Studies on Newtonian chirp from inspiraling compact binaries

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Abstract

In this article we study the gravitational waves produced from inspiral phase of compact binaries in circular orbit using the Newtonian chirp of leading order post-Newtonian(PN) approximation. Expressions governing the Newtonian chirp are useful for obtaining the inspiral waveform, frequency variation plot, merging time value and to plot the amplitude spectral density. We start with the discussion of Newtonian chirp, where we assume the compact binaries to be located at galactic center to obtain the waveform and frequency variation plots. For the singular value of peak frequency seen at end of inspiral phase in Newtonian chirp, the concept of gravitational wave frequency corresponding to inner most stable circular orbit(ISCO) provided by general relativity is described. The qualitative differences observed in chirp duration and frequency evolution of these systems is explained. We also take the GW150914 event parameters reported by LIGO team to obtain the inspiral waveform from Newtonian chirp with F_{ISCO} as inspiral limit and overlay on LIGO discovered full waveform to show the merger and ringdown phase of a real signal. We plot the amplitude spectral density of various compact binaries along with GW150914 and GW151226 events and overlay it on LIGO and aLIGO sensitivity curve; it establishes the merging events and possible distances which can be detected in detectors. We also give a brief discussion on the supermassive black hole binary and slowly orbiting binary systems. We also bring out the contribution of higher order PN terms to number of cycles for various compact binaries. Inferences are drawn from these studies which reveals useful insights and information.

1 Introduction

Einstein's theory of general relativity describes gravity as curvature of space time. Einstein's equation are non-linear whereas the linearised Einstein's equation in free space can be written down as a wave equation [1]. In leading order the gravitational waves are produced by a time varying mass-energy quadrupole moment. This is because for a given mass-energy configuration the monopole and dipole moment corresponds to total energy and total angular momentum of the system respectively. The laws of conservation of mass-energy and angular

momentum prohibits any change in monopole and dipole moment. Binary stars are continuous sources of gravitational waves because its quadrupole moment when calculated for circular and elliptical orbit allows non-vanishing time derivatives [1].

The quadrupole moment is defined as [2],

$$I_{ij} = \int \rho(r) r_i r_j d^3 r \qquad (1)$$

where $\rho(r)$ is mass-energy density.

Consider a binary system where the masses m_1 and m_2 are moving in circular orbit about their common center of mass with an angular velocity ω. We assume the masses to be confined in x-y plane and the distance between them to be a. In the center of mass frame this can be reduced to a one body problem, where a mass μ moves in a circular orbit of radius a. We have $\mu = m_1 m_2/(m_1 + m_2)$ which is the reduced mass and $M = m_1 + m_2$ is the total mass of the binary. The position coordinates are $x = a cos(\theta)$, $y = a sin(\theta)$ and $z = 0$.

$$I_{xx} = \mu x x = \mu a^2 \cos^2(\omega t) = \frac{\mu a^2}{2}(1 + \cos(2\omega t))$$

$$I_{yy} = \mu y y = \mu a^2 \sin^2(\omega t) = \frac{\mu a^2}{2}(1 - \cos(2\omega t))$$

$$I_{xy} = I_{yx} = \mu x y = \mu a^2 \cos(\omega t) \sin(\omega t) = \frac{\mu a^2}{2} \sin(2\omega t)$$

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = I_{zz} = 0$$

The presence of $r_i r_j$ in quadrupole moment I_{ij} results in $\cos^2 \omega t$ function which gives $\cos(2\omega t)$ function. We know that the phase of $\cos(2\omega t)$ is twice than that of $\cos(\omega t)$. Hence the frequency of $\cos(2\omega t)$ is twice of $\cos(\omega t)$.

The time varying quadrupole moment govern the gravitational waves produced, thus for a circular binary in leading order the frequency of gravitational waves comes out to be twice the orbital frequency of binary [2].

2 Newtonian Chirp

Due to the emission of gravitational waves the stars inspiral and the orbital frequency of the system increases as per Kepler's third law. In the leading order post-Newtonian approximation we compute the GW signal. The emitted gravitational waveform is a "chirp" signal due to its increasing amplitude and frequency. We consider the masses to be point objects, hence their size and radius doesn't come into picture in any of the expressions. The essential expressions of this approximation presented in this section is based on Ref. [3] & [4].

The distance D of the binary system from earth, the masses m_1 and m_2 in the binary and the frequency of orbital motion F_{system} are the determining parameters of gravitational waves produced by the system. The gravitational wave parameters like amplitude, frequency and strain are closely related to D, m_1, m_2 and F_{system} parameters of the binary. We usually consider the chirp mass of the system, which is

$$\mathscr{M} = \mu^{3/5} M^{2/5} \qquad (2)$$

The time varying strain of gravitational waves is given by

$$h(t) = A(t) \cos \phi(t) \qquad (3)$$

where $A(t)$ is the amplitude and $\phi(t)$ is the gravitational wave phase.

The strain is a time-varying signal due to the time dependence of amplitude and gravitational wave phase. As the binary system evolves with time its orbital radius decreases and frequency increases. The energy emitted due to gravitational waves is large in smaller radii orbit,

therefore the amplitude and phase is expected to change with time. In the approximation considered the amplitude and phase formula are

$$A(t) = \frac{4\mathcal{M}^{5/3}\pi^{2/3}F(t)^{2/3}}{D} \qquad (4)$$

$$\phi(t) = \phi_0 - 2\left[\frac{1}{256(\pi f_0\mathcal{M})^{8/3}} - \frac{t}{5\mathcal{M}}\right]^{5/8} \qquad (5)$$

where $F(t)$ is the instantaneous gravitational wave frequency and f_0 is the initial frequency of the wave received in detectors. The frequency variation with time is given by

$$F(t) = \frac{(\mathcal{M}f_0^9)^{1/8}}{[(\mathcal{M}f_0)^{1/3} - 256f_0^3\mathcal{M}^2\pi^{8/3}(t/5)]^{3/8}} \qquad (6)$$

By defining the term

$$\tau_m = \frac{5}{256(\pi f_0)^{8/3}\mathcal{M}^{5/3}} \qquad (7)$$

We have the gravitational wave frequency rewritten as,

$$F(t) = f_0\left(1 - \frac{t}{\tau_m}\right)^{-3/8}$$

Starting from an initial frequency at $t = 0$, the frequency of gravitational wave increases with time. This is to be expected since the instantaneous frequency of gravitational wave is twice the instantaneous frequency of the binary system from which it is produced. The separation between the masses is governed by the equation

$$a = a_0\left(1 - \frac{t}{\tau_m}\right)^{1/4}$$

Where at $t = \tau_m$ the separation between the masses becomes zero. This is the merging time at which the corresponding binary frequency $\omega^2 = M/a^3$ goes to ∞, thus we can see that at this instant the frequency of gravitational waves also shoots to ∞.

