Physics Through Problem Solving - XXIV
Poisson Brackets

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Abstract

In this installment we shall do some problems with Poisson brackets. The Poisson brackets can be used to state the equation of motion (i.e., time dependence in the form of a differential equation) of any function of coordinates and momenta (i.e, 'a dynamical variable') in a very elegant manner which emphasizes the role played by the Hamiltonian function and the constants of motion. The problems are meant to demonstrate these aspects of Poisson brackets

Consider a system of n degrees of freedom, whose phase space coordinates are $\mathbf{q} = \{q_1, q_2, \ldots, q_n\}$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}$. The Poisson bracket (PB for short) of two dynamical variable of this system, $u(\mathbf{p}, \mathbf{q}, t)$ and $v(\mathbf{p}, \mathbf{q}, t)$, is defined as

$$[u, v] = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right) \quad (1)$$

The range of summation is from $i = 1$ to $i = n$, and will be assumed in all the following expressions, unless otherwise mentioned. But the index of summation will be always mentioned (no summation convention used anywhere).

Here we summarize some basic properties of Poisson brackets which we will be using in solving the problems of this issue. In the following u, v and w are three dynamical variables and c is a constant.

$$[u,v] = -[v,u] \quad (2)$$
$$[u,u] = 0 \quad (3)$$
$$[cu,v] = [u,cv] = c\,[u,v] \quad (4)$$
$$[u+v,w] = [u,w]+[v,w] \quad (5)$$
$$[u,v+w] = [u,v]+[u,w] \quad (6)$$
$$[uv,w] = u[v,w]+[u,w]v \quad (7)$$
$$[u,vw] = v[u,w]+[u,v]w \quad (8)$$

Problem 1: Find the PBs $[L_x, L_y], [L_y, L_z]$ and $[L_z, L_x]$, where L_x, L_y, L_z are the Cartesian components of the angular momentum of a particle.

Solution: We have $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, and $L_z = xp_y - yp_x$, where p_x, p_y, p_z are canonically conjugate momenta for the Cartesian coordinates x, y, and z respectively. For convenience, let us rename the variables as follows: $\{x, y, z\} \to \{x_1, x_2, x_3\}$, $\{p_x, p_y, p_z\} \to \{p_1, p_2, p_3\}$, and $\{L_x, L_y, L_z\} \to \{L_1, L_2, L_3\}$. With this notation we have $L_1 = x_2p_3 - x_3p_2$, $L_2 = x_3p_1 - x_1p_3$, $L_3 = x_1p_2 - x_2p_1$.

A compact and elegant method of carrying out the algebra to find these PBs is by using Levi-Civita symbol ϵ_{ijk} and the Kronecker delta symbol δ_{ij}. This you can find in some text books, for example in Rana & Joag [1]. Here we shall carry out the algebra without using these neat mathematical devices. In fact, we shall calculate only the first PB $[L_1, L_2]$, and the other two can be readily calculated in the same manner by the reader.

$$[L_1, L_2] = \sum_{i=1}^{3}\left(\frac{\partial L_1}{\partial x_i}\frac{\partial L_2}{\partial p_i} - \frac{\partial L_1}{\partial p_i}\frac{\partial L_2}{\partial q_i}\right)$$

(using eq. 1)

In the above sum the first two terms vanish as $\frac{\partial L_1}{\partial x_1} = \frac{\partial L_1}{\partial p_1} = \frac{\partial L_2}{\partial p_2} = \frac{\partial L_2}{\partial x_2} = 0$. Thus

$$\begin{aligned}[L_1, L_2] &= \left(\frac{\partial L_1}{\partial x_3}\frac{\partial L_2}{\partial p_3} - \frac{\partial L_1}{\partial p_3}\frac{\partial L_2}{\partial x_3}\right) \\ &= x_1p_2 - x_2p_1 = L_3\end{aligned}$$

In the same manner we get $[L_2, L_3] = L_1$ and $[L_3, L_1] = L_2$. Note the cyclical order of the indices. If the order is not cyclical we get negative signs, e.g., $[L_2, L_1] = -L_3$, from the anti-commutative property of PBs as stated in eq. 2. Also, by the property given by eq. 3 (which is actually a corollary of eq. 2) we have $[L_1, L_1] = [L_2, L_2] = [L_3, L_3] = 0$. We note that the quantum analogue of these brackets (commutator brackets) are given by $[L_1, L_2] = i\hbar L_3$ and so forth.

Problem 2: Using the Poisson theorem for PBs show that the angular momentum (about the centre of force) is a constant of motion for the motion of a particle under an inverse square law force.

Solution: Poisson theorem (also called Poisson's first theorem on PBs) states that for a dynamical variable $u(\mathbf{q}, \mathbf{p}, t)$

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \quad (9)$$

This is actually the equation of motion of u. By definition u is a constant of motion if $\frac{du}{dt} = 0$.

