Minkowski's Space Time

Somnath Datta
656, "Snehalata", 13th Main, 4th Stage, T K Layout, Mysore 570009, India
datta.som@gmail.com; http://sites.google.com/site/physicsforpleasure

(Submitted 05-12-2012)

Abstract

In 1908 Minkowski wrote a paper in which he introduced the concept of a 4-dimensional "world" popularly known as space-time, and introduced "world postulate" suggesting that all physical phenomena should be described only in space-time. He showed how to construct 4-vectors, and classified them into two categories, namely time-like and space-like vectors. Some of the examples of these vectors as worked out by him are the velocity and acceleration vectors, the momentum and the force vectors, all in 4-dimensions. He derived the 4-dimensional law of motion from which he also obtained the famous $E = mc^2$ formula. In his final analysis Minkowski demonstrated further application of the world postulate by giving a geometrical construction of the Lienard-Weichert potentials and used it to obtain the force exerted by a moving charge on another moving charge. In this article we have explained Minkowskis work as mentioned above using our own interpretation.

1 Introduction

In 1905 Einstein wrote two revolutionary papers [1] giving the framework of what he later called the Special Theory of Relativity (STR). Eleven years later, i.e., in 1916, he presented his General Theory of Relativity (GTR) which turned out to be a geometrical theory of gravitation[2]. Whereas the STR is mathematically a simple theory originally intended to set right some "asymmetries in Maxwell's Electrodynamics", the GTR involved complex mathematics of Reimannian geometry. It visualized gravitation as a curvature in a four dimensional "world", more familiarly now known as *space-time* and the orbits of planets and satellites (even trajectories of mundane earthly objects like cricket balls) as geodesics marked out on this "curved space-time".

The geometrical ideas that form the bridge between the STR and the GTR were creations of H. Minkowski [3]. In 1908 he geometrized the STR, which under Einstein later culminated in the geometrization of gravity[2]. In his original paper Minkowski fancied a *four-dimensional manifold* of "events" where the time co-ordinate t takes equal status along with the three space co-ordinates (x, y, z), and called it *world*. The points in this "world" are therefore "events" characterized by 4-coordinates (x, y, z, t). The basic physical quantities of classical mechanics. e.g., displacemnent, velocity, momentum, energy and force were conceived as 4-vectors (geometrical objects), or components of 4-vectors, in their corresponding 4-dimenational "worlds".

In 2005, as part of celebration of Einstein's birth cenetennary, we had written an exposition of Einstein's special relativity papers[4] with the following remarks. "Einstein's original papers were terse because they were meant to be read by the leading physicists of that time. We have therefore simplified his work by providing between-the-lines elucidation for many of the concise statements in these papers which many students may find difficult to understand. It is hoped that students and physics teachers may be able to get a full view of Einsteins relativity papers using our article as a guide." In the same spirit we are writing this article to expose the students and teachers of physics to the pioneering work of Minkowski. We have banked on the English version of this paper available in Ref.[3].

The original paper of Minkowski appears to be difficult to understand on first reading. It requires considerable efforts to comprehend his statements which are very brief. We have tried to elucidate these statements, as best as we could understand them, by expanding them into explanatory notes.

In the Appendix of Ref.[3] there are explanatory notes given by Prof. A.Sommerfeld. The reader may also take a look at these notes. However, most of these notes refer to other papers/articles written in German to which we could not get access. Even though we have benefitted from a few of these notes, we have mostly interpreted Minkowski's original paper with our own understanding.

It is our hope that the readers of this article (students and teachers) will find this article useful in strenthening their understanding of the special theory of relativity, in particular their concept of the 4-dimensional representation of classical mechanics, its equations of motion, $E = mc^2$, and the covariant formulation of classical electrodynamics. It may also help them place the works of the founding fathers of the theory of relativity in a proper historical perspective.

In this article we shall frequently use the term *space-time* to mean Minkowski's "world". Many of the new concepts and terms introduced by Minkowski, e.g., *world point, world line, proper time*, pictured in the 4-dimensional "world" are now familiar terms in text books. Even the famous mass-energy equivalence equation $E = mc^2$, which we normally attribute to Einstein, was written in this revolutionary form by Minkowski[1]. See our remarks on page 39.