The singular value of frequency is due to the merging of two point masses in which inspiral continues till $a = 0$. But it has been found in general relativity that not all the orbits are stable till $a = 0$ and the inner most stable circular orbit (ISCO) corresponds to $a = 6M_\odot$ [5]. Hence in post-Newtonian approximation, the inspiral phase is taken to end at this orbital separation and the corresponding gravitational wave frequency at this separation is given by

$$F_{ISCO} \approx \frac{4400}{M}Hz \qquad (8)$$

Inspiral phase is followed by a short lived-merger phase. The merger phase begins when the masses starts to merge into a single object. After this the ringdown phase starts where the new object formed radiates away the deformations resulted from merging [6]. Gravitational waves are produced in all these phases. The merger phase is studied by solving Einstein's equation numerically whereas perturbation theory is required for ringdown phase.

Gravitational waves are continuously produced by binary systems and many of them might reach earth but can be too weak to be noticed in the LASER interferometer detectors. As the inspiral phase of the binary system continues, the frequency of orbital motion increases and if it reaches a value at which the signal is strong enough to be detected, we start seeing the GW from this frequency value (called as initial frequency). Ground based gravitational wave detectors are capable of detecting signals of frequency greater than 10 Hz. At lower frequencies the sources from space is overcome by seismic and other earth-based environmental noise sources [5]. Hence in this study we are interested in the beginning of signal in the detector at $t = 0$ starting at $f_0 = 10$Hz. The period where frequency and amplitude is increasing considerably is called as the chirp duration.

3 Binaries at galactic center

We reside in the milky way galaxy. The galactic center is located at 8 Kpc distance from Earth. We consider Black Hole-Black Hole (BH-BH) binary, Black Hole-Neutron Star (BH-NS) binary, Black Hole-White dwarf (BH-WD) binary, Neutron Star-Neutron Star (NS-NS) binary, Neutron Star-White dwarf (NS-WD) binary and White dwarf-White dwarf (WD-WD) binary being at our galactic center. Chandrasekhar's limit gives the maximum mass of a stable white dwarf star which is about $1.39M_\odot$. Most white dwarf stars are less massive than this value, so we have taken the mass of white dwarf star for Newtonian chirp study to be $1M_\odot$. The theoretical value of maximum mass of a stable neutron star is uncertain [7]. Tolman-Opphenheimer-Volkoff (TOV) limit gives the maximum mass of stable neutron star, the modern estimates range from approximately $1.5M_\odot$-$3M_\odot$. Equation of state for extremely dense matter is not well known and hence there exists an uncertainty in the limit. The observational data suggests that mass of most of the pulsars lies between $1.30M_\odot$ to $1.50M_\odot$ [7].

With this regard we take the mass of neutron star to be $1.4M_\odot$ in this study. Since there exists no well defined limit on mass of neutron star for gravitational collapse we consider the black hole to be of $10M_\odot$.

We assume that at time $t = 0$ the frequency of gravitational wave received in detector to be $f_0 = 10$ Hz. We can find the corresponding orbital frequency of the masses in the binary system, $f_{system} = f_0/2 = 5$ Hz. Gravitational waves takes time to travel from binaries to earth. On receiving in detector at $t = 0$ if the gravitational wave has initial frequency of 10 Hz it doesn't imply that the binaries are currently orbiting at 5 Hz, but it should be thought of as the information encoded about binary system in gravitational waves signal at $t = 0$. The aim is to see for a binary system undergoing 5 orbital rotation per second whose corresponding gravitational wave signal arrives with frequency 10 Hz on Earth, what will be the waveform and frequency variation as they inspiral to merge. We use the binary parameters: masses of two stars, distance from the earth and frequency of orbiting in the Mathematica code to obtain the plots in figure[1,2,3]. The details of Mathematica code which can be downloaded is given in appendix.
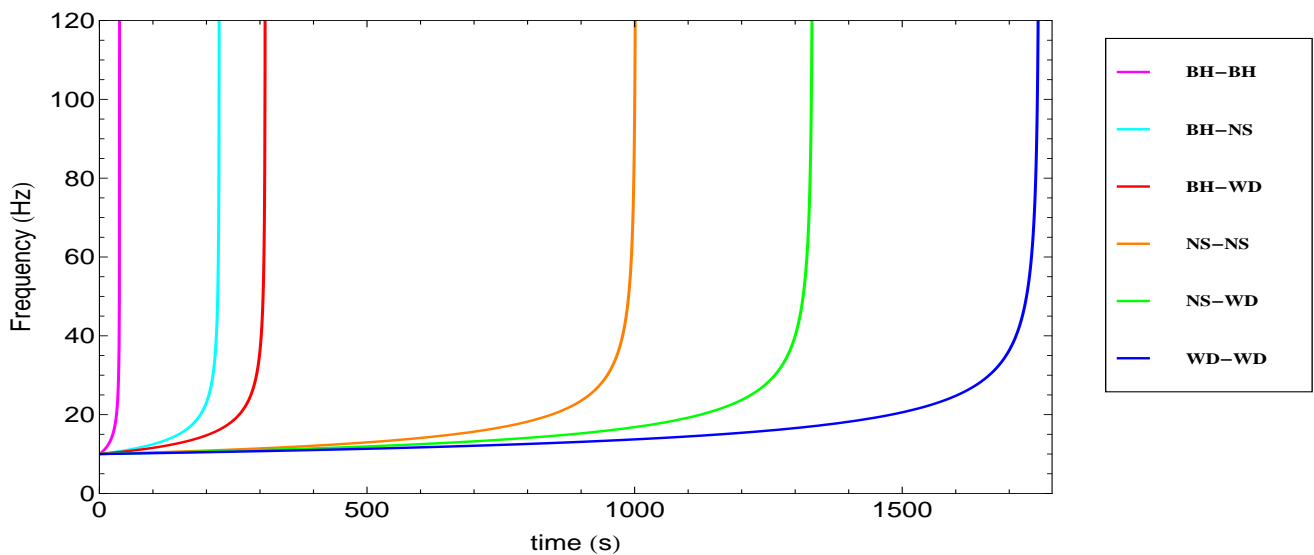

