

Stern-Gerlach Experiments and Complex Numbers in Quantum Physics

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Abstract

It is often stated that complex numbers are essential in quantum theory. In this article, the need for complex numbers in quantum theory is motivated using the known results of tandem Stern-Gerlach experiments.

Keywords: Stern-Gerlach experiment, complex numbers, superposition

1 Introduction

Complex numbers are essential in quantum theory. In classical physics complex quantities are often introduced to aid in solving problems rather than as a necessity. That makes it mysterious for students about the role of complex numbers in quantum theory. In this pedagogical report, it is illustrated that the need for complex numbers in quantum theory can be made plausible

after discussing the results of Stern-Gerlach (SG) experiment. This idea is presented in many texts, for instance, Sakurai[1] or Townsend[2]. Here, we wish to bring this to the notice of physics students and make a simplified presentation.

A SG apparatus is an arrangement to provide a spatially inhomogeneous magnetic field. The purpose of spatial inhomogeneity is to exert force on spins, which are like magnetic moments, so that spins of different ori-

entations are spatially separated. The direction of maximum gradient (a measure of inhomogeneity) is the axis along which spatial separation occurs. If this direction is chosen to be the z -axis, the corresponding SG apparatus is said to be oriented along z -axis and it is denoted by SG_z . If the "spin" is indeed like a classical magnetic moment, then every possible orientation with respect to the orientation of the SG_z is possible and the output beam is expected to be continuously distributed along z direction in space. However, experiments indicated that there were finite number of output streams. Particles in each of the stream is assigned a "spin" value. If there are two outputs, the particles in one of the beams are said to be in up-spin state and those in the other output are said to be in the down-spin state. Such particles are said to be "spin-half" particles. Electrons, protons, neutrons, singly ionized silver atoms are some examples of spin-half systems.

2 Tandem Stern-Gerlach Experiments

The need for introducing complex numbers is easily recognized by knowing the results of experiments using two SG apparatuses in tandem. Consider a beam of spin-half system, for example, singly ionized silver atoms, passing through a SG_z . The output of the apparatus will have two beams that are spatially separated. This indicates that the spin of an atom in the beam has two possible values. In quantum theory this is taken to

mean that the required state space is two-dimensional. Associated with these two possible spin values are two quantum states, namely, $|z+\rangle$ and $|z-\rangle$, corresponding to up-spin and down-spin respectively. . An arbitrary spin state $|\psi_{in}\rangle$ is described by a superposition of the two states,

$$|\psi_{in}\rangle = r_1|z+\rangle + r_2|z-\rangle, \quad (1)$$

where r_1 and r_2 are the superposition coefficients that satisfy $r_1^2 + r_2^2 = 1$. A short notation is used to present these facts. A SG apparatus oriented along the z -axis is denoted by Z enclosed in a box. The experimental fact that an arbitrary beam of spin-half systems will give rise to two output beams is represented by

$$|\psi_{in}\rangle \longrightarrow \boxed{Z} \longrightarrow \{|z+\rangle, |z-\rangle\},$$

where the states corresponding to the two output beams are enclosed in curly brackets. The relative intensities of the output beams decide the magnitude of the superposition coefficients. Let us *assume that the superposition coefficients are real*. According to the Born's rule for statistical interpretation, the relative intensities of the beams corresponding to orthogonal states are the squares of the magnitudes of the respective superposition coefficients. In the case of SG_z experiment with two output beams of equal intensity, the input state is a superposition of the two output states:

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}[|z+\rangle + |z-\rangle]. \quad (2)$$

If the input beam is in the state $|z+\rangle$, there is a single output beam corresponding to the

output stream represented by $|z+\rangle$. In the short notation introduced earlier, it is represented as

$$|z+\rangle \longrightarrow \boxed{Z} \longrightarrow |z+\rangle.$$

That is, $|z-\rangle$ cannot be generated from $|z+\rangle$ using SG_z . Similarly, if the input state is $|z-\rangle$,

$$|z-\rangle \longrightarrow \boxed{Z} \longrightarrow |z-\rangle,$$

implying that $|z+\rangle$ cannot be obtained from $|z-\rangle$. In simple terms, SG_z does not affect $|z+\rangle$ and $|z-\rangle$. Hence, they qualify as "eigenstates" of SG_z . More importantly, the fact that SG_z cannot generate $|z+\rangle$ from $|z-\rangle$ implies that the two states $|z+\rangle$ and $|z-\rangle$ are "orthogonal" to each other. In mathematical terms, orthogonality means the inner product between the two states is zero.

The choice of orientation of the SG apparatus is arbitrary. For instance, if the SG apparatus is oriented along x -direction, then an arbitrary input beam of spin- half particles results in two output beams, separated spatially along the x - direction. The respective states of the particles in the two beams are denoted by $|x+\rangle$ and $|x-\rangle$. As in the case of SG_z , the following are true:

$$|\psi_{in}\rangle \longrightarrow \boxed{X} \longrightarrow \{|x+\rangle, |x-\rangle\},$$

$$|x+\rangle \longrightarrow \boxed{X} \longrightarrow |x+\rangle,$$

and

$$|x-\rangle \longrightarrow \boxed{X} \longrightarrow |x-\rangle.$$

And the conclusion is that the states $|x+\rangle$ and $|x-\rangle$ are orthogonal, eigenstates of SG_x . Similarly, for an experiment with SG_y ,

$$|\psi_{in}\rangle \longrightarrow \boxed{Y} \longrightarrow \{|y+\rangle, |y-\rangle\},$$

$$|y+\rangle \longrightarrow \boxed{Y} \longrightarrow |y+\rangle,$$

and

$$|y-\rangle \longrightarrow \boxed{Y} \longrightarrow |y-\rangle.$$

As in the other cases, the states $|y+\rangle$ and $|y-\rangle$ are orthogonal, eigenstates corresponding to SG_y .