Wherever we shall quote an axact passage from Minkowski's paper, we shall place them within "quotes". Most of these quotes (at least the most important ones of them) will be found in a separate paragraph, placed with an "indent".

Minkowski's paper has 5 sections, labelled with roman numerals I-V. We have coverd them in this article in 17 sections. Our Sections 2-10 cover Sec. I of Minkowski, Section 11 covers Sec.II, Sections 12-14 cover Sec.III, Section 15 covers Sec.IV, and Sections 16-17 cover Sec.V.

Minkowski begins his article with the philosophical prediction, "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

2 Two-fold invariance of Newtonian Mechanics

The concepts of newtonian mechanics revolve around displacement $\mathbf{r}$, velocity $\mathbf{v}$, acceleration $\boldsymbol{a}$. All these quantities are referred to a certain frame of reference S, e.g., a Cartesian system of XYZ axes, having its origin O located somewhere. However, the laws of motion remain unchanged ($\boldsymbol{a}$) "if we subject the underlying system of spatial coordinates to any arbitrary *change of position*", by which Minkowski seems to imply *rotation* of the coordinate axes from XYZ to $X'Y'Z'$, or ($\boldsymbol{b}$) "if we change its state of motion, namely, by imparting to it any *uniform translatory motion*", i.e., switch into another frame of reference S' whose origin O' is moving with respect to O with a constant velocity $\mathbf{u}$.

The totality of all operations ($\boldsymbol{a}$) and ($\boldsymbol{b}$) form two distinct groups of transformation.

> "The two groups side by side, lead their lives entirely apart. Their utterly inhomogenous character may have discouraged any attempt to compound them. But it is precisely when they are compounded that the complete group, as a whole, gives us to think."

Minkowski begins Sec. I of his paper with the proposal to combine the two groups into a single one (which he later calls G_c.) He proceeds towards this goal by introducing the terms *world point, world* and *world-line.*

> "The objects in our perception invariably include places and times in combination. Nobody has ever noticed a place except at a time, or a time except at a place... A point of space at a point of time, that is, a system of values x, y, z, t,

[1] In his relativity papers of 1905 Einstein derived a series of corollaries of his relativity postulates, some of them being (i) transformation under Lorentz transformation of the electromagnetic field, (ii) consequent derivation of the relativistic Doppler formula, (iii) transformation of light energy. Using the last result he established the following. Suppose an object (e.g., an atom) loses energy equal to δw by emission of light. As a consequence the object also loses its mass by an amount δm which satisfies the equality: $\delta w = (\delta m)c^2$. See page 67 of Ref.[4].

I will call a *world point*. The multiplicity of all thinkable x, y, z, t, systems of values we will chirsten the *world*."

We shall elucidate. The term "world point" is synonimous with the term "event" used in special relativity. Having fixed a frame of reference defined by the Cartesian coordinate system XYZ an event Θ takes place at some point P having coordinates (x, y, z) and at a certain time t. We can therefore fancy a 4-dimensional "manifold" $\mathcal{W}$ having four axes X, Y, Z, T in which the event Θ is represented by a corresponding image Θ having coordinates (x, y, z, t) with reference to the above mentioned four axes. This point Θ is a world point. The continuum of all points in this manifold having values of each one of x, y, z, t from $-\infty$ to $+\infty$ is the "world" as defined by Minkowski.

We shall however feel more comfortable to use the more familiar term *space-time* for Minkowski's *world*.

We shall illustrate some of Minkowski's arguments using the diagram in Fig. 1. Since it is not possible to picture the 4-dimensional space time, we have eliminated the Z axis and have presented a kind of picture of space-time on a sheet of paper. In doing this we have taken the XY plane as the base on which we have constructed the superstructure, the space-time, by going vertically up, along the T axis.

Let us think of a material particle moving in some manner such that its location at time t is at some point P having spatial coordinates x, y, z. Minkowski refers to such a particle by the term "substance". We shall find it more natural to use our familiar term "particle" instead of "substance".

Consider two successive events on the path of a material particle.

$\Theta_{\rm P}$	:	The particle is at	$\mathrm{P}(x, y, z)$	at time	t.
$\Theta_{\rm Q}$	:	The particle is at	$\mathrm{Q}(x + dx, y + dy, z + dz)$	at time	$t + dt$.