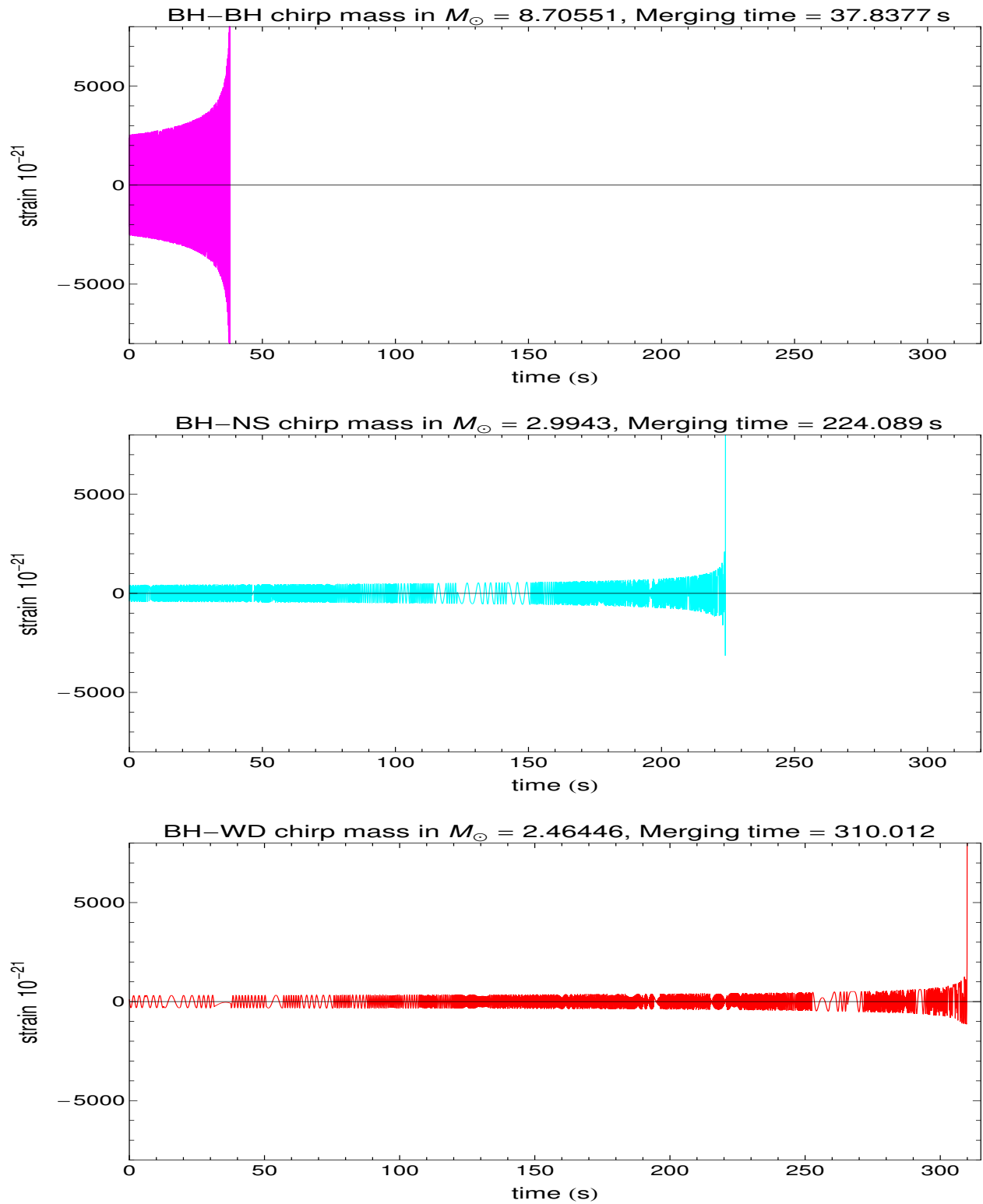Figure 1 : Frequency variation plot of compact binaries

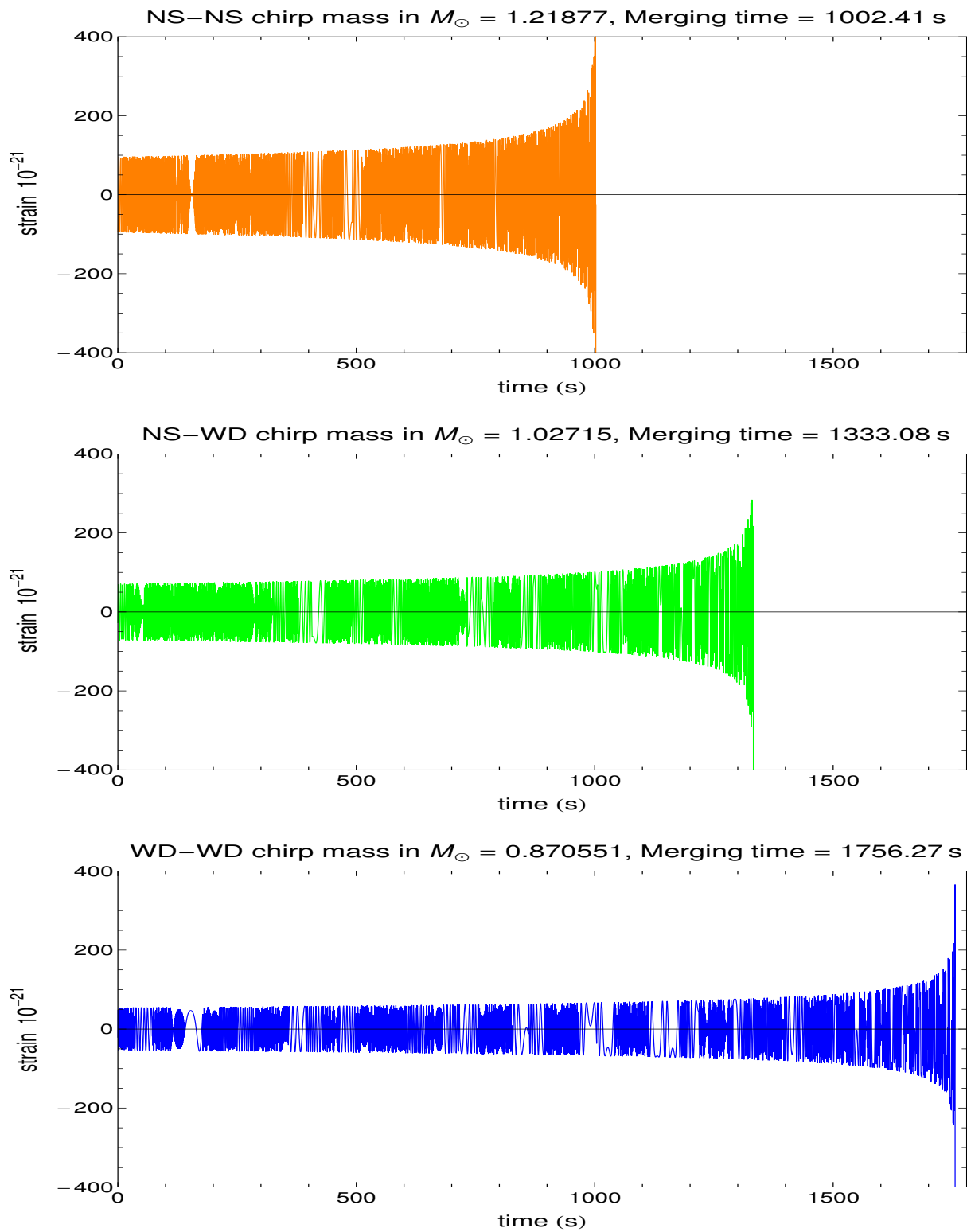Figure 2: Waveforms of BH-BH, BH-NS, BH-WD at galactic center

Figure 3: Waveforms of NS-NS, NS-WD, WD-WD at galactic center

It can be seen that the waveforms produced from Mathematica code starts at $t = 0$ and extends till the merging time τ_m which can be calculated from the formula. The frequency varies from the initial value starting at $t = 0$ and approaches to infinity at τ_m.

