On Bound State of Cooper Pairs in Superconductors

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Abstract

In this pedological article, we discuss about the boundedness of two electrons above the Fermi sphere in three dimensions interacting via an attractive interaction. These two electrons having equal opposite moneta and spin are known as Cooper pairs. According to the spin content of this composite particle, they behave approximately as Bosons and the pairs are essential ingredient to theory of superconductivity. We compute the bound state of energy of the pair when the electrons interact via (i) constant potential (ii) potential separable in two coordinates and (iii) delta function potential above the spherical Fermi surface. With the help of the eigenstate, we proceed further to compute the average radius of the Cooper pair and discuss its implication in theory of superconductivity. We also generalize the form of binding energy of Cooper pair in case of ellipsoidal Fermi surface.

1.Introduction

In nature, all particles can be classified into Bosons and Fermions according to their statistics. Electrons, protons and neutrons are all Fermions. An atom, which contains all three can also be treated as single (composite) particle. Whether the composite is Bosonic or Fermionic depends on the total number of its constituents. For example, He^4 contains two electrons, two protons and two neutrons and thus, it is a Boson. But, the

isotope He^3 is a Fermion. Fermions obey the Pauli exclusion principle while there is no such restriction on Bosons. As a result, a collection of Bosons behaves quite differently from a collection of Fermions. A good example is the dramatic difference between a superconductor and an ordinary metal. The electrical conductivity in ordinary metals can be understood in terms of the properties of Fermions (i.e. electrons); in contrast, superconductivity in terms of Cooper pairs which are Boson-like. Moreover, the response of spinless Bosons in an external magnetic field is diamagnetic [1] while the Fermions show paramagnetic behaviour.

Cooper pair consists of two electrons with equal and opposite momenta and spins. In other words, they have zero total momentum. At first instance, one might wonder that they will violate Heisenberg uncertainty relation. It is worthy to remember that the zero expectation value of momentum does not mean a zero uncertainty of momentum. In fact, there is indeed substantial amount of fluctuation of momentum of these pairs which makes the non-zero uncertainty of the quantity. Moreover, we will explicitly use the uncertainty relation to estimate the size of the Cooper pair. Note that the electrons are fermions having spin $\frac{1}{2}$ (in units of $\hbar$) while Cooper pairs are composite Bosons as its total spin is integer (0 or 1). The wave functions are symmetric under particle interchange and they are allowed to be in the same state. The tendency for all the Cooper pairs in a system is to condense into the same ground state as they are Bosons. This fact is qualitatively responsible for the peculiar properties of the superconductivity. In normal superconductors, Cooper pairs have zero spin while for He^3 system, the spin of the pair is one (in units of $\hbar$). Here, the pairs are formed by two He^3 atoms. The pairs in He^3 is responsible for the superfluidity phenomenon at low enough temperature. In case of normal superconductors, the spins are paired in such a way that the magnetic moment of the electrons cancels and the contribution of the pair to the magnetic properties will in general be a diamagnetic one. This is reflected by the Meissner effect [2, 3] seen in the superconductor.

If these two electrons interact via an attractive potential, above the filled Fermi surface, will they form a bound state? This problem was attacked [4] by L. N. Cooper in 1956 and after his name, this is known as Cooper pairing problem. This problem is a simple two particle problem in quantum mechanics that can reduced effectively to a single particle because of the nature of potential function. However, in contrast to normal hydrogen atom problem, there is another statistical interaction through the Pauli exclusion principle to the filled Fermi sea. By considering this statistical interaction, Cooper showed exactly that these electrons do form a stable bound state and the *normal* Fermi surface becomes unstable. Later on Bardeen, Cooper and Schrieffer [5] used these collection of Cooper pairs to form the stable bound state of the superconductor and showed a gap between the metal and superconducting one. This theory is known as BCS one. The experimental foundations of the so called BCS theory has been beautifully demonstrated by Ginsberg [6]. At the same time, Anderson

[7] also developed another theory based on Pseudo-spin analogy to obtain a finite gap at finite non-zero temperature between metal and superconductor. A lucid account of this feature [8] has been sketched for the graduate students. Apart from normal superconductors, BCS theory is also applicable to other fermionic system such as He^3. Immediately after the tremendous success of BCS theory, Bohr, Mottelson and Pines [9] used the BCS pairing analogy to explain the large gaps in the spectra of even-even nuclei. Thus, BCS theory has been widely used to describe superconductivity in condensed matter and nuclear systems. The nuclear BCS theory has been reviewed recently by Broglia [10]. Besides, the spin pairing of the electrons essentially influences the paramagnetic susceptibility. In fact, the paramagnetic susceptibility should decrease towards zero with temperature $T \rightarrow 0$. The concept of Cooper pairs have been used in discussing the neutrino emission in relation with neutron stars [20]. The anisotropic d-wave Cooper pair wave functions in high T_c superconductors has been considered [21] to explain the spin fluctuations in cuprates.

The question next comes to our mind is: what is the reason of studying these pairs? The formation of pairs is indeed surprising. Because two particles of equal charge should repel each other. But as we have mentioned only the attraction between the electrons will force them to bind them. Therefore, naturally one is eager to know the origin of attraction between these electrons to form Cooper pairs. It is to be noted that two free electrons in a vacuum/free space always repel each other. Thus, it is the medium which is responsible for such an attraction between them. The electrons under consideration are in a dielectric medium of the solid characterized by a dynamic dielectric constant $\epsilon(\omega)$. Under an appropriate condition on the frequency ω, the dielectric constant $\epsilon(\omega)$) can change from a positive value to a negative one making the interacation to be negative. In our modern condensed matter physics, it is the quantized unit of lattice vibration known as *phonon* which is responsible for such an effective attractive interaction between the electrons. It is worthy to mention that like photons, phonons are bosons with zero chemical potential. Unlike photons, phonons have transverse polarization and the wave vectors associated with them are restricted to Brillouin zone only.

When an electron passes through the solid, on account of its negative charge, leaves behind a deformation trail affecting the position of the ion cores . This trail is associated with an increased density of positive charge due to the ion cores, and thus has an attractive effect on a second electron. Therefore, the lattice deformation causes a weak attraction between the pair of electron. Thus, the balance between of electron-phonon interaction [2, 3, 12, 13, 14, 16] and the Coulomb interaction in a material crucially determines whether a given material is superconducting or not. Since, electrons possess both charge as well as spin, during its motion through metal, the effective Coulomb interaction is

screened dynamically[2, 15] as

$$V_{\vec{q}} = \frac{4\pi e^2}{\epsilon(\vec{q},\omega)q^2} \qquad (1)$$

However, with rearrangement of the electrons due to phonon interaction, the effective interaction [2, 15] reduces to

$$V_{\vec{q}}^{eff} = \frac{4\pi e^2}{q^2 + k_{TF}^2} + \frac{4\pi e^2}{q^2 + k_{TF}^2}\frac{\omega_q^2}{\omega^2 - \omega_q^2} \qquad (2)$$

Here, $\vec{q} = \vec{k} - \vec{k'}, \hbar\omega = E_k - E_k'$ with the screening length $\lambda_{TF} = k_{TF}^{-1}$ depending on the density of electrons and the Fermi energy. ω_q describes the spectrum of phonons with dispersion relation $\omega_q = sq$ at long wavelength limit. In the limit $\omega < \omega_q$, the effective interaction becomes negative while in the static limit, the interaction reduces to zero. A schematic view of the formation of Cooper pairs is shown in figure 1. Thus, this interaction is of dynamic nature only. Alternatively, a simple physical intuitive level explanation has been provided for this attractive interaction between two electrons above the Fermi surface by Weisskopf [17]. In fact, it has been pointed out that the motion of Cooper pairs is similar to that of the two nucleons in a deuteron or of the two electrons in the ground state of positronium.