2.1 Experiment I

Are $|z+\rangle$ and $|z-\rangle$ unaffected by SG_x ? To find out, one of the outputs of SG_z , say, the beam of particles corresponding to $|z+\rangle$, is used as input to SG_x . The experimental result is that there are two output beams of equal intensity. So, from $|z+\rangle$, both $|x+\rangle$ and $|x-\rangle$ emerge. Then the following assignment is possible:

$$|z+\rangle = \frac{1}{\sqrt{2}}[|x+\rangle + |x-\rangle], \quad (3)$$

Once this choice is made for $|z+\rangle$, the requirement for orthogonality implies that

$$|z-\rangle = \frac{1}{\sqrt{2}}[|x+\rangle - |x-\rangle]. \quad (4)$$

These expressions are consistent with the requirement that $|z+\rangle$ and $|z-\rangle$ are orthogonal to each other. Note that the superposition coefficients are chosen to be real. It does not matter if the expressions for the states $|z+\rangle$ and $|z-\rangle$ are swapped.

2.2 Experiment II

Let one of the outputs of SG_z be sent through a SG_y . Like the previous case, two output

beams of equal intensity emerge from the apparatus. Arguing as before, the results are

$$|z+\rangle = \frac{1}{\sqrt{2}}[|y+\rangle + |y-\rangle], \quad (5)$$

$$|z-\rangle = \frac{1}{\sqrt{2}}[|y+\rangle - |y-\rangle], \quad (6)$$

where the superposition coefficients have been assumed to be real. There is no inconsistency so far.

2.3 Experiment III

The last piece of information required is a relationship among the states $|x\pm\rangle$ and $|y\pm\rangle$. For this, one of the output beams of SG_x , for instance, the output corresponding to $|x+\rangle$, is fed as input to SG_y . Two output beams of equal intensity emerge. If the input is changed to $|x-\rangle$, there are two output beams of equal intensity. So, the results can be summarized as

$$|x+\rangle = \frac{1}{\sqrt{2}}[|y+\rangle + |y-\rangle], \quad (7)$$

$$|x-\rangle = \frac{1}{\sqrt{2}}[|y+\rangle - |y-\rangle], \quad (8)$$

assuming that the superposition coefficients are real.

3 Analysis of results

What can be inferred from the results of the three experiments described above? First of all, the conclusions from the Experiment III

can be used to rewrite the results of the Experiment II. This yields

$$|z+\rangle = |x+\rangle, \quad (9)$$

$$|z-\rangle = |x-\rangle. \quad (10)$$

This is at variance with the results of the Experiment I which indicate that $|z+\rangle$ and $|z-\rangle$ are linear combinations of $|x+\rangle$ and $|x-\rangle$. Obviously, one of the assumptions used in expressing the results should be wrong. The crucial assumption made is that the input state is expressible as a linear combination of output states with *real* coefficients. Now, it needs to be argued that using complex coefficients yields consistent results. The requirements are that the two output beams are of equal intensity and the corresponding states orthogonal to each other. So, one possibility is to recast the results of Experiment III using complex coefficients to give

$$|x+\rangle = \frac{1}{2}[(1-i)|y+\rangle + (1+i)|y-\rangle], \quad (11)$$

$$|x-\rangle = \frac{1}{2}[(1+i)|y+\rangle + (1-i)|y-\rangle]. \quad (12)$$

where $i = \sqrt{-1}$. The definition of inner product between two states $|\psi_1\rangle = a|z+\rangle + b|z-\rangle$ and $|\psi_2\rangle = c|z+\rangle + d|z-\rangle$ is $\langle\psi_1|\psi_2\rangle = a^*c + b^*d$, where superposition coefficients a, b, c and d are complex numbers, and the superscript $*$ implies complex conjugation. With this definition of inner product, the orthogonality condition is satisfied. Further, the coefficients are of equal magnitude to account for the observation that the output beams are of equal intensity. This specific choice of superposition coefficients ensures that the results of the Experiments I and II need not

be rewritten with complex coefficients, and it concurs with the convention adopted in quantum physics. Other choices such as

$$|x+\rangle = \frac{1}{\sqrt{2}}[|y+\rangle + i|y-\rangle], \quad (13)$$

$$|x-\rangle = \frac{1}{\sqrt{2}}[|y+\rangle - i|y-\rangle], \quad (14)$$

to express the results of Experiment III would require rewriting the results of the Experiment I and Experiment II using complex superposition coefficients.

4 Discussion

Complex numbers are essential in the Hilbert space formulation of quantum theory. Without invoking complex numbers, it is impossible to consistently explain the outcomes of some simple experiments performed with SG

devices in tandem. Another important point to note is that the Schrodinger equation has not been used in the arguments presented here. Even though $\sqrt{-1}$ appears explicitly in the Schrodinger equation which governs dynamics in quantum physics, the requirement for complex numbers is not due to this particular rule of dynamics. It is the linear vector space structure that is crucial in necessitating complex numbers in quantum theory.

References

- [1] J J Sakurai, Introduction to Modern Quantum Mechanics(Addison-Wesley, 1994, New York) p27.
- [2] J. S. Townsend, A Modern Introduction to Quantum Mechanics (McGraw Hill, 1992, Singapore) p17.