Table 1: Compact binaries at galactic center

Name	Masses($M_\odot$)			Peak strain(approx)	Merging time	Peak freq.
	m_1	m_2	$\mathscr{M}$	h_{peak}	τ_m (sec)	F_{ISCO} (Hz)
BH-BH	10	10	8.7005	8000×10^{-21}	37.8377	220
BH-NS	10	1.4	2.9943	2000×10^{-21}	224.089	385.96
BH-WD	10	1	2.46446	1000×10^{-21}	310.012	400
NS-NS	1.4	1.4	1.2187	400×10^{-21}	1002.41	1571.43
NS-WD	1.4	1	1.02715	250×10^{-21}	1333.08	1833.33
WD-WD	1	1	0.870551	200×10^{-21}	1756.27	2200

For binaries at same distance from Earth with same initial orbital frequency in terms of merging time τ_m we can arrange as-

$$\tau_m(BH-BH) < \tau_m(BH-NS) < \tau_m(BH-WD) < \tau_m(NS-NS) < \tau_m(NS-WD) < \tau_m(WD-WD)$$

For binaries at same distance from Earth with same initial orbital frequency in terms of peak strain S_p we can arrange as-

$$S_p(BH-BH) > S_p(BH-NS) > S_p(BH-WD) > S_p(NS-NS) > S_p(NS-WD) > S_p(WD-WD)$$

With point mass consideration in terms of variation of frequency per unit time (δV_f) in 10 Hz to 40 Hz interval we can arrange as-

$$\delta V_f(BH-BH) > \delta V_f(BH-NS) > \delta V_f(BH-WD) > \delta V_f(NS-NS) > \delta V_f(NS-WD) > \delta V_f(WD-WD)$$

Massive binaries lose large amount of energy in the form of gravitational waves hence they inspiral more rapidly which can very well be seen in their frequency-time plot. They rapidly sweep through frequency interval $10Hz-40Hz$ compared to less massive binaries. They also undergo less number of cycles due to rapid inspiral which can be seen in their waveforms and they merge faster. In case of less massive binaries, since they emit small amount of gravitational they inspiral slowly, they sweep through frequency interval 10 Hz-40 Hz slowly undergoing large number of cycles and take comparatively longer time to merge. Since massive binaries emit more gravitational radiation during their inspiral phase than less massive ones their magnitude of strain is always more during the evolution stage and at the end of inspiral stage they attain peak strain larger than less massive ones.

Different binary systems end up their inspiral phase with final orbital frequency given by $F_{ISCO}/2$, which depends inversely upon the total mass of system. This is based upon the assumption that masses are treated as point objects hence in reality the binaries may end up with different orbital frequency before merging,

as their size and radius may cause them to collide and merge prior to reaching peak frequency.

4 Supermassive black holes and slowly orbiting systems

Supermassive black holes can have masses ranging from few hundred solar masses to thousands of solar mass. We have considered a supermassive BH-BH binary with each BH having a mass of $100 M_\odot$ and another such system with each BH having a mass of $200 M_\odot$. The binaries are taken to be located outside our galaxy at 500 Mpc orbiting at 5 Hz. We have obtained its waveform and the frequency variation plot in figure[4] and the results tabulated in table 2.

Table 2 : SBH binary at 500 Mpc

Masses($M_\odot$)			strain	τ_m	F_{ISCO}
m_1	m_2	$\mathscr{M}$	h_{peak}	(sec)	(Hz)
100	100	87.0051	10^{-21}	0.815188	22
200	200	174.11	10^{-21}	0.256768	11

For supermassive black hole binaries with masses $100 M_\odot$ each and $200 M_\odot$ each, orbiting at 5 Hz the merging happens within a second and their F_{ISCO} values being 22 Hz and 11 Hz respectively lies very near to gravitational wave starting frequency 10 Hz. The study of these supermassive black holes reveal that by the time 5Hz is reached in their evolution stage which makes their gravitational signal cross the seismic noise frequency 10 Hz and detectable in ground based interferometer detectors they are at very end of their inspiral stage and merge within a second, hence rather than inspiral phase the merger and ringdown phases are more possible to be detected in aLIGO [8].
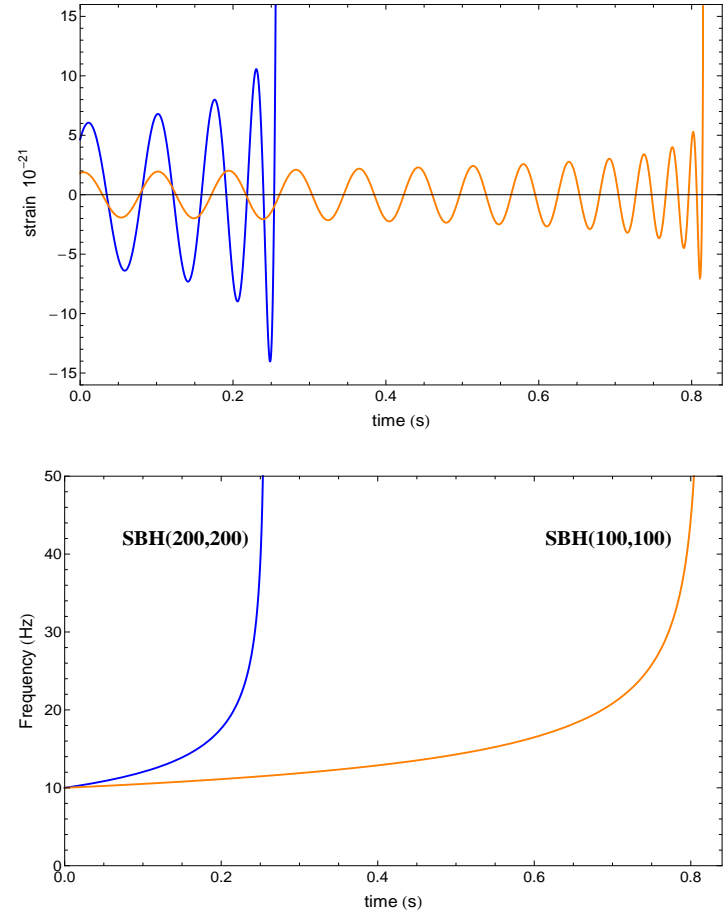

Figure 4: SBH binary at 500 Mpc

For supermassive black hole binary with mass $1000 M_\odot$ the F_{ISCO} comes out to be 4.4 Hz and for $10^6 M_\odot$ the F_{ISCO} comes out to be 4.4×10^{-3}, hence their gravitational waves in inspiral phase never reach 10 Hz to be observed in aLIGO. Considerable inspiral phase of GW from these supermassive black holes can be seen using Laser Interferometer Space Antenna(LISA) which has observable frequency in milli-Hertz [8].

The astrophysical process where merging of supermassive black holes can be seen is in merging of two galaxies. There exists observational evidence that every galaxy has a supermassive black hole at its center and hence the supermassive BH-BH merger is very likely to happen. Strong evidence of supermassive black hole binary exist in NGC 6240 [9], it is considered as a new galaxy formed by the merging of two different galaxies. The two black holes present

in it are currently about 3000 light-years apart. The galaxy spans only 300000 light-years so it is expected that the black holes will merge to form a single black hole.