In this pedagogical article, we would like to discuss the many facets of Cooper pairs as follows. In the next section, we briefly explain the meaning of bound state in quantum mechanics. In section 3 , we state the Cooper pairing problem in spherical Fermi
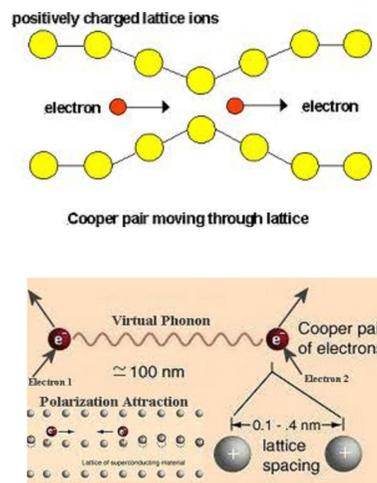

Figure 1: Formation of Cooper pairs inside the lattice.

surface and solve the bound state for various interactions between the electrons. With the help of the eigenfunctions we also compute the average radius of the Cooper pair in section 4. We then generalize in section 5, the bound state problem to ellipsoidal Fermi surface. We give our conclusions in section 6.

2.Bound State Problem in Quantum Mechanics

Classically, it is known that the bound states can exist any time whenever there is a local (or global) minimum in the potential energy. However, a local minimum is insufficient to create a bound state in quantum mechanics. But, in quantum mechanics a global minimum is necessary to allow the existence of bound states. The classical bound states can exist for any value of total energy which

is low enough to keep the particle in the minimum. However, this is not the case in quantum mechanics.

In physics, a bound state generally describes a system where a particle is subject to a potential such that the particle has a tendency to remain localised in one or more regions of space. The potential may be either an external potential, or may be the result of the presence of another particle.

In quantum mechanics with the conservation of number of particles, a bound state is a state (of course in Hilbert space) that corresponds to two or more particles whose interaction energy is less than the total energy of each individual particle, and therefore these particles cannot be separated unless this amount of energy is spent. The energy spectrum of a bound state is discrete, unlike the continuous spectrum of isolated particles. One may have unstable bound states with a positive interaction energy provided that there is an *energy barrier* that has to be tunnelled through in order to decay. In general, a stable bound state is said to exist in a given potential of some dimension if stationary wave functions exist (normalized in the range of the potential). The energies of these wave functions are negative. Bound state implies the *classically expected* state, so bound state energy should be smaller than $V(x \rightarrow \pm\infty)$ or $V(r \rightarrow \infty)$.

Let us illustrate some examples of the bound states. A proton and an electron can move separately and their the total center-of-mass energy is positive, and such a pair of particles can be described as an ionized atom. However, when the electron starts to orbit around the proton, the energy becomes negative, and a bound state so called the hydrogen atom is formed. Only the lowest energy bound state, the ground state is stable. The other excited states are unstable and will decay naturally to bound states with less energy by emitting a photon. A nucleus is a bound state of protons and neutrons (nucleons). A positronium *atom* is an unstable bound state of an electron and a positron. It decays into photons. The proton itself is a bound state of three quarks (two up and one down; one red, one green and one blue). However, unlike the case of the hydrogen atom, the individual quarks can never be isolated.

Any well-behaved eigenfunction ψ of a discrete spectrum should satisfy $\int |\psi|^2 \, d^3r = 1$. The integral is taken over all space. This implies immediately that $|\psi|^2$ must decrease rapidly and approaching zero at infinity. This indicates alternatively that the probabilities of infinite coordinates is rather zero. Thus, the system executes a finite motion and hence, is said to be a *bound state*. For a continuous one, the integral $\int |\psi|^2 \, d^3r$ diverges because $|\psi|^2$ does not tend to zero at infinity. In such a situation, the motion is said to be extended over the infinite space. In solving the Schroedinger equation for a given problem, there are two types of solutions for positive values as well as negative values of the energy E. The negative energy solution $(E < 0)$ is bound one while the other one is termed as scattering state or unbounded state. For harmonic oscillator, the potential

$V(x) \to \infty$ as $x \to \infty$ while for the hydrogen atom $V(r) \to 0$ as $r \to \infty$. However, in both cases, there is bound state. One has to be careful to think whether the motion is confined or not in a given potential. In this respect, there is a remarkable theorem in one dimension about the bound state.

The statement is : *In one dimensional system, there is always a bound state for any attractive potential.* Thus, the potential under consideration have the the following properties; $V(x) \to 0$ as $x \to 0$ and $\int_{-\infty}^{\infty} V(x) \, dx < 0$. Under this criteria, we are looking for a variational estimation of the Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \tag{3}$$

for wave functions

$$\psi(x) = N \exp(-\alpha^2 x^2) \tag{4}$$

$$\psi(x) = N/\cosh(\alpha x) \tag{5}$$

Both these wave functions are normalizable and go to zero as $x \to \pm\infty$. Dimensional analysis [18, 19] can be used to estimate the contribution of the bound state energy eigen values of bound state problem and other related problems. A simple dimensional analysis yields that $N^2 = C\alpha$ with C being a dimensionless quantity and $[\alpha] = M^0 L^{-1} T^0$. It is easy to notice that values of α control the character of the wave function. For example, in the limit of the small values of α, the wave function is very much spread over the space and the state is regarded as a weakly bound. With the help of the wave function, we can now estimate the potential and the kinetic

energy of the above Hamiltonian. For small value of α,

$$
\begin{aligned}
\langle V \rangle &= \int_{-\infty}^{\infty} |\psi|^2 \, V(x) \, dx \\
&\approx |\psi(0)|^2 \int_{-\infty}^{\infty} V(x) \, dx \\
&= -CW\alpha \tag{6}
\end{aligned}
$$

where $W = \int_{-\infty}^{\infty} V(x) \, dx > 0$. Again, on dimensional ground, the kinetic energy can be estimated as $\left\langle \frac{p^2}{2m} \right\rangle = Const. \left(\frac{\alpha^2 \hbar^2}{2m} \right) = D\alpha^2$. Thus,

$$\langle H \rangle = D\alpha^2 - CW\alpha \tag{7}$$

No matter how small V is, we can have $\langle H \rangle < 0$ for small value α. The minimum value of $\langle H \rangle$ is $-\frac{C^2 W^2}{4D}$ for $\alpha = \frac{CW}{2D}$. Essentially, the magnitude of the bound state depends on the shape rather than its strength. The variational proof of the existence of at least one bound state in 1d is firmly rooted to one dimension as shown explicitly on equation (3)-(7) and cannot be generalized to 2d or 3d. In fact, for three dimensional attractive potential problem given by

$$V(r) = -V_0, \quad r \le a; \quad V(r) = 0 \quad r > a \tag{8}$$

there exists a bound state only when the strength of the potential V_0 is greater than the critical strength $V_c = \left(\frac{\hbar^2 \pi^2}{8ma^2} \right)$. In fact, for higher dimensions, the condition $\int V(x) \, d^d x < 0$ does not essentially support [22, 23] always a bound state.