The displacement vector $\overrightarrow{PQ}$ has Cartesian components (dx, dy, dz). Seen in the space-time $\mathcal{W}$ there is a progression of the particle from $\Theta_{\rm P}$ to $\Theta_{\rm Q}$ through four coordinates (dx, dy, dz, dt). The line joining $\Theta_{\rm P}$ to $\Theta_{\rm Q}$ is an element of the *world line* of the particle.

In the 3-dimensional Euclidean space, often denoted as E^3, the particle moves from A to B along a certain path Γ. In the 4-dimensional space-time $\mathcal{W}$ the particle traces out the continuous curve Ω as it progresses from the event $\Theta_{\rm A}$ to the event $\Theta_{\rm B}$. This curve is the *world line* (WL) of the particle.

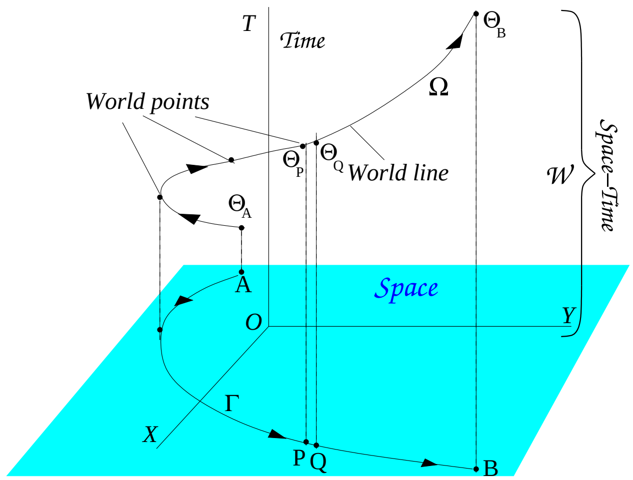

Figure 1: *The trajectory* Γ *of a particle in Space, and its world line* Ω *in Space-time.*

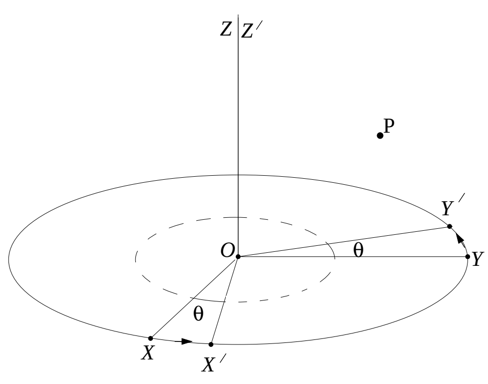

Figure 2: *Rotation about the Z axis.*

3 Freedom of the Time Axis

Let us now get back to the group of transformations (***a***), i.e., rotation of the XYZ axes, called *orthogonal transformations*, and often denoted by the symbol $O(3)$ in the terminology of group theory. The simplest such transformation is the rotation of the XY axes about the Z axis by an angle θ, shown in Fig. 2. The coordinates (x, y, z) of a point change to (x', y', z') in such a transformation and

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta, \\ y' &= -x\sin\theta + y\cos\theta, \\ z' &= z. \end{aligned} \tag{1}$$

In this transformation, in fact in all general orthogonal transformations, the distance of a point measured from the origin remains unchanged, i.e.,

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2. \tag{2}$$

As we have noted, and we repeat it here, the equations representing Newton's laws of motion donot change if the old coordinates (x, y, z) are replaced by the new coordinates (x', y', z') which are obtained from the former by any orthogonal transformation.

The group of transformations (***b***) consists of *Galilean transformations* of "events". We shall abbreviate it as GT. Let us consider a material particle moving along a certain trajectory Γ. It is at a certain point P at time t. Let the spatial coordinates of the point P be (x, y, z) when viewed from an inertial frame S, and (x', y', z') when viewed from another inertial frame S' which is moving with respect to S with a constant velocity $\mathbf{u}$. It is assumed that the Cartesian axes XYZ of S remain parallel to the Cartesian axes XYZ of S', and coincide at $t = 0$. In the parlance of special relativity we often say that the frame S' has a *boost* $\mathbf{u}$ with respect to S.

