Physics Through Problem Solving XXV: Small Oscillations

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Abstract
In this issue we solve some problems on small oscillations.

When a system of particles is disturbed slightly from its equilibrium configuration momentarily and released, it will undergo oscillations about the equilibrium configuration. These oscillations in general can be nonlinear and very complicated and very difficult to analyze. But there is one special category of oscillations where the problem simplifies greatly – and this is the category of 'small oscillations'. If a collection of mutually interacting particles oscillate in such a way that the forces acting on them can be well approximated by Hooke's law forces, they can be considered small oscillations. It so happens that any restoring force can be approximated by a Hooke's law force if the displacement of the particle from the equilibrium position (i.e., the position where the restoring force is zero) is small enough. For a full discussion of the nature of small oscillations the reader should refer any of the standard classical mechanics texts, such as the ones by Goldstein, Poole & Safko [2] or by Rana & Joag [1].

The first two problems below are short, and are meant to illustrate how some arbitrary restoring force acting on a single particle is approximated by a Hooke's law force. The third is a longer problem with two particles, which is meant to illustrate the general method of solving the motion of coupled oscillators. In this issue we will only find the normal frequencies for a system of coupled oscillators. In a future issue we shall consider the problem of finding the amplitudes in terms of normal modes.

Problem 1

Consider the Lennard-Jones Potential, which is used as an approximate model for the potential energy of interaction between two neutral atoms or molecules:

$$V(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right] \tag{1}$$

where r is the separation between the two atoms or molecules, and $\epsilon > 0$ and $\sigma > 0$ are constants. For an atom or molecule moving under this potential, find the frequency of small oscillations about the position of stable equilibrium.

Solution

What we are going to do here is to find an approximate expression of the form $\frac{1}{2}kr^2$ for the potential energy $V(r)$ ($k > 0$ is the spring constant), so that the resulting force is the Hooke's law force $F = -kr$, and the motion is simple harmonic with the frequency $w = \sqrt{k/m}$, where m is the mass of the particle.

At stable equilibrium position $r = r_0$ the potential energy given by eq. 1 is a minimum, which means

$$\begin{aligned} \left(\frac{\partial V}{\partial r}\right)_{r=r_0} &= 0 \\ \Longrightarrow 4\epsilon\left(-\frac{12\sigma^{12}}{r^{13}} + \frac{6\sigma^6}{r^7}\right)_{r=r_0} &= 0 \\ \Longrightarrow 4\epsilon\left(-\frac{12\sigma^{12}}{r_0^{13}} + \frac{6\sigma^6}{r_0^7}\right) &= 0 \\ \Longrightarrow r_0 = 2^{1/6}\sigma & \end{aligned} \tag{2}$$

The spring constant for small oscillations is given by

$$\begin{aligned} k &= \left(\frac{\partial^2 V}{\partial r^2}\right)_{r=r_0} \\ &= 4\epsilon\left(-\frac{12\sigma^{12}}{r_0^{13}} + \frac{6\sigma^6}{r_0^7}\right) \end{aligned}$$

Substituting for r_0 from eq.2 and simplifying we get

$$k = \frac{72\epsilon}{2^{1/3}\sigma^2} \tag{3}$$

Note that $k > 0$, as it should be, because it is the value of the second derivative of $V(r)$ at the minimum. And also we require the spring constant to be positive. The frequency of small oscillations is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{72\epsilon}{2^{1/3}\sigma^2 m}}$$

where m is the mass of the particle.

Problem 2

The following potential, know as Morse potential is sometimes used as an approximate model for the potential energy of interaction between atoms in a diatomic molecule:

$$V(r) = D\left(1 - e^{-\alpha(r-a)}\right)^2 \tag{4}$$

where r is the separation between the two atoms, and $D > 0$, $\alpha > 0$, and $a > 0$ are constants. For an atom moving under this potential, find the frequency of small oscillations about the position of the stable equilibrium.

