On the E-L equation and Snell's law for massive particles : a mathematical revisit

D. N. Basu

Variable Energy Cyclotron Centre
1/AF Bidhan Nagar
Kolkata 700 064, India.
dnb@vecc.gov.in

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Abstract

This article describes: (i) the conditions to check whether Euler-Lagrange equation for extremisation provides minimum or maximum and (ii) the derivation of Snell-Descartes law for massive particles which is in contradiction to that for the light waves.

Keywords: Euler-Lagrange equation; Snell-Descartes law.

1 On the Euler-Lagrange equation

In calculus of variations [1], the Euler-Lagrange equation, Euler's equation or Lagrange's equation, is a differential equation whose solutions are the functions for which a given functional is stationary. The Euler-Lagrange equation is useful for solving optimization problems in which, given some functional, one seeks the function minimizing (or maximizing) it because a differentiable functional is stationary at its local maxima and minima. In Lagrangian mechanics, because of Hamilton's principle of stationary action, the evolution of a physical system is described by the solutions to the Euler-Lagrange equation for the action of the system. In classical mechanics, it is equivalent to Newton's laws of motion, but it has the advantage that it takes the same form in any system of generalized coordinates, and it is better suited to generalizations. However, in other practical problems one, very often, needs to know whether the solutions of Euler-Lagrange equation provides local maxima or minima. In this short note, the condition will be obtained that determines whether these solutions represent local maxima or minima.

Let us consider a fixed end point problem defined by the integral

$$I = \int_{x_1}^{x_2} f(y', y, x)dx \tag{1}$$

where x_1 and x_2 are the fixed end points and $f(y', y, x)$ is an explicit function of the $y' = \frac{dy}{dx}$, $y = y(x)$ and x. The problem is now of extremising the functional I. For this let us make variations as

$$y \to y + \alpha\eta, \quad y' \to y' + \alpha\eta' \tag{2}$$

where $\eta = \eta(x)$, $\eta' = \frac{d\eta}{dx}$ and α is independent of x and demand that the functional I is stationary under such variations brought about by the parameter α such that the functional I attains an extremum at $\alpha = 0$ that is $\frac{\partial I}{\partial \alpha}|_{\alpha=0}$. Taylor's expansion up to first order yields

$$f(y'+\alpha\eta', y+\alpha\eta, x) = f(y', y, x)+\alpha\eta f_y+\alpha\eta' f_{y'} \tag{3}$$

where $f_y = \frac{\partial f}{\partial y}$ and $f_{y'} = \frac{\partial f}{\partial y'}$. The condition for extremum can be obtained using the expansion of Eq.(3)in Eq.(1) to provide:

$$\frac{\partial I}{\partial \alpha}|_{\alpha=0} = 0 = \int_{x_1}^{x_2} [\eta f_y + \eta' f_{y'}]dx. \tag{4}$$

The second term on the right hand side of the above equation can be integrated by parts as

$$\int_{x_1}^{x_2} \eta' f_{y'} dx = [\eta f_{y'}]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{df_{y'}}{dx}\eta dx. \tag{5}$$

Since it is a fixed end point problem, the values of $y(x)$ should not vary at the end points x_1 and x_2 implying $\eta(x_1) = 0 = \eta(x_2)$ which makes the first term on the right hand side of Eq.(5) to vanish and one obtains

$$\int_{x_1}^{x_2} [f_y - \frac{df_{y'}}{dx}]\eta dx = 0. \tag{6}$$

Since the function $\eta(x)$ is quite arbitrary, yields the Euler-Lagrange equation:

$$f_y - \frac{df_{y'}}{dx} = 0 \tag{7}$$

which ensures condition for extremum but does not tell whether it corresponds to minimum or maximum. To find these conditions, let us perform Taylor's expansion up to second order:

$$\begin{aligned} f(y' + \alpha\eta', y + \alpha\eta, x) &= f(y', y, x) + \alpha\eta f_y + \alpha\eta' f_{y'} \\ &+ \frac{(\alpha\eta)^2}{2!} f_{yy} + 2\frac{(\alpha^2\eta\eta')}{2!} f_{yy'} + \frac{(\alpha\eta')^2}{2!} f_{y'y'} \end{aligned} \tag{8}$$

and therefore

$$\frac{\partial^2 I}{\partial \alpha^2}|_{\alpha=0} = \int_{x_1}^{x_2} [\eta^2 f_{yy} + 2\eta\eta' f_{yy'} + \eta'^2 f_{y'y'}]dx. \tag{9}$$