In Fig. 3 we have illustrated a boost in the X direction with velocity u. The GT for this special case is

$$x' = x - ut; \quad y' = y; \quad z' = z; \quad t' = t. \tag{3}$$

In a more general boost, the frame S' will be moving in an arbitrary direction with velocity $\mathbf{u}$ having componets $\mathbf{u} = (u_x, u_y, u_z)$ in the X, Y, Z directions. The GT for this general case will be

$$x' = x - u_x t,\ y' = y - u_y t,\ z' = z - u_z t,\ t' = t. \tag{4}$$

We shall represent GT in space-time as illustrated in Fig. 4. To make the comprehension of the graphical construction easier we shall consider the special case of boost in the X direction, represented by Eqs. (3).

Just as the transformation of the coordinates (x, y, z) represented by the orthogonal transformation (1) induces in the space E^3 the rotation of the XY axes as illustrated in Fig. 2, the transformation GT represented by Eqs. (3) induces a kind of rotation shown in Fig.4(a) in which the time axis T is *rotated* by the angle $\tan^{-1} u$ to become the T' axis, whereas the X', Y', Z' axes remain parallel to the corresponding X, Y, Z axes. To see this clearly we should only note that the equation of the new T' axis should be

$$x' = 0; \quad \text{Or, } x - ut = 0. \tag{5}$$

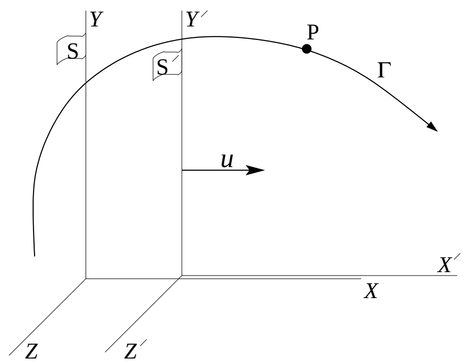

Figure 3: *Boost in the X direction*

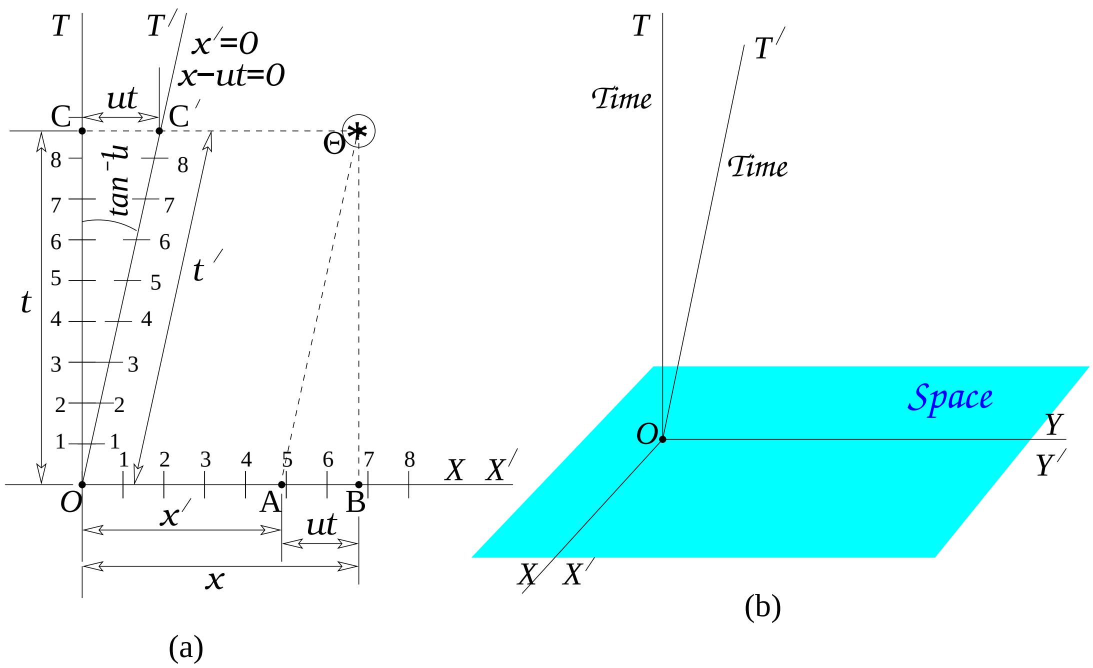

Figure 4: *Boost seen in Space-time*

We can get back the GT given by Eqs. (3) from the above diagram in the following way. Consider an arbitrary event Θ represented in space-time. Draw three lines ΘA, ΘB and ΘC parallel to the axes T', T and X respectively. The first two intercept the X axis at A and B respectively, and the third one intercepts the T axis at C, and the T' axis at C$'$. Fix the scale of the X and X' axes 1,2,3,... at equal intervals by taking unit length along the X' axis to be unit length along the X axis. Similarly fix the time scale 1,2,3,.. on the T axis at equal intervals. Draw lines 11,22,33,.. from the T axis, parallel to the X axis, to fix the length scale on the T' axis as shown. In other words if the intercept O1 on the T axis represents unit length on the T axis then the intercept O1 on the "slanted" T' axis represents unit time on this axis. Using these scales it is then seen that

$$\begin{aligned} \widehat{\mathrm{OC'}} &= \widehat{\mathrm{OC}}, \qquad \text{Or,} \quad t' = t. \\ \widehat{\mathrm{OA}} &= \widehat{\mathrm{OB}} - ut, \quad \text{Or,} \quad x' = x - ut. \end{aligned} \tag{6}$$

In the above equation we have used "wide-hat" $\widehat{......}$ to represent the measure of a given segment indicated by the capital letters. We have thus retrieved the same GT as given in Eqs. (3) by a graphical method.

It is then seen that just as the XYZ axes are not unique due to invariance of Newton's laws of motion, under the group ($\boldsymbol{a}$) of transformations, the time axis is not unique due to invariance of Newton's laws of motion under the ($\boldsymbol{b}$) group of transformations.

In Fig. 4(b) we have pictured a *rotated* time axis T' under a more general GT.