Solution

Once again we first find the equilibrium position $r = r_0$, which is the minimum position for the potential energy $V(r)$ of eq. 4.

$$\left(\frac{\partial V}{\partial r}\right)_{r=r_0} = 0$$

$$\Longrightarrow 2D\alpha\left((r-a)e^{-\alpha(r-a)^2}\right)_{r=r_0} = 0$$
$$\Longrightarrow 2D\alpha(r_0-a)e^{-\alpha(r_0-a)^2} = 0$$
$$\Longrightarrow r_0 = a \qquad (5)$$

The last equation follows by noting that $e^{-\alpha(r-a)^2}$ is never zero for finite r and therefore $(r_0 - a) = 0$. And the spring constant is

$$\begin{aligned} k &= \left(\frac{\partial^2 V}{\partial r^2}\right)_{r=a} \\ &= 2D\alpha\left(e^{-\alpha(r-a)^2} + 2\alpha(r-a)^2 e^{-\alpha(r-a)^2}\right)_{r=a} \\ &= 2D\alpha \end{aligned} \qquad (6)$$

Once again k is positive, as it must be. The frequency of small oscillations on the equilibrium position, for a particle of mass m is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2D\alpha}{m}}$$

Problem 3
Find the normal frequencies of oscillations for a double pendulum undergoing small oscillations in a single vertical plane. Consider the special case of equal masses and the lengths.

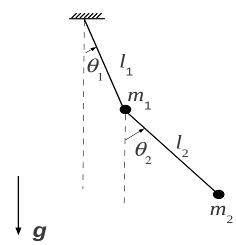

Figure 1: A double pendulum. See the text for description.

Solution Let me briefly describe what are ‘normal frequencies’. In a system of couple oscillators, even when the oscillations are small and the forces can be approximated by Hooke’s law forces, the general motion is not simple harmonic or even periodical. But it can always be represented as a superposition of simple harmonic motions. The frequencies of these simple harmonic motion are called normal frequencies. It is possible for the system to oscillate in such a way that all of the particles undergo simple harmonic motions with a single frequency, which is one of the normal frequencies. Such a motion is called a normal mode. In following we shall find the normal frequencies for the double pendulum and we shall find the corresponding normal modes in another issue.

The figure 1 shows the the double pendulum – where m_1, m_2 are the masses, l_1, l_2 are the two string lengths, and θ_1, θ_2 are the two angular displacements from the equilibrium position, which is $\theta_1 = \theta_2 = 0$. **g** is the acceleration due to gravity. This is a system with two degrees of freedom, and we can use θ_1, θ_2 as our generalized coordinates. Note that for the analysis of small oscillations it is convenient if we choose generalized coordinates such that they take values zero at the equilibrium position, and that is what we have done here. Finding the potential and kinetic energies in generalized coordinates for a double pendulum is a very routine exercise in classical mechanics courses, and so we omit the details and directly write down the expressions. We hope the reader can easily verify them. Taking the potential energy as zero at $\theta_1 = \theta_2 = 0$. **g**, we get

$$V(\theta_1, \theta_2) = -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \tag{7}$$

$$T(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) = \frac{1}{2} m_1 l_1^2 \dot\theta_1{}^2 + \frac{1}{2} m_2 \left[l_1^2 \dot\theta_1{}^2 + l_2^2 \dot\theta_2{}^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2) \right] \tag{8}$$

The above expressions 7 and 8 for the potential and the kinetic energies of the system are valid for all value of generalized positions θ_1, θ_2 and the generalized velocities $\dot\theta_1, \dot\theta_2$. But we need to approximate these expressions for small values of the positions and velocities. These expressions take the following form:

$$V = \frac{1}{2} \sum_i \sum_j V_{ij} \theta_i \theta_j \tag{9}$$

$$T = \frac{1}{2} \sum_i \sum_j T_{ij} \dot\theta_i \dot\theta_j \tag{10}$$

where $i, j = 1, 2$, and the coefficients V_{ij} and T_{ij} are constants (i.e, not functions of coordinates or velocities) to be determined. In the case of potential energy these coefficient are nothing but the Taylor series expansion coefficients about the equilibrium position (compare with the problems 1 and 2 above). Thus V_{11} is the coefficient of the first non-vanishing term in the Taylor series expansion in the variable θ_1:

$$\begin{aligned} V_{11} &= \left(\frac{\partial^2 V}{\partial \theta_1^2} \right)_{\theta_1 = 0} \\ &= (m_1 g l_1 \cos\theta_1 + m_2 g l_1 \cos\theta_1)_{\theta_1 = 0} \\ &= (m_1 + m_2) g l_1 \end{aligned}$$