Not all binary systems orbit very rapidly to have a orbital frequency of 5 Hz, so we consider slowly orbiting binaries for our discussion. The orbital frequency of known double neutron star systems comes out to be about 10^{-6} Hz so we assume that initially the double neutron star system possesses this frequency. We also consider WD-WD binary orbiting at the same frequency. Hence, the arriving gravitational waves is seen with initial frequency of 2×10^{-6} Hz. We assume both the systems to be located at 1pc distance. The waveform and frequency variation plot can be obtained for these systems, the results are tabulated in table 3.

Table 3 : Slowly orbiting binary system at 1 pc

System	Masses($M_\odot$)			Peak strain order	Merging time	Peak freq.
	m_1	m_2	$\mathscr{M}$	h_{peak}	τ_m (sec)	F_{ISCO} (Hz)
NS- NS	1.4	1.4	1.21877	10^{-21}	7.32765×10^{20}	1571.43
WD-WD	1	1	0.870551	10^{-21}	1.28384×10^{21}	2200

The merging time of slowly orbiting NS-NS system comes out to be 7.32765×10^{20} sec and WD-WD system comes out to be 1.28384×10^{21} sec whereas the age of our universe is 13.799×10^9 yrs $= 4.35165 \times 10^{17}$ sec. Hence compact binary systems which at present are slowly orbiting their merger is not likely to be seen in aLIGO anytime in future. The detections of NS-NS merger, NS-WD merger and WD-WD merger if is spotted in aLIGO then those systems must have evolved in cosmic times to reach this late inspiral stage or they had a very less orbital separation and relatively large orbital frequency at the time of formation or due to any other factor.

5   LIGO discovery

LIGO collaboration discovered gravitational waves produced from merging events. The events detected by LIGO are named as "GW" followed by date in YYMMDD format. Two of the events are the most promising merger events detected in year 2015 run of LIGO detectors. When GW150914 was received in detectors it had a strain which attained peak order of 10^{-21}, this signal was seen to increase in frequency from 35Hz and reached 250 Hz in about 0.2 seconds. It was seen that in 8 cycles the signal increased from 35 Hz to 150 Hz [10]. The gravitational wave frequency of 150 Hz corresponds to the orbital frequency of 75 Hz for the binary system. For Newtonian point mass binary this gives 350 Km as the orbital separation. The WD-WD, NS-NS or NS-BH merger should occur at a much larger distance than this value hence LIGO collaboration suggested that the signal to be from a BH-BH binary. This was the first direct observation of black holes and their existence. The masses of binary system could be computed from data analysis and estimated to be $36M_\odot$ and $29M_\odot$ [10]. The luminous distance of this system was estimated to be 410 Mpc. The second event GW151226 when received in detectors it had a peak strain of the order 10^{-22}, this signal was seen for about a second in which its frequency increased from 35 Hz to 450 Hz in about 55 cycles [11]. The binary parameters predicted from analysis was reported to be $14.2M_\odot$ and $7.5M_\odot$. The luminous distance of this system was estimated to be 440 Mpc. We

take the prameters of GW150914 and GW151226

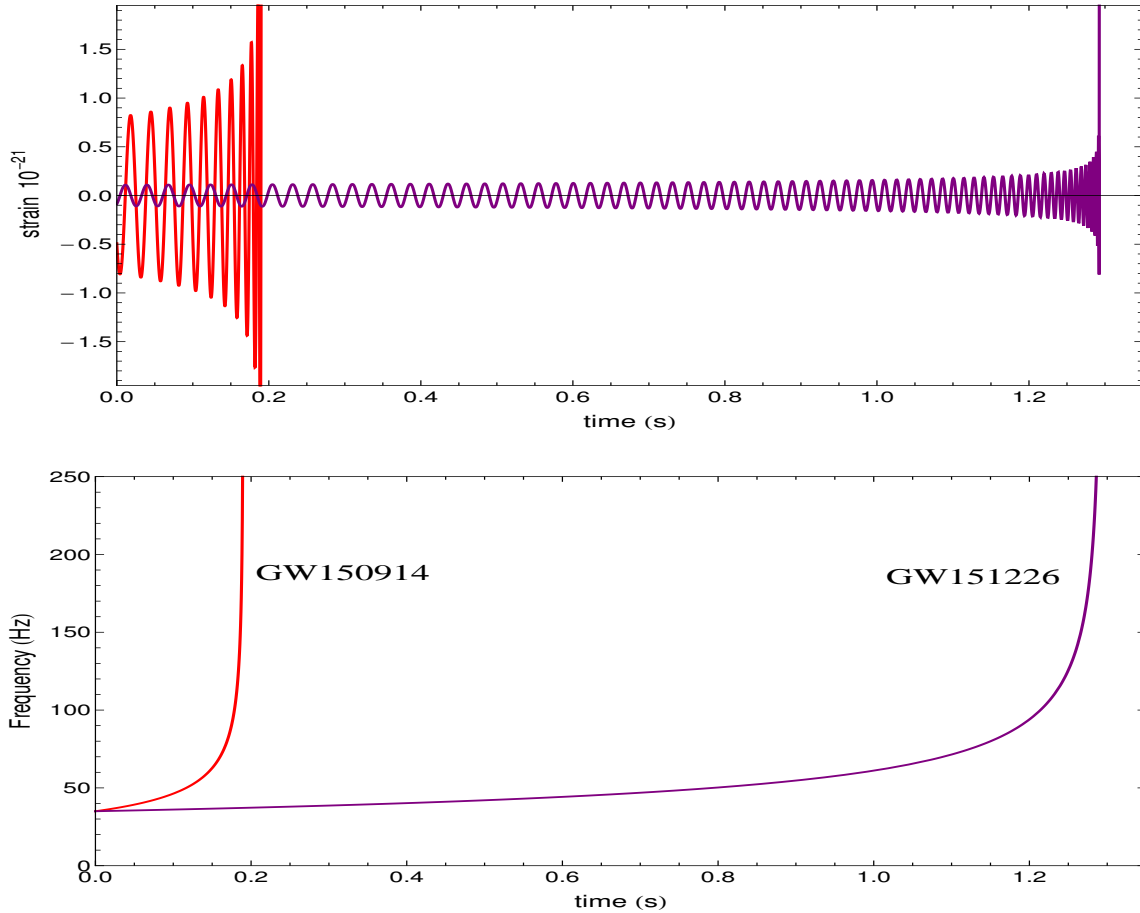to obtain the Newtonian Chirp plots in figure[5], the results are tabulated in table 4.