Minkowski observes, which we write as the following proposition

> "Hence we may give to the time axis whatever direction we choose towards the upper half of the world, $t > 0$."

He next poses the question,

> "Now what has the requirement of orthogonality in space to do with the perfect freedom of the time axis in an upward direction?"

4 The Group G_c

The central theme of Minkowski's paper is *invariance of the laws of physics under a group of transformation which he denoted as G_c.* It consists of all linear transformations in space-time which change the co-ordinates (x, y, z, t) of an "event" to a new set of co-ordinates (x', y', z', t') in such a way that the expression

$$F(x, y, z, t) \stackrel{\text{def}}{=} c^2t^2 - x^2 - y^2 - z^2 \tag{7}$$

remains invariant, i.e.,

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2. \tag{8}$$

In Minkowski's words

> "Let us take a positive parameter c, and consider a graphical representation of
>
> $$c^2t^2 - x^2 - y^2 - z^2 = 1.$$
>
> It consists of two surfaces separated by $t = 0$, on the analogy of a hyperboloid of two sheets. We consider the sheet in

> the region $t > 0$, and take those homogeneous linear transformations of x, y, z, t into four new variables x', y', z', t', for which the expression for this sheet in the new variables is of the same form. ... It is evident that the rotations of space about the origin pertain to these transformations."

It seems to us that by "rotations of space" Minkowski means "rotations of space-time". Minkowski then proposes a graphical construction of this transformation simplifying it by keeping the y, z variables unchanged.

> "Thus we gain full comprehension of the rest of the transformations simply by taking into consideration one among them, such that y and z remain unchanged."

We shall explain this construction in the next section. We shall however proceed with the following remarks. A trivial example of G_c is pure rotation in space, i.e., the group of orthogonal transformations denoted by $SO(3)$ and discussed in the previous section. It is obtained from Eq. (8) by setting $t = 0$. Our interest however lies in *Lorentz transformation* proper which had been derived by Einstein as a corollary to his relativity postulates. In our discussion to follow we would like to be clear that we are talking about Lorentz transformation, or the group of Lorentz transformations (even though Minkowski is silent about it) by mentioning G_c. We shall often use the abbreviation LT to mean Lorentz transformation.

Let S and S' be two "inertial frames of reference" whose axes X, Y, Z and X', Y', Z' are parallel and coincide at $t = t' = 0$. In the rest of this article we shall prefer the term *Lorentz frame* to mean an inertial frame in the context of special relativity. Let $\mathbf{u} = \boldsymbol{\beta} c$ be the velocity with which S' is moving relative to S. Here c is the speed of light, so that $\boldsymbol{\beta}$ is a dimensionless velocity having magnitude less than unity. The "event" co-ordinates (x, y, z, t) and (x', y', z', t') mentioned above are with reference to S and S' respectively.