3.Bound state of single Cooper pair

The ground state of a non-interacting Fermi gas of electron in a potential well corresponds to the situation where all the electron states with wave vector $\vec{k}$ within the Fermi sphere $[E_F^0 = \frac{\hbar^2 k_F^2}{2m}$ at T=0K] are filled and all states with $E > E_F^0$ are unoccupied. We consider a simple model [2, 3, 4] of two electrons added to just above the filled Fermi surface characterized by the Fermi energy E_F^0. A weak attractive interaction $V(\vec{r_1}, \vec{r_2})$ between these two electrons is switched on resulting from the phonon exchange as discussed in the introduction section. All other electrons in the Fermi sea are assumed to be non-interacting, and, on account of the Pauli exclusion principle, they exclude a further occupation of states with $|k| < k_F$. The added two electrons do feel a statistical interaction through Pauli exclusion principle with those inside the filled Fermi sphere. Hence, the most simple bound state energy eigenvalue equation reads as

$$
\begin{aligned}
H\Psi_0(\vec{r}_1, \vec{r}_2) &= E\Psi_0(\vec{r}_1, \vec{r}_2) \\
H &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(\vec{r}_1, \vec{r}_2) \quad (9)
\end{aligned}
$$

Now, if the interaction $V(\vec{r_1}, \vec{r_2})$ depends only on the magnitude of the relative coordinate $\vec{r} = \vec{r_1} - \vec{r_2}$, then the lowest energy state will correspond to the zero momentum of centre of mass. This is true because we have a translational invariant system ($[H, P] = 0; \vec{P} = \vec{p_1} + \vec{p_2}$). This fact also can be viewed from the phonon mechanism. Due to phonon ex-

change the two additional electrons continually exchange their wave vector, where by, however, momentum must be conserved.

$$
\vec{k}_1 + \vec{k}_2 = \vec{k}_1' + \vec{k}_2' = \vec{K} \qquad (10)
$$

We also assume that the interaction is short-ranged and in k-space is restricted to a shell with an energy thickness of $\hbar\omega_D$ (ω_D= Debye frequency) above E_F^0. The strength of the attractive interaction is maximum for $\vec{K}$=0.

It is therefore sufficient in what follows to consider the case $\vec{k}_1 = -\vec{k}_2 = \vec{k}$, i.e. electron pairs with equal and opposite wave vectors. This suggest the orbital wave function to be

$$
\Psi_0(\vec{r_1}, \vec{r_2}) = \sum_{\vec{k}} g_{\vec{k}} e^{i\vec{k}.\vec{r_1}} e^{-i\vec{k}.\vec{r_2}} \qquad (11)
$$

It should be noted here that zero momentum of the pair does not mean that the uncertainty in momentum (Δp) of the pair is zero.

Taking into account the antisymmetry [2, 24] of the total wave function with respect to exchange of the two electrons, Ψ_0 is converted either to a sum of products of $\cos(\vec{k}.(\vec{r}_1 - \vec{r}_2))$ with the antisymmetric singlet spin function $(\alpha_1\beta_2 - \beta_1\alpha_2)$ or to a sum of products of $\sin(\vec{k}.(\vec{r}_1 - \vec{r}_2))$ with one of the symmetric triplet spin function $\alpha_1\alpha_2$, $(\alpha_1\beta_2 + \beta_1\alpha_2)$,$\beta_1\beta_2$, where α= up spin state, β= down spin state. However, because of an attractive interaction, we expect the singlet coupling to have lower energy because the cosinusoidal dependence of its orbital wave function on $(\vec{r}_1 - \vec{r}_2)$ gives a larger probability amplitude for the electrons to be near each

other. Hence, we consider a two electrons singlet wave function of the form

$$
\begin{aligned}
\Psi_0(\vec{r}_1, \vec{r}_2) &= [\sum_{\vec{k}>\vec{k}_F} g_{\vec{k}} \cos \vec{k}.(\vec{r}_1 - \vec{r}_2)] \\
&\times \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2) \quad (12)
\end{aligned}
$$

This kind of pairing of electrons is known as S-wave pairing wave as g_k will depend on the magnitude of the wave vector $\vec{k}$. The summation is confined to pairs with $\vec{k}=\vec{k}_1=-\vec{k}_2$, which, because of the interaction is restricted to the region $\hbar\omega_D$, must obey the condition

$$
E_F^0 < \frac{\hbar^2 k^2}{2m} < E_F^0 + \hbar\omega_D
$$

The quantity $|g_k|^2$ is the probability of finding one electron in state $\vec{k}$ and the other in $-\vec{k}$, that is, the electron pair in $(\vec{k}, -\vec{k})$. Due to Pauli exclusion principle and above condition, we have

$$
g_k=0 \quad \text{for} \left\{ \begin{array}{l} k < k_F \\ k > \sqrt{2m(E_F^0 + \hbar\omega_D)/\hbar^2} \end{array} \right.
$$

With the help of the above Hamiltonian depicted in equation (9), the eigen value equation can be rewritten in terms of the coefficients g_k as

$$
(E - 2\epsilon_{\vec{k}})g_{\vec{k}} = \sum_{\vec{k}'} g_{\vec{k}'} V_{\vec{k}\vec{k}'} \quad (13)
$$

In this expression, the $\epsilon_{\vec{k}}$ are unperturbed plane wave energies and $V_{\vec{k}\vec{k}'}$ are the matrix element of the interacting potential obtained

through Fourier transformation of $V(\vec{r})$ is given by

$$
V_{\vec{k}\vec{k}'} = \Omega^{-1} \int V(\vec{r})e^{i(\vec{k}'-\vec{k}).\vec{r}}d\vec{r} \quad (14)
$$

where $\vec{r}$ is the distance between the two electrons and Ω is the normalization volume. This $V_{\vec{k}\vec{k}'}$ characterizes the strength of the potential for scattering a pair of electrons with momenta $(\vec{k}', -\vec{k}')$ to momenta $(\vec{k}, -\vec{k})$. If a set of $g_{\vec{k}}$ satisfying equation (13) with $E < 2E_F^0$ can be found, then a bound pair exists. Since it is hard to analyse this situation for general $V_{\vec{k}\vec{k}'}$, below we consider three forms of the interaction for which we can solve the equation (13) exactly.