Figure 5: LIGO events Newtonian chirp

Table 4: Newtonian chirp of GW150914 and GW151226

Data source	Masses($M_\odot$)			Peak strain order	Merging time	Peak frequency
	m_1	m_2	$\mathscr{M}$	h_{peak}	τ_m (sec)	F_{ISCO} (Hz)
GW150914	36	29	28.0956	10^{-21}	0.19	67.69
GW151216	14.2	7.5	8.89442	10^{-22}	1.29	202.76

GW151226 chirp has a smaller strain amplitude, undergoes large number of cycles and its signal energy is spread over a longer time interval than GW150914 chirp.

The strain recorded by LIGO's Livingston

detector for GW150914 is plotted along with the inspiral part from Newtonian chirp with F_{ISCO} as inspiral limit in figure[6]. The F_{ISCO} comes out to be 67.69 Hz and the time corresponding to it can be found from frequency-time plot of

Newtonian chirp. For a signal starting at 0.25 seconds this corresponds to 0.407 seconds. The Newtonian chirp doesn't mimic exactly the real signal's amplitude and phase observed, but its worth appreciating that the strain order and evolution during inspiral phase has resemblance. This is to be expected as higher order terms

of Post-Newtonian approximation if considered bring changes in phase and amplitude. We can also see that merger and ringdown part of the actual signal is not captured by the Newtonian chirp. The sources for LIGO detectors observed strain data and details of Mathematica code which can be downloaded is given in appendix.
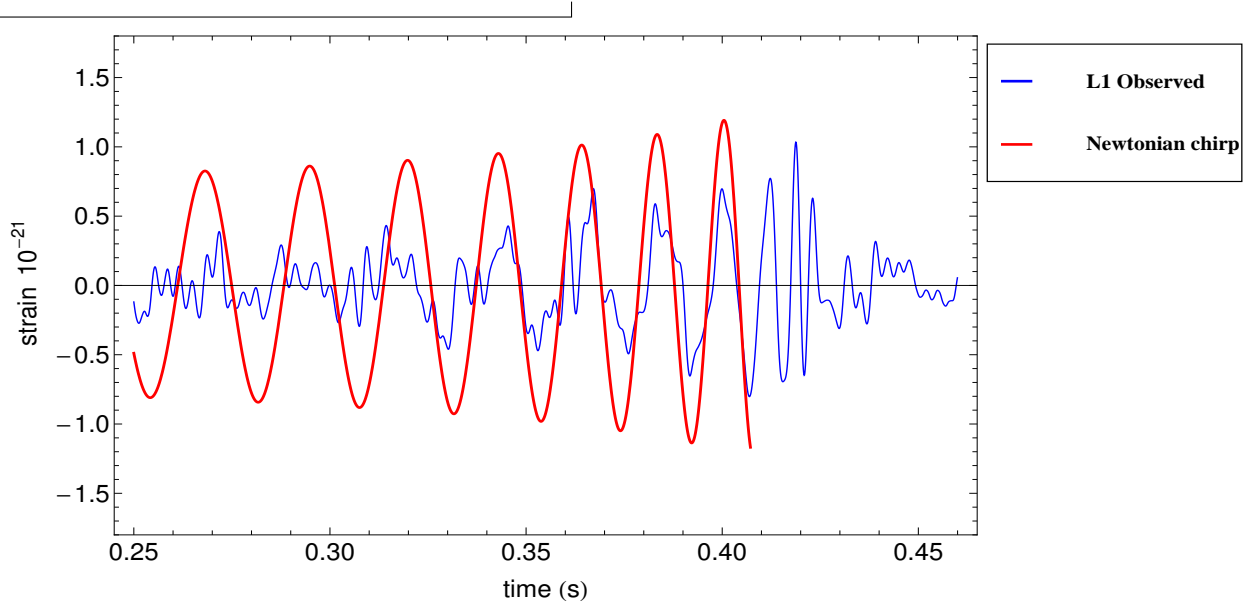

Figure 6: GW150914 strain observed in Livingston with Newtonian chirp

6 Sensitivity curve

Sensitivity curve is used to determine whether a source is detectable by the detector. The power spectral density S_h is the power per unit frequency. The square root of the power spectral density gives the amplitude spectral density defined as $h_f = \sqrt{S_h}$ [2]. If h_f of the source lies above the sensitivity curve of the detector then the source can be detected [2]. For a source the amplitude spectral density is given by [12],

$$\sqrt{S_h} = h_c f^{-1/2} = 2f^{1/2}\left|\tilde{h}_c\right| \qquad (9)$$

where h_c is characteristic strain and $\tilde{h}_c$ is the Fourier transform of signal.

For inspiraling binaries in the leading order it has been found that [3],

$$\tilde{h}_c = A(f)e^{i\psi}$$

where

$$A(f) = \sqrt{\frac{5}{24}}\frac{\mathscr{M}^{5/6}}{D\pi^{2/3}}f^{-7/6} \qquad (10)$$

$$\psi = 2\pi f t_0 - \phi_0 - \pi/4 + \frac{3}{128}(\pi\mathscr{M}f)^{-5/3}$$

Using the above expressions we get,

$$\sqrt{S_h} = 2f^{1/2}\left|\tilde{h}_c\right| = \sqrt{\frac{5}{6}}\frac{\mathscr{M}^{5/6}}{D\pi^{2/3}}f^{-2/3} \qquad (11)$$

In inspiral a gravitational wave signal spends greater number of cycles in lower frequency whereas lesser number of cycles in higher frequency. Since the amplitude spectral density h_f is inversely related to frequency, with increase in frequency h_f decreases. Detectors sensitivity curve is governed by various noises present in the detector. We plot the h_f vs frequency plot for the various chosen compact binaries and also for the two LIGO events along with the LIGO design sensitivity curve and aLIGO design sensitivity curve to see if the signal could be detected. The area between the signal curve and detector's noise curve indicates the signal to noise ratio [13].
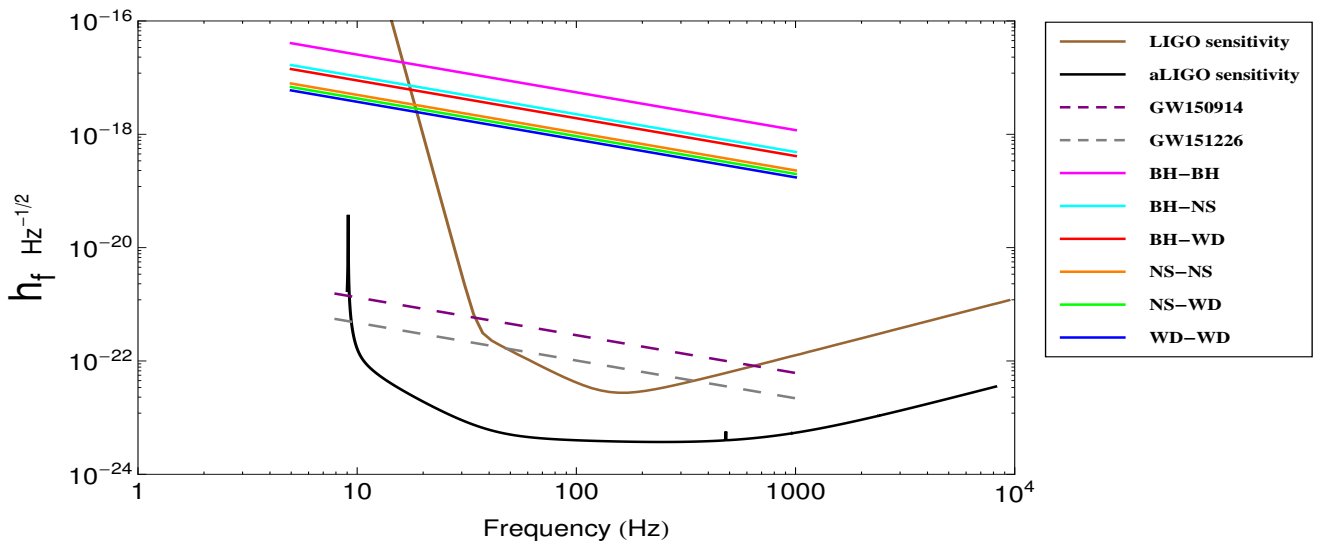

