A simple way to solve the brachistochrone problem with resistance

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Abstract

Variational problems are ubiquitous in physics. But an introductory course on the calculus of variations is typically restricted to solving a few standard problems like the classical brachistochrone. Several experiments have clearly shown this theory to be inadequate, because any actual physical situation involves resistance, but no attempt has been made so far to reconcile experiment with theory. Adding resistive forces to the problem makes analytical solutions intractable. We show how such hard variational problems can be easily solved using a simple numerical approach. This allows a large variety of variational problems to be solved at an introductory level and the solution checked against simple experiments. We illustrate this by solving the brachistochrone problem with Coulomb friction and fluid resistance. We outline an experiment which could be used to check the result.

1 Introduction

Variational principles are ubiquitous in physics. Yet an introductory course on the calculus of variations treats very few problems. There is also no simple numerical method (as with ordinary differential equations) through which a larger variety of problems can be examined. The classical brachistochrone problem is the standard problem solved in introductory textbooks [1]. However, the addition of any kind of resistance makes the problem much harder to solve. The addition of Coulomb friction was first examined in [2] and requires a constrained variational technique. An examination of [3] reveals that simple models of fluid resistance lead to very involved algebra.

There have been several experiments on the brachistochrone performed at the undergraduate/high school level and all of them have shown significant deviations from the expected result. Moreover, because the theory with resistance is too complicated, there has been no attempt to incorporate it. As an example, take the isochronous property of the cycloid which is also the brachistochrone without resistance. Introductory physics courses teach that the simple pendulum has an amplitude dependent time period which makes it unsuitable as a clock. The cycloidal pendulum is proposed as the solution to this problem on the belief that it is isochronous. In a recent experiment with high school students however [4], it was found that that a real cycloidal path is not truly isochronous and a definite amplitude-dependence was observed, as is to be expected (however, no theoretical examination was attempted). The obvious cause of the experimental deviations is that a real cycloidal pendulum (as opposed to an idealised one) involves resistance.

Similarly, an experiment with 'Hot-Wheels' cars found the cycloid to be the fastest path among those that were tried, but the authors did not examine the difference in the experimental and theoretical time [5]. Another experiment with undergraduates again found significant difference in the theoretical and experimental time values [6] but did not investigate the possibility that the cycloid is no longer the brachistochrone when friction is included.

We present a simple numerical method which can be used to solve any variation of the problem. In particular, it can be used to *quantitatively* examine how resistive forces affect the solution and hence obtain agreement between theory and experiment. Using this, one can even ask more complicated questions, like, what is the shortest path underwater? Is it still a cycloid? This, too, can be directly linked to a simple experiment (as we explain later).

The brachistochrone problem with friction has been considered by other authors [7, 8, 9, 10]. The numerical approach found in these references is mostly limited to obtaining numerical solutions to the Euler equation. Numerical solutions to partial differential equations are well known. The real difficulty is to first formulate these problems variationally. To get over this difficulty, we use the fact that these situations are simple from a Newtonian point of view. This makes it easy enough to be used in introductory courses.

In a different context, there is a numerical approach to variational problems in mechanics [11]. The algorithm used in [11] does not directly apply to our problem because we seek to minimize the time of descent in the presence of non-conservative forces. Further, our numerical algorithm is useful not only as an educational tool, but can also be applied to solve a wide variety of variational problems where the analytical solution is not feasible. We demonstrate this by solving the brachistochrone problem with fluid resistance and Coulomb friction.

2 Algorithm

The mathematical problem at hand becomes much easier if seen as a physical problem (like a bead sliding down a pipe filled with water). We begin by discretizing the x axis into some N points. Specifying the y values at those points completely defines the curve. We now need to minimize the time it takes for the bead to slide down. It is simple to formulate this from a Newtonian point of view. We need to minimize the time it takes for the bead to travel down a path. The time of travel is obtained by solving the equations of motion. While in most cases, an analytical solution will not exist, it is easy to solve the equations of motion numerically.

To calculate the time in this way we need, first, to construct a path, given the y coordinates at the N points. One could use a straight line between the points. However, this is not a very good choice for the present problem from a numerical and algorithmic point of view because the lines do not join smoothly, and differentiability fails at those points. A better choice is a smoothened polynomial. Let us for the moment say that the path is given by a function $y(x)$. The forces involved are the force of gravity, the buoyant force, Coulomb friction and fluid resistance. For the fluid resistance, we assume a resistance proportional to the square of the velocity. The coefficient will depend on the nature of the fluid and the object. Then the equations of motion for the system are

$$m\ddot{x} = mg_e \sin\theta\cos\theta - \mu m g_e \cos^2\theta - kv\dot{x}, \tag{1}$$

$$m\ddot{y} = -mg_e \sin^2\theta + \mu m g_e \cos\theta\sin\theta - kv\dot{y}, \tag{2}$$

$$\theta = -\tan^{-1} f'(x), \tag{3}$$

$$g_e = \frac{(m - \frac{4}{3}\pi\rho r^3)g}{m}. \tag{4}$$

Here m is the mass, ρ is the density of the fluid, μ is the coefficient of friction, k is the drag coefficient and $v = \sqrt{\dot{x}^2 + \dot{y}^2}$. The dot denotes derivative with respect to time whereas the prime denotes derivative with respect to x. g is the acceleration due to gravity whereas g_e is the effective acceleration due to gravity when the buoyant force is taken into account. The buoyant force has been calculated for a spherical object.