3.1 Constant potential in k-space

Cooper [4] introduces the approximation that all $V_{\vec{k}\vec{k}'}$ =-V for $\vec{k}$ states out to a cut off energy $\hbar\omega_D$ away from E_F, and that $V_{\vec{k}\vec{k}'}$=0 beyond $\hbar\omega_D$. The right hand side of equation (13) is a constant, independent of $\vec{k}$, and we have

$$
g_{\vec{k}} = V\frac{\sum_{\vec{k}'} g_{\vec{k}'}}{2\epsilon_{\vec{k}} - E} \quad (15)
$$

Summing both sides and cancelling $\sum g_{\vec{k}}$ we obtain,

$$
\frac{1}{V} = \sum_{\vec{k}}(2\epsilon_{\vec{k}} - E)^{-1} \quad (16)
$$

Note that a constant V in the reciprocal space means a spatially varying in real space. In figure 2, we show the schematic variation of the varying potential in real space.
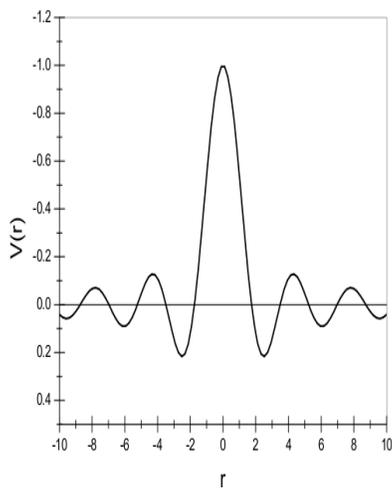
Figure 2: Schematic view of potential in real space.

When we replace the summation by an integration, with $N(\epsilon)$ denoting the density of states at the Fermi level for electrons, then the above equation reduces to

$$\frac{1}{V} = \int_{E_F^0}^{E_F^0+\hbar\omega_D} \frac{N(\epsilon)d\epsilon}{(2\epsilon - E)} \qquad (17)$$

Since $N(\epsilon)$ in three dimensions is proportional to $\sqrt{\epsilon}$ and $\hbar\omega_D \ll E_F$, then we can assume that $N(\epsilon)$ does not change appreciably over the energy interval and can be replaced by the density of states at the Fermi energy $(N(E_F^0))$. Thus, we get a very simple equation connecting the strength of interaction as

$$\frac{1}{V} = N(E_F^0) \int_{E_F^0}^{E_F^0+\hbar\omega_D} \frac{d\epsilon}{2\epsilon - E} \qquad (18)$$

$$1 = \frac{1}{2}N(E_F^0)V \ln\frac{2E_F^0 - E + 2\hbar\omega_D}{2E_F^0 - E} \qquad (19)$$

or,

$$2E_F^0 - E = 2\hbar\omega_D \frac{e^{-2/N(E_F^0)V}}{1 - e^{-2/N(E_F^0)V}} \qquad (20)$$

Now $(\hbar\omega_D << E_F^0)$ and weak-coupling approximation is $(N(E_F^0)V << 1)$, then the solution can be written as

$$E = 2E_F^0 - 2\hbar\omega_D e^{-2/N(E_F^0)V} \qquad (21)$$

There thus exists two electron bound state, whose energy is lower than that of the fully occupied Fermi sea $(T = 0K)$ by an amount $E - 2E_F^0 < 0$. In reality the instability leads to the formation of a new lower energy ground state.

It is interesting to note that from equation (21) that the strength of the interaction is inversely proportional to the density of electrons at the Fermi surface. Moreover, the binding energy is finite for any arbitrary value of the interaction potential. This brings out the open question of the impossibility of forming the binding state in three dimensions. This can be however argued in the following way as illustrated in [11]. The particles under consideration are not isolated particles but quasi-particles related with the filled Fermi sphere. This eventually leads to the reduction of three dimensional problem $(\int d^3k = N(E_F^0) \int dE)$ to two dimensional $(\int d^2k = m \int dE)$ one. Any attractive interaction in two dimensions will be sufficient enough for the formation of bound state. In fact this also indicates the key role played by the filled Fermi sea in the formation of Cooper pair. Physically speaking, in one or two dimensions, the motion of the particle is

restricted to a straight line or a plane. In such a situation, any order of magnitude is enough to produce a bound state.

An important conclusion [12] can be drawn from equation (18). The equation (18) has a solution for an arbitrarily weak potential if and only if $N(E_F^0) \neq 0$. Note that in 3d, $N(\epsilon) \sim \sqrt{\epsilon}$ and hence $N(\epsilon) = 0$ at $\epsilon = 0$. Thus, in the above integral, if the lower cut-off were zero instead of E_F^0, then there would not be a solution for any arbitrary weak coupling V. This points out the important key role played by Pauli exclusion principle.

Suppose we split the Fermi sphere by a magnetic field so that $E_{F\uparrow} \neq E_{F\downarrow}$. In this situation, for a pair with $\vec{K} = 0$, the minimum excitation energy relative to free particles can be computed as $\Delta E_F = E_{F\uparrow} - E_{F\downarrow}$. Hence, the gap connecting equation reads as

$$\frac{1}{V} = N(E_F^0) \int_{E_F^0}^{E_F^0 + \hbar\omega_D} \frac{d\epsilon}{2\epsilon + \Delta E_F - \tilde{E}} \quad (22)$$

Here, $|\tilde{E}| = |E| - \Delta E_F$. If ΔE_F is greater than the binding energy of the state for $|\Delta E_F| = 0$, no binding state is possible. Suppose $\Delta E_F = 0$, but we would like to have a finite center of mass momentum $\vec{K}$ relative to Fermi sea. In that situation, it is clear that if the minimum value of ϵ_k is of the order $\hbar v_F K$, bound state solution disappears when $|\vec{K}| > \frac{E}{\hbar v_F}$. In that sense, it is really interesting to note that Cooper treated the two electrons as really special and the rest as only blocking states in Fermi sea.

If we measure the binding energy from the Fermi energy, it is evident from the equation (21) that the magnitude of the binding energy

is given by

$$|E_B| = 2\hbar\omega_D e^{-2/N(E_F^0)V} \quad (23)$$

Moreover, as an order of magnitude estimation, one can approximate $|E_B|$ to be order of $k_B T_c$. In that situation, we can get a simple relation between microscopic interaction and the observed critical temperature T_c in the weak coupling limit as

$$T_c = 2\theta_D \exp\left[-\frac{2}{N(E_F^0)V}\right] \quad (24)$$

Here θ_D is the corresponding Debye temperature of the phonon frequency. This equation (23) can be compared with the many body calculation involving many Cooper pairs in BCS theory[5]

$$T_c = 1.14\theta_D \exp\left[-\frac{1}{N(E_F^0)V}\right] \quad (25)$$

Again $\theta_D \propto \frac{1}{\sqrt{M}}$, M being the mass of the ion and $N(E_F^0)V$ is assumed to be independent of the mass of the ion, then we notice from the above equation (23) that $T_c \propto \frac{1}{\sqrt{M}}$. This dependence on the ion's mass is known as isotope effect [2, 24] in superconductivity a milestone in realizing the importance of role of lattice or ion. Furthermore this equation (23) can be utilized to compute the various parameters for the microscopic theory such as binding energy, size of the Cooper pairs.