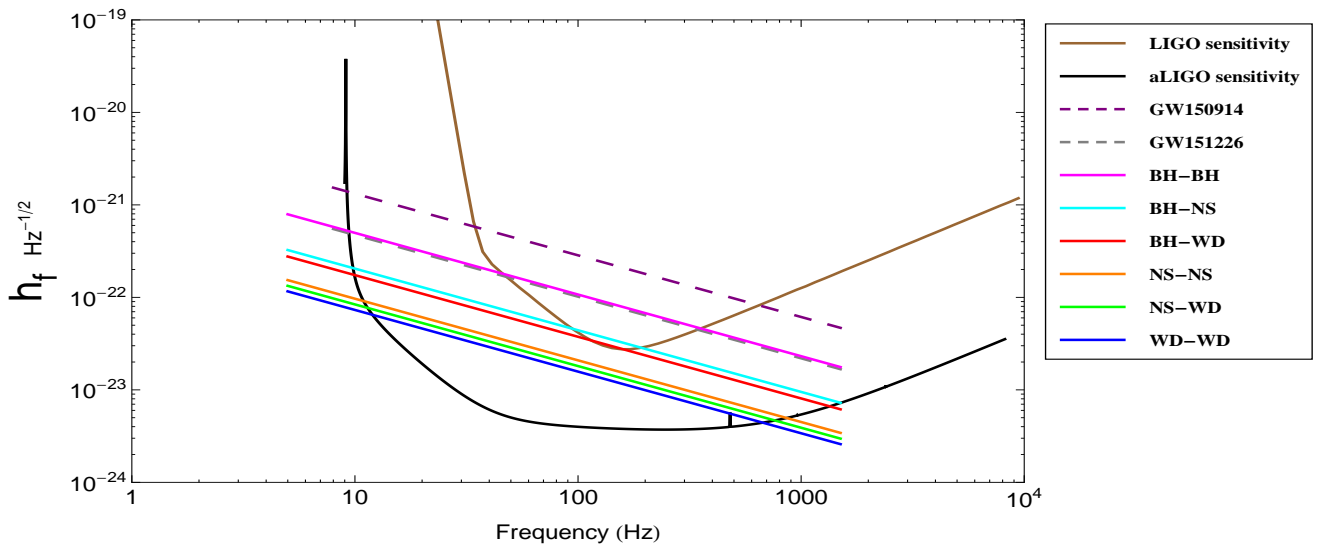Figure 7: Compact binaries at 8kpc along with GW150914 and GW151226

Figure 8: Compact binaries at 410kpc along with GW150914 and GW151226

We first consider the compact binaries in their location at galactic center and then at 410 Mega parsec (the distance from which GW150914 originated). We keep the distance of GW150914 event and GW151226 their respective ones to obtain the plots in figure[7,8]. The details of data obtained for LIGO and aLIGO sensitivity curve along with Mathematica code for the plot is given in appendix.

In the plot with Newtonian chirp context we have extended the frequency of these systems till 1000 Hz but the inspiral phase of each compact binary will end at their respective peak frequency F_{ISCO}. The amplitude spectral density curves of GW150914 and GW151226 during the inspiral, merger and ringdown phases reported by LIGO and VIRO collaboration can be seen from Ref [13].

The amplitude spectral density curve of compact binaries at galactic center (8kpc) lie above LIGO and aLIGO sensitivity curve, hence they are within the detector detection capabilities. When the same binaries are located at 410 Mpc only the BH-BH is well within the detection capability of LIGO whereas BH-NS and BH-WD systems lie very slightly above the LIGO curve in a small frequency range but extracting the signal from noise is hard. The NS-NS, NS-WD and WD-WD systems lie below the sensitivity curve of LIGO at all frequency values. Even

when located at 410 Mpc NS-NS, NS-WD and WD-WD systems are in detection capabilities of aLIGO. Hence the merger of these systems can be expected to be observed in future. GW150914 signal lies well above the aLIGO detector sensitivity curve than GW151224; whereas the NS-NS, NS-WD and WD-WD binary curve located at 410 Mpc lies over the aLIGO curve lowest among the all. Hence sophisticated data analysis is required to pull out such signals from noise.

7 Higher order Post-Newtonian terms

In Post-Newtonian approximation the expressions for orbital frequency, phase of gravitational wave and waveform is in the form of expansion of PN order contributions. The 0 PN order term (neglecting higher order contributions) gives the orbital frequency, phase ϕ and h describing the Newtonain chirp. The higher order terms gives a significant contribution to phase and amplitude of gravitational wave. The nomenclature of the nPN order is for terms proportional to x^n [6]. Where,

$$x = (M\omega)^{2/3} \tag{12}$$

and $\omega = \pi F$ is the angular velocity, F is the gravitational wave frequency.

The expression for the phase as a function of x is given by the equation [6],

$$\phi = \frac{-x^{\frac{-5}{2}}}{32\eta}[C_0 + C_1 x + C_{1.5}\ x^{\frac{3}{2}} + C_2\ x^2 + C_{2.5}\ x^{\frac{5}{2}} + C_3 x^3 + C_{3.5} x^{\frac{7}{2}}]$$

The coefficients are-

$$C_0 = 1 \qquad\qquad C_1 = \left(\frac{3715}{1008} + \frac{55}{12}\eta\right) \qquad\qquad C_2 = \left(\frac{15293365}{1016064} + \frac{27145}{1008}\eta + \frac{3085}{144}\eta^2\right)$$

$$C_{1.5} = -10\pi \qquad C_{2.5} = \left(\frac{38645}{1344} - \frac{65}{16}\right)\pi \ln(\frac{x}{x_0}) \qquad C_{3.5} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\eta - \frac{74045}{6048}\eta^2\right)\pi$$

$$C_3 = \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}C - \frac{856}{21}\ln(16x) + \left(\frac{-15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\eta + \frac{76055}{6912}\eta^2 - \frac{127825}{5184}\eta^3$$

C=0.577 called as the Euler-Mascheroni constant.

