this apparent asymmetry, it is known that it is enough to introduce an imaginary time coordinate $i\tau$ transforming Lorentz relations in orthogonal transformations with a rotation angle $\alpha = arctgi\beta$. In this abstract case the problem is that it is not possible any graphic representation and, therefore, is not useful from a didactic point of view.

4. Real rotation matrix

From the second relation of (3) we have

$$ct' = ct\sqrt{1-\frac{v^2}{c^2}} - \frac{v}{c}x'$$

and substituting in the first of (3) we get

$$x = \left(\frac{v}{c}\right)ct + \sqrt{1-\frac{v^2}{c^2}}\,x'$$

If we pose

$$\frac{v}{c} = sin\alpha$$

we can write

$$\begin{cases}\tau' = \tau cos\alpha - x' sin\alpha \\ x = \tau sin\alpha + x' cos\alpha\end{cases} \qquad (4)$$

obtaining the standard clockwise rotation of a real angle. This was observed for the first time by Enrique Loedel Palumbo and it is an excellent tool for teaching special relativity. The peculiarity of this rotation is the mix between the axes of the two systems: indeed it is not a rotation of the (ct, x) frame with respect to the (ct', x') frame, but the rotation of (ct', x) with respect to (ct, x'). The main benefit of this approach is that unlike in Minkowski diagrams, the scales of both axes of both frames are identical and we do not have apparent preference of one of the inertial frames.

5.Invariance of unit vectors for Loedel observers

If we have, in primed frame, $A'(1,0)$, then we obtain

$$\begin{cases}0 = \tau cos\alpha - sin\alpha \\ x = \tau sin\alpha + cos\alpha\end{cases}$$

getting

$$\begin{cases}\tau = tg\alpha \\ x = \dfrac{sin^2\alpha}{cos\alpha} + cos\alpha = \dfrac{1}{cos\alpha}\end{cases}$$

Therefore from Pythagorean Theorem

$$OA' = \sqrt{\frac{1}{cos^2\alpha} - tg^2\alpha} = \sqrt{\frac{1-sin^2\alpha}{cos^2\alpha}} = 1$$

With similar reasoning, if we have $B'(0,1)$ we obtain

$$\begin{cases}1 = \tau cos\alpha \\ x = \tau sin\alpha\end{cases}$$

and

$$\begin{cases}\tau = \dfrac{1}{cos\alpha} \\ x = tg\alpha\end{cases}$$

getting, also on the (x, τ) plane, $OB' = 1$.

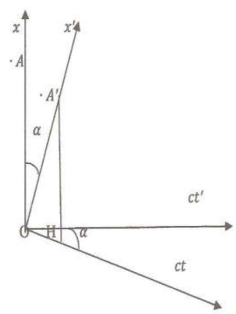

Fig.1: Loedel observers

6.Conclusions

Special relativity theory makes several predictions, some of which seem counter intuitive. The author of this paper has always tried to inspire students so that they could themselves infer such phenomena as time dilation and lenght contraction and requesting the explanation of some " relativistic paradoxes". For example the Ladder Paradox is a famous thought experiment that shows why you can fit objects into spaces that are too small for them. Students generally have great difficulty and I have verified that they are able to find the right solution, in much more simple manner, through Loedel approach that is often neglected in standard textbooks.

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