A simple and special case of this transformation is when the *boost* is in the x direction, so that $\mathbf{u} = \beta c\, \mathbf{i}$ as illustrated in Fig. 3. The intended transformation is then:

$$\begin{aligned} x' &= \gamma(x - \beta ct), & (a) \\ y' &= y; \quad z' = z. & (b) \\ ct' &= \gamma(ct - \beta x). & (c) \\ \text{with} \quad \gamma &= \frac{1}{\sqrt{1-\beta^2}}. & (d) \end{aligned} \tag{9}$$

5 Graphical Construction of Lorentz Transformation

We shall now present Minkowski's construction of Lorentz Transformation, analogous to the graphical construction of Galilean transformation presented in Sec. 3. Imagine the hyperboloid hyper-surface

$$F(x, y, z, t) = 1, \tag{10}$$

carved out in space-time, where the expression for $F(x, y, z, t)$ was defined in Eq. (7). For the sake of pictorial representation of this hyperboloid on this sheet of paper we shall suppress the y and z dimensions. This will reduce the hypersurface to a two dimensional curve, viz., the hyperbola

$$c^2t^2 - x^2 = 1. \tag{11}$$

In Fig. 5(a) we have plotted the above *hyperbola* and its *asymptotes* OE ($t = \frac{x}{c}$) and OF ($t = -\frac{x}{c}$). Its vertex V lies at ($x = 0, t = \frac{1}{c}$).

Draw any "radius vector" OA from the origin O to any point A on the hyperbola. *This line extended indefinitely becomes the new T' axis.* At A draw a *tangent* to the hyperbola, intersecting the asymptote OE at B. The parallelogram ABCO is completed by drawing BC and OC parallel to AO and AB respectively and intersecting at C. *The straight line OC is now extended to represent the new X' axis.*

We now *specify the scales of the new axes* as follows. The intercept OC is to measure unity on the X' axis and the intercept OA to measure $\frac{1}{c}$ on the T' axis. We have reproduced this scale in Fig. 5(b).

Let us consider an arbitrary event P having co-ordinates (x_P, t_P) with respect to the $X - T$ axes. The event P is now projected to the pair of points (R,Q) on the $X' - T'$ axes. With the scales just defined the space-time co-ordinates (x'_P, t'_P) of P with respect to the $X' - T'$ axes are now given by the following ratios.

$$x'_P = \frac{\widehat{OR}}{\widehat{OC}}, \quad t'_P = \frac{1}{c}\left(\frac{\widehat{OQ}}{\widehat{OA}}\right). \tag{12}$$

In the above equation we have used "wide-hat" to represent the length of a given segment indicated by the capital letters. This completes Minkowski's graphical transformation of (x, t) to (x', t').

It now needs to be seen that the transformation of co-ordinates given above meets our requirement, which we write in the form of the following statement.

Theorem 1 *The co-ordinates (x', t') of an arbitrary event P referred to the new $X' - T'$ axes are related to their old co-ordinates (x, t) referred to the $X - T$ axes in such a way that*

$$F(x, t) \stackrel{\text{def}}{=} c^2t^2 - x^2 \tag{13}$$

remains invariant. That is,

$$c^2t'^2 - x'^2 = c^2t^2 - x^2. \tag{14}$$

Instead of proving the above theorem directly we shall now prove the following equivalent lemma.

Lemma: 1 *The co-ordinates (x', t') of an arbitrary event P referred to the new $X' - T'$ axes are related to their old co-ordinates (x, t) referred to the $X - T$ axes in such a way that they satisfy Lorentz transformation given in Eq. (9).*

It is a simple exercise (the reader must have done this himself as part of his homework in special relativity) to prove that the transformation (9) satisfies the invariance required by Eq. (14).