Let us say we want to find the brachistochrone between points (0, 0) and (a, b). The algorithm starts by calculating the time it takes to cover this distance through some initial path (like a straight line). This path is defined by a set of N points (x_i, y_i) between $(0, 0)$ and (a, b), where the x_i points are taken to be fixed. Now, the algorithm proceeds by sequentially updating the y_i points by changing them by a specified small amount. So it starts by increasing (decreasing) y_1. This gives a new curve $y(x)$. The equations of motion are solved again to obtain a new time of descent. If this time of descent is smaller than the previous one, y_1 is increased (decreased) again. This is continued till a change in y_1 leads to an increase in the time of descent. Then the algorithm proceeds to y_2 and repeats the same process. After reaching y_N the algorithm comes back and updates y_1 again. It stops when no possible step leads to a decrease in time.

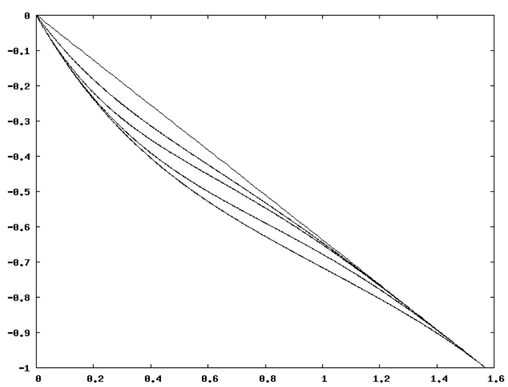

Figure 1: The plot shows typical curves the algorithm tries before reaching the solution.

3 Results

For the present problem, we use a Bezier curve to interpolate between the points. Even though we are solving 2 equations of motion, the curve is actually defined by 1 parameter, say t. We use a Runge-Kutta 4th order solver to solve the equations of motion. Though the equations of motion implicitly constrain the object to move along the curve, it is possible for numerical errors to develop. Hence, for each step the ODE solver takes, rather than computing the derivative of the curve (which is required in the equations of motion) from either the current x or y position of the object, the x, y position is first mapped to t using a simple linear search. The derivative of the curve is then calculated at point t. We show typical steps in the algorithm in Figure 1. If needed, a more sophisticated optimization algorithm can also easily be applied to the problem as formulated. As an example, the simulated annealing algorithm can be used since the problem has a cost function as well as a specified way to change its state. We show simulation results for different values of $\frac{k}{m}$ in Figure 2. The least time curves obtained are between the cycloid and the straight line. As the drag coefficient increases, the curves start resembling a straight line. For a large enough drag coefficient, the least time curve is the straight line.

To check this independently, it is possible to perform a simple experiment for a fluid (say, water) which we briefly describe. This experiment requires only a flexible pipe filled with water. By fastening the pipe at some appropriate points it can be made to resemble a smooth curve passing through those points. Even though the exact shape of the pipe might be difficult to ascertain, the shape of the pipe can be approximated by a smoothened polynomial through those points. The time a ball bearing takes to slide down the pipe can be measured and hence the time it takes to slide down different curves can be experimentally compared.

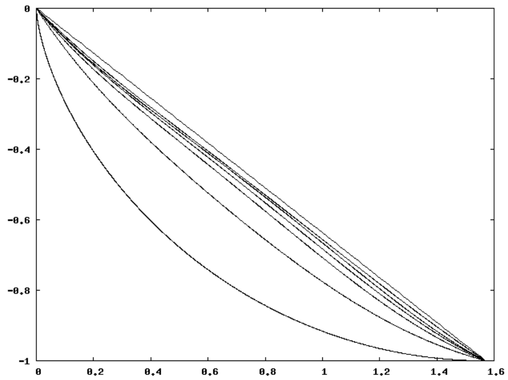

Figure 2: The plot shows the brachistochrone between points (0, 0) and ($\frac{\pi}{2}$, -1). The effective gravity because of the buoyant force is 8.72 m/s^2. We have $\mu = 0.1$. Starting from above the plots are that of a straight line, the curve obtained for $\frac{k}{m} = 11$, $\frac{k}{m} = 7$, $\frac{k}{m} = 5$, $\frac{k}{m} = 3$, $\frac{k}{m} = 1$ and the cycloid between the two points.

4 Concluding remarks

The brachistochrone problem has earlier been suggested to be the best introduction to variational calculus. On this note, an earlier project with undergraduates tried to analyze this problem in detail using both theory and experiment [6]. The difference in theory and experiment (due to resistance) could not be addressed since their numerical method was limited to evaluating the time integral of the classical brachistochrone problem for different curves. We have shown how this problem, which is hard even to formulate analytically, from a variational point of view, can be easily solved using a simple numerical scheme. Moreover, the solution can be checked with experiments easy enough to perform in the classroom.

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