Now $\int_{x_1}^{x_2} \eta\eta' f_{yy'} dx = \int_{x_1}^{x_2} [\frac{d}{dx}(\frac{\eta^2}{2})] f_{yy'} dx$ which upon integration by parts gives $[f_{yy'}\frac{\eta^2}{2}]_{x_1}^{x_2} - \int_{x_1}^{x_2} [\frac{d}{dx}(f_{yy'})]\frac{\eta^2}{2} dx$ and since $\eta(x_1)$=0=$\eta(x_2)$, it is just $-\frac{1}{2}\int_{x_1}^{x_2} \eta^2 [\frac{d}{dx}(f_{yy'})]$. Thus

$$\frac{\partial^2 I}{\partial \alpha^2}|_{\alpha=0} = \int_{x_1}^{x_2} [\eta'^2 f_{y'y'} + \eta^2 \{f_{yy} - \frac{d}{dx}(f_{yy'})\}]dx. \quad (10)$$

We can choose $\eta(x)$ to be any arbitrary sawtooth function. Sawtooth function is a continuous function of x but its derivative $\eta'(x)$ is not, rather, η' is alternately $+m$ and $-m$ where m is a constant. But as η'^2 appears in the above equation, we have the advantage that $\eta'^2 = m^2$ which is always positive and remains fixed. Also, the sawtooth function η can be chosen as small as possible ($|\eta| < \epsilon$) while keeping η' same ($= \pm m$). Thus for $|\eta|$ arbitrarily small, it follows that $\frac{\partial^2 I}{\partial \alpha^2} > 0 \;\Rightarrow\; m^2 \int_{x_1}^{x_2} f_{y'y'} dx > 0$, and since m^2 is positive and the preceding arguments hold for any arbitrary fixed end points x_1 and x_2, the inequality

$$f_{y'y'} > 0 \quad (11)$$

represents the necessary condition that the extremisation by Eq.(7) provides minimum and *vice versa*. Therefore, whenever it is necessary to ascertain whether extremisation by Eq.(7) provides minimum or maximum, $f_{y'y'} > 0$ or $f_{y'y'} < 0$ should be checked.

2 Snell's law for waves and massive particles

Snell's law (also known as Snell-Descartes law and the law of refraction) is a formula used to describe the relationship between the angles of incidence and refraction, when referring to light or other waves passing through a boundary between two different isotropic media. It states that the ratio of the sines of the angles of incidence and refraction is equivalent to the ratio of phase velocities in the two media, or equivalent to the opposite ratio of the indices of refraction (with respect to vacuum) resulting in bending of a ray towards the normal (to the boundary separating the two media) for the medium in which velocity of light or other waves is less. The indices of refraction of the media, labeled n_1, n_2 etc. represent the factor by which a light (or other) ray's speed decreases when traveling through a refractive medium as opposed to its velocity in vacuum. Snell's law for waves can be readily proved by Fermat's principle of least time (taken by the light or other waves for traveling from one point to the other across a boundary) or derived from wave nature of light (or other waves) using Huygen's construction [2].

Let us now consider the case of a massive particle (such as neutron) which is incident (from vacuum) with kinetic energy E on a nucleus of radius R which offers a uniform potential $-V$ (where V is positive) to the incident particle. Although massive, if this particle is treated like wave and as its velocity inside the nucleus and vacuum are proportional to $\sqrt{E+V}$ (from energy conservation) and $\sqrt{E}$, respectively, it would result in bending of the ray (as described above) away from the normal (which is along the radius of the nucleus) inside the nucleus where velocity is more. This would lead to the existence of critical angle $sin^{-1}\sqrt{E/(E+V)}$ beyond which there is no transmission (even in an attractive potential of $-V$) and a refractive index n for the nuclear medium less than that of vacuum (which is unity) which are physically unacceptable. Thus, the case of a massive particle can not be treated as wave. For deriving Snell's law for massive particle one can use the principle of angular momentum conservation. If b (the impact parameter) and x are the lengths of the perpendiculars on the incident and deflected paths from the centre of the nucleus, the angular momentum conservation provides the relation $b\sqrt{E} = x\sqrt{E+V}$ where initial (incident) angular momentum is equated to the final angular momentum of the deflected particle. This immediately shows that $x < b$ implying that the refracted (deflected) particle bends towards the centre of the nucleus and the refractive index $n = \sqrt{E+V}/\sqrt{E}$ which is greater than unity. These physically correct results, which were also the intuitive results, are just the opposite of those if the particle were considered as a wave.

3 Summary and Conclusion

In this short note, the necessary conditions of minimum and maximum for extremisation by Euler-Lagrange equation are obtained and the Snell-Descartes law for massive particles is derived which is in contradiction to that for the light waves.

References

[1] H. Goldstein, Classical Mechanics, Chapter 2, Addison-Wesley Publishing Company Inc. (1950).

[2] F. A. Jenkins and H. E. White, Fundamentals of Optics, McGraw-Hill, New York, (1976).