Similarly

$$\begin{aligned} V_{12} &= \left(\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} \right)_{\theta_1 = \theta_2 = 0} \\ &= 0 \\ V_{21} &= V_{12} = 0 \end{aligned}$$

$$V_{22} = \left(\frac{\partial^2 V}{\partial \theta_2^2}\right)_{\theta_2=0} = m_2 g l_2$$

Thus the potential energy of eq. 9 is

$$V = \frac{1}{2}\left[(m_1+m_2) g l_1 \theta_1^2 + m_2 g l_2 \theta_2^2\right] \quad (11)$$

We do a similar exercise to the kinetic energy in the form of eq. 10. The idea is to take the Taylor series expansion of the kinetic energy given in eq. 8 such that only the terms quadratic in velocities (i.e, the terms $\dot{\theta}_1^{\,2}, \dot{\theta}_2^{\,2}, \dot{\theta}_1\dot{\theta}_2$) are retained. We can see that eq. 8 is already in that form, except for the factor $\cos(\theta_1-\theta_2)$ in the last term. Therefore we expand this factor in Taylor series and retain the first non-zero term, which is 1. Thus, the kinetic energy in eq. 10 is given by

$$\begin{aligned} T(\dot{\theta}_1,\dot{\theta}_2) &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^{\,2} + \frac{1}{2} m_2\left[l_1^2\dot{\theta}_1^{\,2} + l_2^2\dot{\theta}_2^{\,2} + 2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\right] \\ &= \frac{1}{2}\left[(m_1+m_2) l_1^2\dot{\theta}_1^{\,2} + m_2 l_2^2 \dot{\theta}_2^{\,2} + 2 m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\right] \end{aligned} \quad (12)$$

And thus $T_{11} = (m_1+m_2) l_1^2, T_{12} = T_{21} = m_2 l_1 l_2, T_{22} = m_2 l_2^2$ (You have to note that $2m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 = m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_2\dot{\theta}_1$ to correctly compare eqs. 10 and 12 and identify T_{12} and T_{21}).

Thus eqs. 11 and 12 give us the two matrices:

$$\begin{aligned} \mathbb{V} &= \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \\ &= \begin{bmatrix} (m_1+m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \mathbb{T} &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \end{aligned} \quad (14)$$

At this point we consider the special case of equal masses and lengths, i.e. $m_1 = m_2 = m$ and $l_1 = l_2 = l$. Then the two matrices simplify to

$$\mathbb{V} = \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix} \quad (15)$$

And

$$\mathbb{T} = \begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix} \quad (16)$$

The normal frequencies ω are given by the solutions to the equation

$$\begin{aligned} det\left(\mathbb{V} - \omega^2\mathbb{T}\right) &= 0 \\ det\begin{bmatrix} 2mgl - 2ml^2\omega^2 & -ml^2\omega^2 \\ -ml^2\omega^2 & mgl - ml^2\omega^2 \end{bmatrix} &= 0 \end{aligned}$$

We can simplify the above equation substituting $g = l\omega_0^2$, and canceling out ml^2 throughout (also note that $\omega_0 = \sqrt{g/l}$ happens to be the frequency of a simple pendulum of length l), to get

$$det \begin{bmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2 - \omega^2) \end{bmatrix} = 0$$

Expanding the determinant and simplifying we get

$$\omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$$

which is a quadratic equation in ω^2 and has two solutions

$$\omega_1^2 = (2 + \sqrt{2})\omega_0^2$$

$$\omega_2^2 = (2 - \sqrt{2})\omega_0^2$$

Thus ω_1 and ω_2 are the normal frequencies.

References

[1] N C Rana and P S Joag (1991). Classical Mechanics, Tata McGraw Hill, New Delhi

[2] H Goldstein & C P Poole and J Safko Classical Mechanics, Pearson, New Delhi