Lorentz transformations without light

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Abstract:

A six-step derivation is given for the Lorentz transformation which, without any reference to light and without resorting to advanced group-theory arguments, should avoid any misunderstanding about the connection of *light* with relativity theory.

When presenting the special theory of relativity, very appropriately one usually starts with two postulates: (1) Galilean relativity, according to which fundamental physics laws must be invariant upon going from one inertial frame to another, and (2) the universality of the speed of light, according to which the speed of light is independent of the motion of its source, i.e., any inertial frame, upon measuring the speed of the *same* photon, reports the same value of $c = 3 \times 10^8$ m/s . Following these two postulates, the space-time coordinate transformations from one inertial frame to another – Lorentz transformations (LT) – are derived.

However, light *per se* does not have much to do with relativity, besides the fact that it just happens to travel at the limiting speed – c, for *celeritas* – prescribed by the theory. The second postulate is then a convenient way of positing the existence of such a limiting speed – convenient because we have at our disposal something that experimentally certifies that this is the way the world is constituted: the reference to light is then due to the historical role that electromagnetism has played in the discovery of special relativity.[1]

Here is shown a six-step derivation of the LT which, without any reference to light and without resorting to advanced group-theory arguments,[2] should avoid any misunderstanding about the connection of *light* with relativity theory. This holds not because the speed of light is invariant, but because space-time is so specially constituted: the experimentally verified invariance of the speed of light is just one of many experiments confirming that LT are the correct space-time coordinate

transformations between two equivalent frames.

Two options

The framework on which physics is built is that of a time which is uniform and a space which is homogeneous and isotropic, i.e., the fundamental laws of physics are invariant under translation in time, and translation and rotation in space. We also require invariance upon going from one frame, say F, to another, say F', moving with constant velocity, $\mathbf{V}$, with respect to F. This is the usual first postulate of relativity, Galileo's postulate. In particular, the space–time coordinate transformations from F to F' must have the same functional form of those from F' to F, a circumstance which we shall refer to as *reciprocity*. For the time being, we do not require the speed of light to be constant. We now proceed to seeking the transformations between the space-time coordinates from F to F' and *vice versa*. This is done in six steps. In what follows, we shall put a prime to any quantity evaluated in the frame F'.

First step. From reciprocity, it is sufficient that the sought transformations be linear, which implies that the inverse transformations have the same functional form. The transformations are then of the type

$$x_{\mu'} = \sum_{\nu} L_{\mu'\nu} x_\nu + k_{\mu'} , \qquad (1)$$

where x_μ and $x_{\mu'}$ are the space-time coordinates in F and F', and $L_{\mu'\nu}$ and $k_{\mu'}$ are quantities independent of those

coordinates. Later on, we shall specify on what they *may* depend. The indices vary from 1 to 4, with $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = t$.

Second step. The symmetry properties of space and time allow us to choose the space and time origin and the orientations of the coordinate axes at will. Designating a given set of the four space-time coordinates as an 'event,' we make the choice that the event labeled as $x = y = z = t = 0$ in F is labeled as $x' = y' = z' = t' = 0$ in F'. Likewise, we may as well choose $y' = y$ and $z' = z$ at $t = 0$ and the positive direction of both the x-and x'-axes along the positive direction of $\mathbf{V}$. From $x_{\mu'} = 0 = x_\mu$ for all μ it follows that $k_{\mu'} = 0$ for all μ.

Third step. We determine the transformation for the *longitudinal* space-coordinate component, i.e., with the choice made, the x- coordinate. Since $x' = 0 = t'$ when $x = 0 = t$, then $L_{1'2} = L_{1'3} = 0$, as can be seen from Eq. (1). The x- coordinate transformation equation must then be of the form $x' = \gamma x + \delta t = \gamma(x + \frac{\delta}{\gamma} t)$. At all times, the first spatial coordinate of the origin O' are $x' = 0$ and $x = Vt$, i.e. $\delta/\gamma = -V$. The transformation equation for the x-coordinate is therefore

$$x' = \gamma(x - Vt) . \qquad (2a)$$

At this point we can say that $\gamma > 0$, from the choice we have made of the coordinate axes, according to which $\mathrm{sgn}(x') = \mathrm{sgn}(x)$ at $t = 0$. Also, γ *may* depend on V and, if so, the

isotropy of space requires that $\gamma(-V) = \gamma(V)$, reciprocity requires that $|V'| = |V|$, and the chosen direction of the x- and x'- axes requires that $V' = -V$. Hence, the inverse transformation is obtained by the replacements $x' \leftrightarrow x$ and $-V \leftrightarrow V$:

$$x = \gamma(x' + Vt') . \qquad (2b)$$

Fourth step. We now find the transformations for the time coordinate. From Eq. (2b), isolating t' and inserting Eq. (2a), we get