<u>*Proof of lemma 1*</u> : Let us first note that the X' and T' axes must satisfy $t' = 0$ and $x' = 0$ respectively. By Eq. (9) they must therefore to be represented by the following straight lines

$$\begin{array}{llll} X' \text{ axis}: & t & = & \frac{\beta}{c}x \quad (a) \\ T' \text{ axis}: & t & = & \frac{1}{\beta c}x \quad (b) \end{array} \tag{15}$$

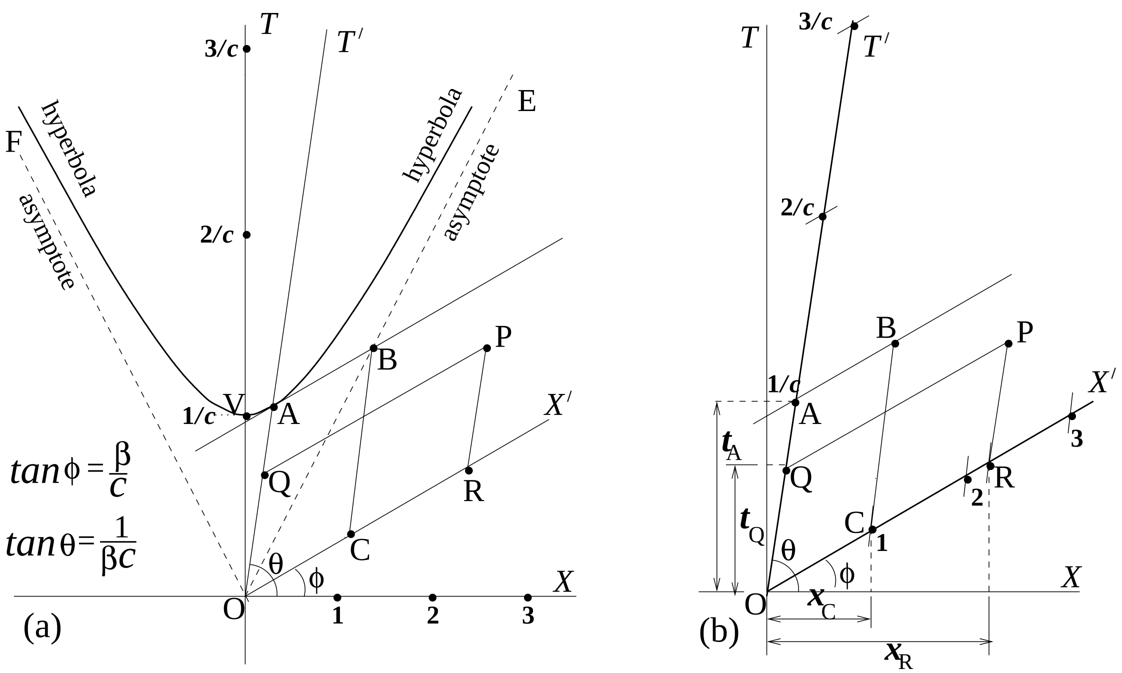

Figure 5: *Graphical construction of Lorentz transformation.*

Let the angle subtended by the line OA (which represents the T' axis) with the X axis be θ. Then by Eq. (15b) $\tan\theta = \frac{1}{\beta c}$.

Note that a tangent $\frac{dt}{dx}$ to all hyperbolas of the form

$$c^2t^2 - x^2 = k, \text{ satisfies } \frac{dt}{dx} = \frac{x}{c^2 t}. \quad (16)$$

Hence the tangent AB drawn at A must have the slope $\tan\phi = \frac{x}{c^2t} = \frac{\beta ct}{c^2t} = \frac{\beta}{c}$, since the point A lies on the T' axis. Consequently the line OC which is parallel to AB, will have the same equation as that of the X' axis given by Eq. (15a). We are thus satisfied that the lines OC and OA truly represent the transformed $X' - T'$ axes.

In summary, the angles θ, ϕ that the T', X' axes make with the X axis are given as

$$\tan\theta = \frac{1}{\beta c}; \quad \tan\phi = \frac{\beta}{c}. \quad (17)$$

We shall now convert the segment ratios given in Eq. (12) into algrebraic expressions in terms of the co-ordinates (x_P, t_P) of the event P. For this purpose we shall obtain the (x, t) co-ordinates of the points A, B, C, R, Q on the $x-t$ diagram. Each of these points lies at the intersection of two curves or straight lines. The reader should work out the equations of the straight lines OA, AB, CB, OC, QP, RP and the co-ordinates of the intersection points to tally with the results tabulated below.