The angular momentum **L** is a vector, and to show that it is conserved to we have to show that all the three components L_x, L_y and L_z are conserved. These components have no explicit time dependence, so the partial derivatives $\frac{\partial L_x}{\partial t} = \frac{\partial L_y}{\partial t} = \frac{\partial L_z}{\partial t} = 0$. Thus, from eq. 9 the three components are conserved if the three PBs $[L_x, H] = [L_y, H] = [L_z, H] = 0$. We shall prove one of them, i.e, $[L_z, H] = 0$, and the reader can easily prove the other two in the same manner. Once again for the components of position, momentum and angular momentum we shall use the notation used in problem 1.

We shall use Cartesian coordinates with the centre of force at the origin. The Hamiltonian is given by

$$\begin{aligned} H(x_1, x_2, x_3, p_1, p_2, p_3) &= \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) - \frac{k}{r} \\ &= \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) - \frac{k}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \end{aligned} \quad (10)$$

where k is a constant, positive for attractive force and negative for repulsive force.

$$[L_3, H] = \sum_{i=1}^{3}\left(\frac{\partial L_3}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial L_3}{\partial p_i}\frac{\partial H}{\partial x_i}\right) \quad (11)$$

We have the partial derivatives

$$\frac{\partial L_3}{\partial x_1} = p_2,\ \frac{\partial L_3}{\partial x_2} = -p_1,\ \frac{\partial L_3}{\partial x_3} = 0,$$

$$\frac{\partial L_3}{\partial p_1} = -x_2,\ \frac{\partial L_3}{\partial p_2} = x_1,\ \frac{\partial L_3}{\partial p_3} = 0,$$

$$\frac{\partial H}{\partial p_1} = \frac{p_1}{m},\ \frac{\partial H}{\partial p_2} = \frac{p_2}{m},\ \frac{\partial H}{\partial p_3} = \frac{p_3}{m},$$

$$\frac{\partial H}{\partial x_1} = \frac{kx_1}{r^{3/2}},\ \frac{\partial H}{\partial x_2} = \frac{kx_2}{r^{3/2}},\ \text{and}\frac{\partial H}{\partial x_3} = \frac{kx_3}{r^{3/2}}$$

Using these in eq. 11 we have $[L_3, H] = 0$.

Problem 3: Show that for a free particle moving in one dimension, the function $F = x - \frac{pt}{m}$ and $\frac{\partial F}{\partial t}$ are constants of motion. Here x, p, and m are position, momentum and mass of the particle. Do this by direct calculation of total time derivatives of F as well as $\frac{\partial F}{\partial t}$, and by using Poisson's first and second theorem about PBs.

Solution: Note that F is explicitly a function of time, but nevertheless it is a constant of motion. This is quite trivial to show by taking the total time derivative of F. We use that fact that for a free particle momentum p is a constant.

$$\begin{aligned} \frac{dF}{dt} &= \frac{dx}{dt} - \frac{p}{m} \\ &= \frac{p}{m} - \frac{p}{m} = 0. \end{aligned}$$

And

$$\frac{d}{dt}\left(\frac{\partial F}{\partial t}\right) = \frac{d}{dt}\left(-\frac{p}{m}\right) = 0$$

Now we use Poisson's first theorem stated above in the previous problem. Note that the Hamiltonian for the free particle is $H = \frac{p^2}{2m}$.

$$\begin{aligned} \frac{dF}{dt} &= [F,H] + \frac{\partial F}{\partial t} \\ &= \left(\frac{\partial F}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial x}\right) - \frac{p}{m} \\ &= \left(\frac{p}{m} - 0\right) - \frac{p}{m} = 0 \end{aligned}$$

And, as $\frac{\partial F}{\partial t} = -\frac{p}{m}$ is not an explicit function of time, if it is a constant of motion its PB with H must be zero, as we can see:

$$\begin{aligned} \left[\frac{\partial F}{\partial t}, H\right] &= \left[-\frac{p}{m}, \frac{p^2}{2m}\right] \\ &= -\frac{1}{2m^2}\left[p, p^2\right] \\ &= -\frac{1}{2m^2}\left(p\,[p,p] + [p,p]\,p\right) = 0 \end{aligned}$$

In the above we have used PB properties given by eqs. 3, 4 and 8 for illustration, though here it is equally simple to take the partial derivatives. We can also illustrate Poisson's second theorem on PBs, which states that the PB of two constants of motion is itself a constant of motion. Now that we know F is a constant of motion, we can take its PB with H, which is also a constant of motion (because the particle is free). This PB we already evaluated above, i.e., $[F,H] = \frac{p}{m} = -\frac{\partial F}{\partial t}$. So by Poisson's second theorem $\frac{\partial F}{\partial t}$ must be a constant of motion, as we already verified.