$$t' = \gamma\left(t - \frac{\varepsilon}{V}x\right), \qquad (3)$$

where we have set

$$\varepsilon \equiv 1 - \frac{1}{\gamma^2} , \qquad (4a)$$

by which we must have $\varepsilon < 1$. From Eq. (4a),

$$\gamma^2 \equiv \frac{1}{1 - \varepsilon} . \qquad (4b)$$

Fifth step. We determine the transformations for the *transverse* space-coordinate components, i.e., the y- and z- coordinates. Having chosen $y' = y$ and $z' = z$ at $t = 0$, we have $L_{2'1} = L_{2'3} = 0 = L_{3'1} = L_{3'2}$ and $L_{2'2} = 1 = L_{3'3}$. However, from Eq. (3) we see that the time-coordinate transformation does not involve the transverse spatial components, so that $L_{2'4} = 0 = L_{3'4}$. The transformations for the transverse spatial components are then
$$y' = y \qquad \text{and} \qquad z' = z . \qquad (5)$$

Sixth step. We determine the dimensionless quantities γ and ε. The crucial point is that there are two options here. Either they are constant, or they depend on V. The matter has to be resolved experimentally, unless some extra assumption comes into play.

In the former case, γ may be taken equal to 1: any other value would simply imply, as can be seen from Eq. (2), a change in scale of the units chosen. With $\gamma = 1$, we have $\varepsilon = 0$, and
$$x' = x - Vt \qquad \text{and} \qquad t' = t . \qquad (6)$$

Equations (5) and (6) are the first-option transformations, i.e., Galileo transformations. Notice that the Galilean velocity-composition rules readily follow from Eqs. (5) and (6):

$$u_{x'} \equiv \frac{dx'}{dt'} = \frac{dx'}{dt} = u_x - V \qquad (7a)$$

$$u_{y'} \equiv \frac{dy'}{dt'} = \frac{dy}{dt} = u_y \qquad \text{and}$$

$$u_{z'} \equiv \frac{dz'}{dt'} = \frac{dz}{dt} = u_z . \qquad (7b)$$

If the second option holds, being dimensionless quantities, γ and ε must rather depend on V/c, where c is some V-independent quantity (i.e., some universal constant) with the dimensions of velocity, whose value must be determined from experiments. Let us then determine γ and ε. The velocity-composition rules that follow from Eqs. (2) and (3) are

$$u_{x'} \equiv \frac{dx'}{dt'} = \frac{dx - Vdt}{dt - \frac{\varepsilon}{V}dx} = \frac{u_x - V}{1 - \frac{\varepsilon}{V}u_x} \qquad (8a)$$

$$u_{y'} \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{\varepsilon}{V}dx\right)} = \frac{u_y}{\gamma\left(1 - \frac{\varepsilon}{V}u_x\right)} \qquad (8b)$$

And

$$u_{z'} \equiv \frac{dz'}{dt'} = \frac{u_z}{\gamma\left(1 - \frac{\varepsilon}{V}u_x\right)} \quad . \qquad (8c)$$

Squaring and adding these velocity components, and making use of Eq. (4b), one obtains

$$\left(u'\right)^2 = \frac{\left(u_x - V\right)^2 + \left(u^2 - u_x^2\right)\left(1 - \varepsilon\right)}{\left(1 - \frac{\varepsilon}{V}u_x\right)^2} \quad . \qquad (9)$$

With c as an invariant velocity, setting $u^2 = c^2 = \left(u'\right)^2$, Eq. (9) becomes, after rearrangement,

$$c^2 = \frac{c^2\left(1 - 2\frac{V}{c^2}u_x + \frac{\varepsilon}{c^2}u_x^2\right) + V^2 - \varepsilon c^2}{\left(1 - \frac{\varepsilon}{V}u_x\right)^2} \quad . \qquad (10)$$

With $\varepsilon < 1$, the only physically meaningful solution to this equation is

$$\varepsilon = \frac{V^2}{c^2} \equiv \beta^2 \quad . \qquad (11)$$

The other solution, $\varepsilon = \left(V/u_x\right)\left(2 - V/u_x\right)$, has to be discarded on physical grounds, for ε would otherwise depend on the orientation of frames in space.

We then see that $0 \leq \varepsilon < 1$, whereby $V < c$ (no frame can travel at $V \geq c$), and $\gamma = \dfrac{1}{\sqrt{1 - \beta^2}} > 1$. $\qquad (12)$

Equation (3) then becomes

$$t' = \gamma\left(t - \frac{V}{c^2}x\right) , \qquad (13a)$$

and its inverse is

$$t = \gamma\left(t' + \frac{V}{c^2}x'\right) \quad . \qquad (13b)$$

Equations (2), (5), (12) and (13) are the sought second-option transformations, i.e., Lorentz transformations.

Notice, from Eq. (9), that $u < c \Rightarrow u' < c$ and $u = c \Rightarrow u' = c$, where u and u' are the speeds of a particle in F and F': not only is c an invariant, but it is also a limiting speed.