Table 5: Contribution of each PN term to the number of gravitational wave cycles

PN order	BH-BH	BH-NS	BH-WD	NS-NS	NS-WD	WD-WD
0	602.712	3582.14	4956.29	16056.7	21354.6	28134.8
1	59.3521	213.313	280.495	441.305	526.559	618.958
1.5	-51.3811	-181.362	-242.645	-210.884	-241.22	-265.799
2	4.06297	9.80028	12.4367	9.91219	10.733	11.3563
2.5	-7.14497	-20.0051	-26.426	-11.6897	-12.4027	-12.4674
3	2.17844	2.28362	2.26033	2.55632	2.56639	2.58872
3.5	-0.81811	-1.81626	-2.33096	-0.90533	-0.93078	-0.91179

We take $F_1 = f_0$ and $F_2 = F_{ISCO}$, the value of f_0 is related to lowest possible frequency observable by the detector. The contribution of nth PN order to number of cycles can be found using [14],

$$N_{cyles} = \frac{\phi(F_2) - \phi(F_1)}{\pi} \qquad (13)$$

Using ϕ expression with $n = 0$ order term alone and neglecting the higher orders, we can find the contribution to number of cycle due to 0 PN order. The phase of 0 PN order is given by ,

$$\phi_{n=0} = \frac{-(x)^{-5/2}}{32\eta} = \frac{-(M\pi F)^{-5/3}}{32\eta}$$

$$N_{cycles} = \frac{-(M\pi)^{-5/3}}{32\eta}\left(\frac{F_2^{-5/3} - F_1^{-5/3}}{\pi}\right)$$

In similar way the contribution due to higher order terms can also be found. For the compact binaries taken in this study we have found contribution to number of cycles due to various PN orders. The results are tabulated table 5. The Mathematica code to find contributions due to various orders is given in Appendix.

8 Conclusion

Newtonian chirp in the leading order Post-Newtonian approximation is based upon the assumption that masses are point objects, which is not true for real systems but still is a good approximation. The amplitude spectral density of Newtonian chirp is used to determine whether

the inspiral phase of a binary can be observed in ground base detectors and hence the study of it for binaries with the aLIGO and LIGO sensitivity curves helps us to understand the detector's capabilities and the possible developments that could be made to make the detectors more efficient.

The real gravitational wave signals obtained from the detectors are more complicated, when we try to match the Newtonian chirp waveform against the real waveform obtained from LIGO Livingston detector for GW150914 we could see that it didn't exactly match with the real signal's amplitude and phase observed. It needs the study of the higher order terms (Post-Newtonian terms) to describe the real wave's inspiral exactly.

We see that the formula for peak frequency of gravitational waves together with the phase formula having the higher order PN terms, could determine the number of cycles contributed by each higher order in addition to the leading order for the real gravitational wave signal from the binary.

This study brings out the essential features of GW produced by different compact binary systems and the possibilities of detecting them ; also helps us to understand the insights which Newtonian chirp convey regarding the actual gravitational wave signals which are complex in nature.

Appendix

All expressions are taken in geometrized units where G=c=1. Mass and distance have units of seconds. In geometrical units, $1M_\odot = 4.92549095 \times 10^{-6}$ sec and 1pc=$1.0292712503 \times 10^{8}$sec. Physical units can be obtained by replacing a mass $\mathscr{M}$ by $G\mathscr{M}/c^3$ and a distance D by cD.

The Newtonian chirp waveform, frequency-time plot and amplitude spectral density plot

can be obtained using the Mathematica code (or any programming language). LIGO event data and sensitivity curves can be plotted in 2-D using GNU, R program, Mathematica or any other tool. We provide the Mathematica code nb file and pdf file along with required txt files for the plots used in the paper.

1. Waveform and frequency variation

Mathematica code for waveform and frequency-time plot be downloaded as

nb file : https://drive.google.com/open?id=0B1PM7VcZPwY2eWZwQWdSRk1mREU

pdf file : https://drive.google.com/open?id=0B1PM7VcZPwY2XONzVWxHZkxiQXM

Mathematica code for frequency-time plot for all compact binaries plotted together can be downloaded as

nb file : https://drive.google.com/open?id=0B1PM7VcZPwY2bVRNWi1EeVNaN3c

pdf file : https://drive.google.com/open?id=0B1PM7VcZPwY2T1ljM2oOT2FxROk

Mathematica code for single column waveform plot can be obtained as nb file : https://drive.google.com/open?id=0B1PM7VcZPwY2QUUzUFB2LXZxbUE .

2. LIGO events

GW150914 event strain data observed in Livingston detector can be obtained from the LIGO website : https://losc.ligo.org/events/GW150914/ . Alternatively can also be downloaded as txt file from : https://drive.google.com/open?id=0B1PM7VcZPwY2WFhOVkFzSOMOTOU .

Mathematica code to plot the strain of GW150914 and overlay the inspiral part of Newtonian chirp can be downloaded as

nb file : https://drive.google.com/open?id=0B1PM7VcZPwY2cU9WXzh5M2lWX1E

pdf file : https://drive.google.com/open?id=0B1PM7VcZPwY2RVJ1VXJmT2NOZjQ

3. Strain spectral amplitude

The LIGO design sensitivity curve data can be obtained from LIGO website : https://dcc.ligo.org/LIGO-E950018/public , where the file named "SRD-strain-4k.txt" is the required txt

file. Alternatively as txt file from :
`https://drive.google.com/open?id=0B1PM7VcZPwY2VUhYd1dsQzBUVDA`.

The aLIGO design sensitivity curve data can be obtained from LIGO website : `https://dcc.ligo.org/LIGO-T0900288/public` , where the file named "ZERO-DET-high-P.txt" is the required txt file. Alternately as txt file from :
`https://drive.google.com/open?id=0B1PM7VcZPwY2SlFPa3JrWWRmd00` .

Mathematica code for plotting sensitivity curves and amplitude spectral density of compact binaries(at galactic centre) can be downloaded as
nb file : `https://drive.google.com/open?id=0B1PM7VcZPwY2TlhJTUpKRzAwUDQ`
pdf file : `https://drive.google.com/open?id=0B1PM7VcZPwY2alNyQzBjdEVOeWs`

Mathematica code for plotting LIGO sensitivity curves and amplitude spectral density of compact binaries binaries(at 410 Mpc) can be downloaded as
nb file : `https://drive.google.com/open?id=0B1PM7VcZPwY2SnJFTkkwQ1FDRDA`
pdf file : `https://drive.google.com/open?id=0B1PM7VcZPwY2M0lNZllJMGZBOVE`

4. Higher order PN terms

The code to calculate the contribution of higher order PN terms to number of cycles can be obtained as
nb file : `https://drive.google.com/open?id=0B1PM7VcZPwY2ZWhnNUo1cDl6dzg`
pdf file : `https://drive.google.com/open?id=0B1PM7VcZPwY2U0tsYkFFbEM3aEU`

All the Mathematica nb files, pdf files and txt files can be downloaded in single folder as zip file from : `https://drive.google.com/open?id=0B1PM7VcZPwY2UlNpR3ZENHJtTm8` .

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