This demonstrates one valuable application of Poisson's second theorem: If we have two constants of motion, we can take their PB to construct one more constant of motion, which might of interest. But it can also turn out some function of already known constants of motion, which can hardly be of any interest. Consider this example. Here we have two constants of motion, F and $\frac{\partial F}{\partial t}$, and their PB is

$$\begin{aligned} \left[F, \frac{\partial F}{\partial t}\right] &= \left[x - \frac{pt}{m}, -\frac{p}{m}\right] \\ &= \left[x, -\frac{p}{m}\right] + \left[-\frac{pt}{m}, -\frac{p}{m}\right] \\ &= -\frac{1}{m}[x,p] - \frac{t}{m^2}[p,p] = -\frac{1}{m} \end{aligned}$$

(using the properties of PBs given by eqs. 3, 4 and 8, and $[x,p] = 1$), which is obviously a constant of motion, and not a terribly interesting one, as all it means is that mass remains constant during the motion.

Problem 4: Consider the following functions of position q and momentum p of a one-dimensional harmonic oscillator (m is the mass and ω angular frequency) :

$$a = \sqrt{\frac{m\omega}{2}}\left(q + \frac{ip}{m\omega}\right) \qquad (12)$$

and its complex conjugate

$$a^* = \sqrt{\frac{m\omega}{2}}\left(q - \frac{ip}{m\omega}\right) \qquad (13)$$

Write down and solve the equations of motion for a and a^* in terms of PBs of these functions with the Hamiltonian, and from these solutions find the solutions $q(t)$ and $p(t)$.

Solution: The equations of motion for a and a^* are (from Poisson's first theorem discussed above)

$$\frac{da}{dt} = [a, H] \tag{14}$$

and

$$\frac{da^*}{dt} = [a^*, H] \tag{15}$$

Note that the partial derivatives $\frac{\partial a}{\partial t}$ and $\frac{\partial a^*}{\partial t}$ are absent in the above equations of motion as both are zero, because the functions a and a^* are not explicitly time-dependent. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{16}$$

We can evaluate the the PBs $[a, H]$ and $[a^*, H]$ by using the general definition given in eq. 1 or by using the properties of PBs listed in eqs. 2 – 8. We use the latter method, because that will also allow the reader to compare the functions a and a^* with their quantum mechanical analogues – the lowering and raising operators a and $a^\dagger$. Consider first the product

$$\begin{aligned} aa^* &= \sqrt{\frac{m\omega}{2}}\left(q + \frac{ip}{m\omega}\right)\sqrt{\frac{m\omega}{2}}\left(q - \frac{ip}{m\omega}\right) \\ &= \frac{m\omega}{2}\left(q^2 + \frac{p^2}{m^2\omega^2}\right) \end{aligned}$$

Comparing this with eq.16 we immediately get

$$H = \omega a a^* \tag{17}$$

Let us also get

$$\begin{aligned} [a, a*] &= \frac{\partial a}{\partial q}\frac{\partial a^*}{\partial p} - \frac{\partial a}{\partial p}\frac{\partial a^*}{\partial q} \\ &= \frac{m\omega}{2}\left(-\frac{i}{m\omega}\right) - \frac{m\omega}{2}\left(\frac{i}{m\omega}\right) \\ &= -i \end{aligned} \tag{18}$$

Now

$$\begin{aligned} [a, H] &= [a, \omega a a^*] \quad \text{(using eq. 17)} \\ &= \omega\,[a, aa^*] \quad \text{(using eq. 4)} \\ &= \omega\,(a\,[a, a^*] + [a, a]\,a^*) \ \text{(using eq. 8)} \\ &= -i\omega a \end{aligned} \tag{19}$$

In the last line we used

$$[a, a] = 0 \text{ and } [a, a^*] = -i.$$

Thus we have the equation of motion for a (eq. 14)

$$\frac{da}{dt} = -i\omega a \tag{20}$$

which can be immediately integrated to give

$$a = a_0 e^{-i\omega t} \tag{21}$$

where a_0 is the constant of integration.

Similar calculations give us $[a^*, H] = i\omega a^*$ and using it in the equation motion of eq. 15, and integrating we get

$$a^* = a_0^* e^{i\omega t} \tag{22}$$

where a_0^* is the constant of integration.

Now to find the solutions $q(t)$ and $p(t)$, we solve eqs. 12 and 13 for q and p to get

$$q = \sqrt{\frac{1}{2m\omega}}(a + a*) \tag{23}$$

and

$$p = -i\sqrt{\frac{m\omega}{2}}(a + a*) \tag{24}$$

Now using eqs. 21 and 22 in eqs. 23 and 24 we have

$$q = \sqrt{\frac{1}{2m\omega}}\left(a_0 e^{-i\omega t} + a_0^* e^{i\omega t}\right) \tag{25}$$

$$p = -i\sqrt{\frac{m\omega}{2}}\left(a_0 e^{-i\omega t} - a_0^* e^{i\omega t}\right) \tag{26}$$

References

[1] N. C. Rana and P. S. Joag *Classical Mechanics* (Tata McGraw-Hill, New Delhi (1991).