Deciding between the options: The tragedy of a muon

Which one is the option to pick is an experimental matter (unless other assumptions are added). Of course, the fact that photons *do* travel at an invariant speed tells us that the world is constituted according to the second option, and the speed of light has to be identified with the constant c . This is the path historically taken. However, if light *did not* travel with an invariant speed – or, for that matter, if *nothing* traveled at an invariant speed – the second option could not be discarded, and should have been (and it has been) answered by appropriate experimental results, as it should be recalled at this point.

There are, among others, two remarkable consequences of LT: length contraction and time dilation. According to *any* frame, length-of-a-stick means the distance between the stick end-points when their spatial coordinates are evaluated *at the same time*. A stick at rest in frame F, along the x- axis, and with end-point spatial coordinates x_A and x_B, would have length

$\lambda \equiv \Delta x = \left| x_B - x_A \right|$ in frame F. From Eq. (2b), the end-points have coordinates $x_A = \gamma(x'_A + Vt'_A)$ and $x_B = \gamma(x'_B + Vt'_B)$. In order for $\left| x'_B - x'_A \right|$ to be the length $\Delta x'$ of the stick in frame F', in the above equations $t'_A = t'_B$ should hold, whereby $\lambda = \left| x_B - x_A \right| = \gamma \left| x'_B - x'_A \right| = \gamma \Delta x'$. Therefore,

$$\Delta x' = \frac{\lambda}{\gamma} \, , \tag{14}$$

i.e., a stick in F' is shorter than in F (where it is at rest and with length λ) by a factor γ, a circumstance called *length contraction*.

Similarly, if two events happen in F at the same place $\left(x_A = x_B \right)$ and with a lag in time given by $\tau = \left| t_B - t_A \right|$, the lag in time $\left| t'_B - t'_A \right| \equiv \Delta t'$ between them, according to F', is obtained from Eq. (13a): $t'_A = \gamma \left(t_A - \frac{V}{c^2} x_A \right)$ and $t'_B = \gamma \left(t_B - \frac{V}{c^2} x_B \right)$. With $x_A = x_B$, one obtains

$$\Delta t' = \gamma \tau \, , \tag{15}$$

i.e., in F' the time interval between two events is longer by a factor γ than in F (where the two events happen at the same place), a circumstance called *time dilation*.

Now, we recall – in a simplified version, i.e., not realistic but adapted to our purposes – a remarkable experiment.[3] In a laboratory on Earth, it is possible to detect a muon, born at, say, $15\,\text{km}$ up in the troposphere, striving to reach us at the speed $V = 2.997 \times 10^8$ m/s, but dying just before touching the Earth's surface, after a life

$50.05\,\mu$s long. From the reference frame of the muon, once born, she sees the planet Earth heading against her at the speed $V = 2.997 \times 10^8$ m/s; however, she dies after only $2.2\,\mu$s, just before experiencing the crush. The two events – birth and death of the muon – have a different time lag in the two frames: undoubtedly, the world must be constituted according to the second option. The value of the limiting invariant speed can now be evaluated. From Eq. (15), with $\tau = 2.2\,\mu$s and $\Delta t' = 50.05\,\mu$s, γ turns out to be $\gamma = 22.75$, and from Eqs. (11) and (12), $\beta \equiv V/c = 0.999$, whereby $c = 3 \times 10^8$ m/s. (Once again we stress the simplified version given of the actual experiment.)

Of course, if the muon had lived long enough to survive the crush, she could have evaluated how far was our planet at the time she was born: $V\tau = 659\,\text{m}$, a result in agreement with what would be obtained from Eq. (14), with $\lambda = 15\,\text{km}$ and $\gamma = 22.75$.

Conclusions

We would like to stress once again that c, usually called the speed of light in vacuum, is rather the invariant (and limiting) speed built in the geometry obeyed by our space-time. This is a notion that could (and should) be conveyed right at the beginning when LT are derived. Due to reciprocity, they have to be of the form of Eq. (1), with the coefficients that, without any further assumption, either depend or do not depend on V: *tertium non datur*. In the former case, the invariant speed must also be a limiting speed, regardless of whether or not there exist particles travelling at that speed.

The development of the theory provides for a free particle with mass m and linear momentum p, an energy E given by $E^2 = m^2c^4 + p^2c^2$: unlike classical mechanics, relativity allows zero-mass particles, in which case $E = pc$. However, according to the theory, $E = mc^2\gamma_u$, where $\gamma_u \equiv \left(1 - u^2/c^2\right)^{-1/2}$: the last two relations for the energy of a zero-mass particle are compatible only if $u = c$, i.e., massless particles must travel at speed c.

The fact that ubiquitous light travels at speed c has been indeed a lucky occurrence, without which, everything else being equal, the realization of the space-time real structure might have waited some longer time.

References

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