Let us pause for a moment to estimate the order of magnitude of this energy gap. For Hg, assuming the transition temperature $T_c = 4K$, we find that the gap is 0.001 eV. Note that the typical Fermi energy or the average energy of free electron states is of the

order 3-10 eV and the typical thermal energy of a particle at room temperature (300 K) is roughly 0.025 eV. The binding energy of an electron in the ground state of hydrogen atom is 13.6 eV which is almost 10,000 times that of the energy gap seen for Hg. Thus, the minute magnitude of this energy gap can mislead one to use perturbation theory. However, the previous analysis by Cooper has shown clearly that it is indeed a non-perturbative effect. We will elaborate this point later.

What we have achieved till now is the following: If one starts from a degenerate free electron gas and switch on the attractive interaction V, then one notices that the electrons above the Fermi sea pair themselves and go below the Fermi surface to form a bound state. Thus, the normal state becomes unstable. Remarkably, this instability continues whatever may be the magnitude of the interaction strong or weak. Only thing required is the attractiveness of the interaction. However, there exists a contrasting nature from other two particle states coupled by some attractive finite range potential. They may not form always a bound state the boundedness of the state depends on some critical value of the interaction. For repulsive potential, we will have no bound state as the energy is always greater than the Fermi energy to form continuum state. This can be visualized [24] as follows. Consider the integral in equation (17) as

$$ F(E) = V \int \frac{N(\epsilon)\, d\epsilon}{-E + 2\epsilon}; \ \epsilon = \frac{\hbar^2 k^2}{2m} - E_F \quad (26) $$

For any general two body problem, we have $k_F = 0$ but $N(\epsilon) \sim \sqrt{\epsilon}$ in three dimensions.

Then, it is easy to notice that the above integral converges for $E = 0$. For small V, $F(0) < 1$ and for $E < 0$, one can convince that $F(E) < F(0) < 1$. Hence, $F(E) - 1$ is not realized for bound state. Following the same logic, we can see that for Copper pairing problem that $N(\epsilon)$ is constant over the energy interval and $F(0) \to \infty$ and there exists always a bound state value of $E < 0$ such that $F(E) = 1$.

It is also evident for constant $V_{kk'}$, $g_{\vec{k}}$ depends on the magnitude of the wave vector. This implies immediately that the spatial part of the wave function is symmetric in nature $(\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1))$. Hence, according to Pauli exclusion principle, the spin part of the wave function must be antisymmetric. Therefore, the spin part must have the following antisymmetric form

$$ \phi_{spin} = \frac{1}{\sqrt{2}} \left(\alpha_1 \beta_2 - \beta_1 \alpha_2 \right) \quad (27) $$

where α and β are the spin up and spin down eigenstates of spin 1/2 Pauli spin matrices. The total spin operator S_{tot} and the z component of the spin operator S_z acting on the above eigenstate ϕ_{spin} gives identically zero eigenvalues only. This is consistent with the argument given in the section for writing down the variational form of the wave function for the two particles.

With the help of antisymmetric spin wave function, we can now construct the full wave function of two electrons as

$$ \psi(\vec{r}_1, \sigma_1 : \vec{r}_2, \sigma_2) $$
$$ = \frac{1}{\sqrt{2}} \left(\alpha_1 \beta_2 - \beta_1 \alpha_2 \right) $$

$$\times \sum_{k} g_{\vec{k}} e^{i\vec{k}\cdot(\vec{k}_1 - \vec{r}_2)} \qquad (28)$$

with $g_{-\vec{k}} = g_{\vec{k}}$. This wave function, however can be further simplified as

$$\psi(\vec{r}_1, \sigma_1 : \vec{r}_2, \sigma_2)$$
$$= \frac{1}{\sqrt{2}} \sum_{k} g_{\vec{k}} \left(\alpha_1 \beta_2 \, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} \right.$$
$$\left. - \beta_1 \alpha_2 \, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} \right)$$
$$= \frac{1}{\sqrt{2}} \sum_{k} g_{\vec{k}} \left((\vec{k}\uparrow)_1 (-\vec{k}\downarrow)_2 \right.$$
$$\left. - (\vec{k}\uparrow)_2 (-\vec{k}\downarrow)_1 \right)$$
$$= \sum_{k} g_{\vec{k}} c^{\dagger}_{\vec{k}\uparrow} c^{\dagger}_{-\vec{k}\downarrow} |vac> \qquad (29)$$

In the last step, we have used the formalism of second quantization with fermionic creation and annihilation operators. A quick inspection to the form of wave function reveals that

$$\psi(\vec{r}) = \mathcal{N} \int_{FS} d^3k \, e^{i\vec{k}\cdot\vec{r}}$$
$$= \mathcal{N} \frac{\sin(k_F r) - k_F r \cos(k_F r)}{r^3} \quad (30)$$

Here, $\mathcal{N}$ is the normalization constant. In Figure 3, we show the variation of the wave function with radial distance for two different values of the Fermi wave vector. We notice the drastic difference of the behaviour of the wave function for large Fermi wave vector at small values of distance. It is clear that the wave function is spatially symmetric and the radial probability density $P(r)$ varies as r^{-4} as $r \to \infty$. This indicates immediately that the mean square radius $< r^2 >= 4\pi \int P(r) r^2 dr$ diverges.
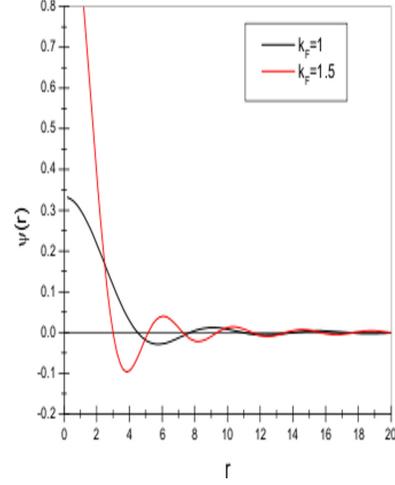

Figure 3: Schematic view of the wave function for two different values of the Fermi wave vector.

The variation of the probability density can be visualized in another way. Using the value $g_{\vec{k}}$, we can rewrite the wave function as

$$\psi(r) = \frac{1}{r} \int_{k_F}^{k_c} dk \, \frac{k \sin(kr)}{2\epsilon_k + |E|}$$
$$= -\frac{1}{r} \frac{d}{dr} \left\{ \cos(k_F r) \int_{0}^{\epsilon_c} \frac{\cos(\frac{\epsilon r}{\hbar v_F}) \, d\epsilon}{\epsilon + |E|/2} \right\} \quad (31)$$

Here we have used $k - k_F \approx \frac{\epsilon_k}{\hbar v_F}$. If we represent the integral as $J(r)$, then it is noticed that the wave function is nothing but that of two free scattering particles ($\frac{1}{r} \sin(k_F r)$ times the integral $J(r)$. A careful look reveals that $J(r)$ is roughly constant for $r < \frac{\hbar v_F}{|E|}$ and beyond that it falls of as $\frac{1}{r}$. Therefore, the wave function varies as $\frac{1}{r^2}$ in the limit $r \to \infty$. This fact thus justifies the statement about the radial probability density in a natural way.

Now one may ask the following question: Is

this spatial symmetric wave function unique for any form of interaction $V_{kk'}$? In fact, the nature of the wave function depends strongly on the nature of the interaction. If the interaction depends on the angle between $\vec{k}$ and $\vec{k}'$ (such as $\lambda(\vec{k} \cdot \vec{k}')$), then several bound states [24] may arise. Moreover, spatially anisotropic solutions leading to complicated spin dependences rather than the simplistic as suggested above may be found. However, it has been noticed that in most of the normal superconductors do not show a strong angular dependence of $V_{kk'}$ on $\vec{k}$ and $\vec{k}'$.

3.2 Potential separable in two coordinates

Note that in the original Cooper pair problem, the interaction potential $V(\vec{r}_1, \vec{r}_2) = V(|\vec{r}_1 - \vec{r}_2|) = V(r)$. That's why the potential $V(\vec{r}_1, \vec{r}_2)$ is separable in relative coordinate r only. There is no part of the centre of mass coordinate $\vec{R}$. If the potential could depend on the individual coordinates rather than the distance between the particles, then we cannot write $V(\vec{r}_1, \vec{r}_2)$ as $f(r)g(R)$. As in the case of constant potential, here also we define the matrix element $V_{\vec{k}\vec{k}'}$ as

$$V_{\vec{k}\vec{k}'} = < \vec{k}, -\vec{k}|V|\vec{k}', -\vec{k}' > \qquad (32)$$

There is no general solution to equation (13) unless $V_{\vec{k}\vec{k}'}$ is assumed to be separable as functions of $\vec{k}$ and $\vec{k}'$. So now we choose the separable potential [22] given by $V_{\vec{k}\vec{k}'} = -\lambda\omega_{\vec{k}}\omega_{\vec{k}'}^*$. We consider the potential V in relative coordinate is spherically symmetric and hence,

$\Psi(\vec{r}_1, \vec{r}_2)$ is an eigenfunction of angular momentum with angular momentum quantum numbers l and m_1.

$V_{\vec{k}\vec{k}'}$ can be expanded into partial wave components

$$V_l(|\vec{k}|, |\vec{k}'|) = \lambda_l\omega_{\vec{k}}^l\omega_{\vec{k}'}^{*l} \qquad (33)$$

Thus, the eigen value equation in this case for each value of l of $V_l(|\vec{k}|, |\vec{k}'|)$ reads as

$$(B_{lm_1} - 2\epsilon_{\vec{k}})g_{\vec{k}} = -\lambda_l\omega_{\vec{k}}^l\sum_{\vec{k}'}\omega_{\vec{k}'}^{*l}g_{\vec{k}'} \qquad (34)$$

with

$$g_{\vec{k}} = -\frac{\lambda_l\omega_{\vec{k}}^l C}{(B_{lm_1} - 2\epsilon_{\vec{k}})} \qquad (35)$$

Where the constant C is defined as

$$C = \sum_{\vec{k}'}\omega_{\vec{k}'}^{*l}g_{\vec{k}'} \qquad (36)$$

Repeating the same arguments as done in the previous section, we obtain the desired equation

$$\begin{aligned} 1 &= -\lambda_l\sum_{\vec{k}}|\omega_{\vec{k}}^l|^2\frac{1}{(B_{lm_1} - 2\epsilon_{\vec{k}})} \\ &= -\lambda_l\Phi(B_{lm_1}) \qquad (37) \end{aligned}$$

determining the energy eigenvalues B_{lm_1}. Before we proceed for the energy eigen value, we would like to study the analytic structure [25] of the equation (32). The poles of $\Phi(B_{lm_1})$ occurs at $B_{lm_1} = 2\epsilon_{\vec{k}}$. As $B_{lm_1} \rightarrow 2\epsilon_{\vec{k}}$ from below, $\Phi(B_{lm_1}) \rightarrow -\infty$ while just above $B_{lm_1} \rightarrow 2\epsilon_{\vec{k}}^+$, $\Phi(B_{lm_1}) \rightarrow +ve$. For all the values of $B_{lm_1} \ll 2E_F^0$, $\Phi(B_{lm_1}) = -ve$. Therefore, a bound state forms when

$\Phi(B_{lm_1})$ crosses $-\frac{1}{\lambda_l}$. Thus, the intersection of $\Phi(B_{lm_1})$ with the straight line $-\frac{1}{\lambda_l}$ determines the bound state solutions for the Cooper pair in such a solution.

For the simple case $\omega_{\vec{k}}^l = 1$ for $0 < \epsilon_{\vec{k}} < \hbar\omega_D$ and $\omega_{\vec{k}}^l = 0$ for otherwise, and $\lambda_l < 0$, the binding energy $|B_{lm_1}|$ of the pair in the split off state is given by

$$\frac{1}{\lambda_l} = \frac{N(E_F^0)}{2} \log[\frac{|B_{lm_1}| + 2\hbar\omega_D}{|B_{lm_1}|}]$$

$$|B_{lm_1}| = \frac{2\hbar\omega_D}{\exp(2/N(E_F^0)\lambda_l) - 1} \qquad (38)$$

We have assumed the density of states $N(\epsilon)$ is slowly varying in the interval $0 < \epsilon_{\vec{k}} < \hbar\omega_D$ and have approximated it by $N(E_F^0)$, the density of single electron states of one spin orientation, evaluated at the Fermi surface. From above equation one has in the weak coupling limit,

$$|B_{lm_1}| = 2\hbar\omega_D \exp(-2/N(E_F^0)\lambda_l) \qquad (39)$$

Thus, like constant potential case, we notice in this situation also that the binding energy is an extremely sensitive function of the coupling strength for weak coupling; however, a bound state exists for weak coupling so long as the potential is attractive near the Fermi surface. One more remark is in order. In contrast to the previous constant potential case, instead of single bound state, here we get many bound states as indicated in the figure 4. The bound state does not exist for repulsive interaction between the electrons above the filled Fermi surface. It is interesting to note that the form of the binding energy is not analytic at $V = 0$ or $\lambda = 0$. In
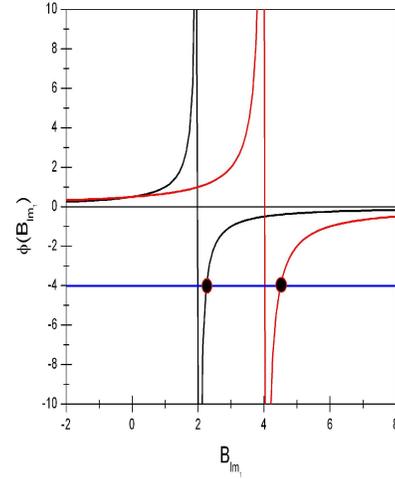

Figure 4: Schematic view of bound states for potential separable in two coordinates.

other words, the energy cannot be expanded in powers of V or λ, Hence, the result cannot be obtained by perturbation theory. This points out the importance of variational principle as adopted here. This is one of very few occasions in condensed matter physics, where the final result cannot be obtained from the perspectives of perturbation theory. The failure of perturbation theory can be traced back in the frame work of change of different symmetry associated with the system [26].

We can also generalize the above analysis to

$$V_{\vec{k}\vec{k}'} = -V_0\delta(\vec{k} - \vec{k}') \qquad (40)$$

to obtain exactly the energy eigen value. So, we see that for all three above cases we get bound state. Now it is almost impossible to get a general analytic solution from any $V_{\vec{k}\vec{k}'}$. The exact analytic solution exists only for three forms of $V_{\vec{k}\vec{k}'}$ as discussed above. Of course any linear combination of the three

forms of $V_{\vec{k}\vec{k}'}$ will also form a bound state and can be found exactly. It has been shown by Randeria et al [27] that in two spatial dimensions, the many body ground state of a dilute gas of fermions interacting through an arbitrary pair potential is unstable to s-wave pairing. This work has a remarkable implications in high-T_c superconductivity because of the effective two dimensions of the materials used.

3.3 Binding Energy and Center of Mass Momentum

Dimensionally, one may argue that the binding energy depends on the center of mass momentum corordinate $\vec{P}$ as P^2. This is reasonably true because the wave function contains a part of the order $\exp\left(\frac{i\vec{P}\cdot\vec{R}}{\hbar}\right)$. However, we will demonstrate below that the presence of the Fermi surface will force the binding energy to depend only linearly [25] on the magnitude of the center of mass momentum $\vec{P}$.

The equation (16) in such a case is modified as

$$1 = -V \sum_k \frac{1}{E - \epsilon_{\vec{k}+\frac{\vec{P}}{2}} - \epsilon_{-\vec{k}+\frac{\vec{P}}{2}}} \qquad (41)$$

Now, for small $\vec{P}$, neglecting the terms of the order P^2, we can rewrite the above equation in an integral form

$$1 = -V \int_{E_F^0 + \frac{v_F P}{2}}^{E_F^0 + \frac{v_F P}{2} + \hbar\omega_D} \frac{N(\epsilon)\, d\epsilon}{E - 2\epsilon} \qquad (42)$$

to obain the final expression of the binding energy in terms of the Fermi velocity v_F and other relevant parameters

$$E = 2E_F^0 + Pv_F - \frac{2\hbar\omega_D}{\exp\left(\frac{2}{N(E_F^0)V}\right) - 1} \qquad (43)$$

Thus, we notice that the inclusion of the center of mass energy in the expression of the binding energy for the relative motion reduces its order of magnitude and can even break the pair. If we measure the energy from the Fermi surface, we can set E_F^0 to zero; then the appropriate value of the magnitude of center of mass momentum for which the breaking of Cooper pair can be found as

$$Pv_F = \frac{2\hbar\omega_D}{\exp\left(\frac{2}{N(E_F^0)V}\right) - 1} \sim k_B T_c \qquad (44)$$

If we now divide this center of mass momentum by Planck's constant, we get an inverse of length scale [$\frac{P}{\hbar} \sim \frac{k_B T_c}{\hbar v_F} = 10^4 \ cm^{-1}$]. This length scale ξ is known as Pippard coherence length [2, 13, 24] and the estimated order of magnitude turns out as $\xi \sim \ 10^{-4} \ cm = 1000\mathring{A}$. Interestingly, ξ, the coherence length or the average size of the Cooper pair is seen to be larger than the inter-atomic distance between the particles. Such a large coherence length is the signature of the electron-phonon interaction.

4. Average Size of Cooper pairs

In the previous section, we have used simple dimensional argument to estimate the size of the Cooper pair. This has been done from the

energy gap at the Fermi energy. That's the key point for using this dimensional analysis. This can be also be dome via uncertainty principle [2]. In this section, we would like to compute the size of the pair from the Cooper pair wave function through the expectation value [13, 24].

Cooper pair wave function can be written as

$$\psi(\vec{r}_1, \vec{r}_2) = \sum_{\vec{k}} g_{\vec{k}} e^{i\vec{k}.\vec{r}} \tag{45}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$, therefore

$$|\psi(\vec{r}_1, \vec{r}_2)|^2 = \sum_{\vec{k}} \sum_{\vec{k}'} g_{\vec{k}} g_{\vec{k}'} e^{i(\vec{k}-\vec{k}').\vec{r}} \tag{46}$$

Now the expectation value of the Cooper pair radius squared is given by

$$< r^2 > = \frac{\int |\psi(\vec{r}_1, \vec{r}_2)|^2 r^2 d\vec{r}}{\int |\psi(\vec{r}_1, \vec{r}_2)|^2 d\vec{r}} \tag{47}$$

putting the value of equation(31), we get

$$< r^2 > = \frac{\sum_{\vec{k}} \sum_{\vec{k}'} \int r^2 g_{\vec{k}} g_{\vec{k}'} e^{i(\vec{k}-\vec{k}').\vec{r}} d\vec{r}}{\sum_{\vec{k}} \sum_{\vec{k}'} \int g_{\vec{k}} g_{\vec{k}'} e^{i(\vec{k}-\vec{k}').\vec{r}} d\vec{r}} \tag{48}$$

or,

$$< r^2 > = \frac{\sum_{\vec{k}} \sum_{\vec{k}'} (\nabla_{\vec{k}}.\nabla_{\vec{k}'} \int e^{i(\vec{k}-\vec{k}').\vec{r}} d\vec{r}) g_{\vec{k}} g_{\vec{k}'}}{\sum_{\vec{k}} \sum_{\vec{k}'} \delta(\vec{k} - \vec{k}') g_{\vec{k}} g_{\vec{k}'}} \tag{49}$$

therefore,

$$< r^2 > = \frac{\sum_{\vec{k}} \sum_{\vec{k}'} \nabla_{\vec{k}}.\nabla_{\vec{k}'} \delta(\vec{k} - \vec{k}') g_{\vec{k}} g_{\vec{k}'}}{\sum_{\vec{k}} \sum_{\vec{k}'} \delta(\vec{k} - \vec{k}') g_{\vec{k}} g_{\vec{k}'}} \tag{50}$$

or, we can write

$$< r^2 > = \frac{\sum_{\vec{k}} |\nabla_{\vec{k}} g_{\vec{k}}|^2}{\sum_{\vec{k}} |g_{\vec{k}}|^2} \tag{51}$$

Now, from previous section we know that

$$g_{\vec{k}} = \frac{C}{2\epsilon_{\vec{k}} - E} \tag{52}$$

where C be the constant. Therefore,

$$g_{\vec{k}} = \frac{C}{2(\frac{\hbar^2 k^2}{2m} - E_F - w/2)} \tag{53}$$

where $w = -2\hbar\omega_D e^{-2/N(0)V}$ is the binding energy of the Cooper pair. Now,

$$\nabla_{\vec{k}} g_{\vec{k}} = -\frac{C\frac{\hbar^2 k}{m}}{2(\frac{\hbar^2 k^2}{2m} - E_F - w/2)^2} \tag{54}$$

therefore,

$$|\nabla_{\vec{k}} g_{\vec{k}}|^2 = \frac{C^2 \frac{\hbar^4 k^2}{m^2}}{4(\frac{\hbar^2 k^2}{2m} - E_F - w/2)^4} \tag{55}$$

and

$$|g_{\vec{k}}|^2 = \frac{C^2}{4(\frac{\hbar^2 k^2}{2m} - E_F - w/2)^2} \tag{56}$$

Substituting this value of $|g_k|^2$ in equation (46), we get

$$< r^2 > = \frac{2\hbar^2}{m} \frac{\int_{E_F}^{E_F + \hbar\omega_D} \frac{N(\epsilon)\epsilon d\epsilon}{(\frac{\hbar^2 k^2}{2m} - E_F - w/2)^4}}{\int_{E_F}^{E_F + \hbar\omega_D} \frac{N(\epsilon)d\epsilon}{(\frac{\hbar^2 k^2}{2m} - E_F - w/2)^2}} \tag{57}$$

One can approximate those integrals as

$$\int_{E_F}^{E_F+\hbar\omega_D} \frac{N(\epsilon)\epsilon d\epsilon}{\left(\frac{\hbar^2 k^2}{2m} - E_F - w/2\right)^4}$$

$$\sim N(E_F^0)E_F \int_{E_F}^{\infty} \frac{d\epsilon}{\left(\frac{\hbar^2 k^2}{2m} - E_F - w/2\right)^4} \quad (58)$$

or,

$$\int_{E_F}^{E_F+\hbar\omega_D} \frac{N(\epsilon)\epsilon d\epsilon}{\left(\frac{\hbar^2 k^2}{2m} - E_F - w/2\right)^4} = \frac{-8N(E_F^0)E_F}{3w^3} \quad (59)$$

similarly,

$$\int_{E_F}^{E_F+\hbar\omega_D} \frac{N(\epsilon)d\epsilon}{\left(\frac{\hbar^2 k^2}{2m} - E_F - w/2\right)^2} = \frac{-2N(E_F^0)}{w} \quad (60)$$

Incorporating these values of the integrals, we obtain finally,

$$<r^2> = \frac{4\hbar^2 v_F^2}{3w^2} \quad (61)$$

where v_F is the Fermi velocity given by $v_F = \frac{\hbar k_F}{m} = \frac{\hbar}{m}(3\pi^2 n)^{1/3}$ with n being the density of electrons.

Now the order of binding energy is 10^{-4} eV. So we get,

$$<r^2>^{1/2} = 10^{-4} cm \quad (62)$$

So the extension of Cooper pair is much greater than mean interatomic separation. So, such a high density of such electron pairs create a new lower energy ground state which is known as BCS ground state.

Below we estimate the size of the Cooper pairs in Table I for different systems assuming the binding energy $w \sim k_B T_c$. Note that

Table 1: Size of Cooper pair

Metal	T_c	$v_F(\times 10^6 m/s)$	Size (μm)
Al	1.2	2.03	14.92
Cd	0.56	1.62	25.51
Ga	1.09	1.92	15.54
Zn	0.9	1.83	17.93
Sn	3.72	1.90	4.50
Pb	7.2	1.83	2.24
Nb	9.26	1.37	1.30
Hg	4.15	1.58	3.36
In	3.4	1.74	4.51

as $V_0 \to 0$, $w \to 0$ rendering the average radius to diverge. Kadin [28] has beautifully illustrated the real space physical picture for the Copper pair consistent with BCS theory.

5. Cooper pair above the ellipsoidal Fermi surface

Suppose we put two electrons above the ellipsoidal Fermi surface rather than the original spherical Fermi surface as done by Cooper himself. Will they still form a bound state like the spherical one? We know that the ellipsoidal Fermi surface can be represented by the equation

$$E = \frac{\hbar^2}{2m_x^*}k_x^2 + \frac{\hbar^2}{2m_y^*}k_y^2 + \frac{\hbar^2}{2m_z^*}k_z^2 \quad (63)$$

This equation can be rearranged in the following form

$$1 = \frac{\hbar^2}{2m_x^* E}k_x^2 + \frac{\hbar^2}{2m_y^* E}k_y^2 + \frac{\hbar^2}{2m_z^* E}k_z^2 \quad (64)$$

The three axis of the ellipsoid can be read of as

$$a = \frac{\sqrt{2m_x^* E}}{\hbar}$$
$$b = \frac{\sqrt{2m_y^* E}}{\hbar}$$
$$c = \frac{\sqrt{2m_z^* E}}{\hbar} \qquad (65)$$

Now, the total number of electronic state be

$$n = \frac{\frac{4}{3}\pi abc}{\left(\frac{2\pi}{L}\right)^3} \qquad (66)$$

Putting the value of a, b, c into equation (61), the value of n is given by

$$n = \frac{\frac{4}{3}\pi\sqrt{8m_x^* m_y^* m_z^*}}{\hbar^3\left(\frac{2\pi}{L}\right)^3} E^{3/2} \qquad (67)$$

This, the density of state is

$$N(E) = \frac{dn}{dE} = C\sqrt{m_x^* m_y^* m_z^*} E^{1/2} \qquad (68)$$

where C be the constant. Thus, like the spherical case, here $N(E)$ is also proportional to $\sqrt{E}$. This can be understood from the fact that by defining $\tilde{k}_x = \frac{k_x}{\sqrt{m_x^*/m}}$, $\tilde{k}_y = \frac{k_y}{\sqrt{m_y^*/m}}$ $\tilde{k}_z = \frac{k_z}{\sqrt{m_z^*/m}}$, the ellipsoidal Fermi surface defined in equation (58) can be transformed to spherical one. So it is clear that, the form of binding energy of Cooper pair does not change for ellipsoidal Fermi surface. Only the magnitude of the effective mass is modified in such a situation as $m_d = (m_x^* m_y^* m_z^*)^{1/3}$. Thus, the equations (20) or (33) remain valid with suitable replacement of the density of

states at the Fermi surface as shown above. Although the density of states will not change shape, however, it is to be noted that the Fermi velocity at different k-points on the Fermi surface will be different. This in turn will indicate the electron-phonon coupling will necessarily depend on k and hence, the gap calculated above becomes anisotropic instead of isotropic.

6. Conclusions

To conclude, we have discussed the energetics of a single Cooper pair which eventually forms an important ingredient for the development of BCS theory which is nothing but the condensate of interacting many Cooper pairs. The meaning of bound state along with the inapplicability of perturbation theory have been addressed here. The original bound state of Cooper pairs in spherical Fermi surface has been generalized to ellipsoidal Fermi surface with various forms of the interacting potentials. The average radii of the Cooper pairs for many superconducting materials have been computed and it has been argued that they are quite larger than the typical average distance between the quantum particles (here electrons).

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