Straight line # 1	Curve/straight line # 2	Their intersection
OA : T' axis: $t = \frac{1}{\beta c}x$	Hyperbola ; $c^2t^2 - x^2 = 1$	A: $(x_A, t_A) = \left(\beta\gamma, \frac{\gamma}{c}\right)$
AB : $t = \frac{1}{c}\left(\frac{1}{\gamma} + \beta x\right)$	OE :$x = ct$	B: $(x_B, t_B) = \frac{1}{\gamma(1-\beta)}\left(1, \frac{1}{c}\right)$.
OC : X' axis : $t = \frac{\beta}{c}x$	BC : $t - \frac{1}{c\gamma(1-\beta)} = \frac{1}{\beta c}\left(x - \frac{1}{\gamma(1-\beta)}\right)$	C: $(x_C, t_C) = \left(\gamma, \frac{\beta\gamma}{c}\right)$
OC : X' axis : $t = \frac{\beta}{c}x$	PR : $t - t_P = \frac{1}{c\beta}(x - x_P)$	R: $x_R = \gamma^2(x_P - \beta c t_P)$
OA : T' axis : $t = \frac{1}{\beta c}x$	PQ : $t - t_P = \frac{\beta}{c}(x - x_P)$	Q: $t_Q = \gamma^2(t_P - \frac{\beta}{c}x_P)$
Ox : X axis : $t = 0$	BC : $t - \frac{1}{c\gamma(1-\beta)} = \frac{1}{\beta c}\left(x - \frac{1}{\gamma(1-\beta)}\right)$	D: $x_D = \frac{1}{\gamma}$

(18)

The last row in the table, though not relevant to the present exercise, will be useful in the next section.

We can now compute the ratios suggested in Eq (12) from the geometrical constructions given in Fig. 5 and the values of the co-ordinates obtained in Eqs. (18).

$$x'_P = \frac{\widehat{OR}}{\widehat{OC}} = \frac{x_R}{x_C} = \gamma\left(x_P - \beta c t_P\right);$$
$$t'_P = \frac{1}{c}\left(\frac{\widehat{OQ}}{\widehat{OA}}\right) = \frac{1}{c}\frac{t_Q}{t_A} = \gamma\left(t_P - \tfrac{\beta}{c}t_P\right). \tag{19}$$

which are same as the Lorentz Transformation equations of (9).

Q.E.D.

We have thus completed a graphical construction of Lorentz transformation for arbitrary "boost" βc in the X direction. With $0 \leq \beta < 1$ as continuous parameter the set of all such boosts constitute a 1-parameter group. It is a subgroup of a larger 3-parameter group in which the boost velocity $\boldsymbol{\beta}$ has arbitrary direction. It is this group that Minkowski has identified with the symbol G_c.

We shall summarize the essential features of Minkowski's graphical construction in the following proposition.

Proposition 1 *1. A straight line drawn from the origin O of the $X - T$ plane to any point A of the hyperbola given in Eq. (11) can become a new time axis which we can represent as T'.*

2. Let AB be a tanegent to this hyperbola at A. Then a straight line OR drawn from the origin O and parallel to the straight line AB can be the new space axis which we can represent as X'.

3. Projection of an event P projected on the new X', T' axes according to the rules and scales defined in this section will give the new (x', t') coordinates of this event under a boost $c\beta$ along the X axis.

6 The Group G_∞

We have just seen that G_c is associated with Lorentz transformation. When $c \to \infty, \gamma \to 1, \beta c = u$, and G_c becomes G_∞. In this limiting case the Lorentz transformation Eq. (9) reduces to the old *Galilean Transformation* as given in Eq. (3).

In Fig. 6 we have illustrated graphically the metamorphosis of $G_c \to G_\infty$ as $c \to \infty$. However, $c\beta = u$ remains unchanged, so that $\beta \to 0$. As a consequence the inclination angle $\theta = \tan^{-1} u$ of the axis T' remains unchanged. However, since $\frac{\beta}{c} \to 0$, the angle $\phi = \tan^{-1}\left(\frac{\beta}{c}\right) \to 0$, and the axis X' merges with the X axis.

In Fig. 6(a) the line TT$'$ is parallel to the X' axis, so that the intercepts $\widehat{OT}$ on the T axis and $\widehat{OT'}$ on the T' axis give the same measure of time interval with respect to the frames S and S' respectively. The same is true in Fig. 6b, with the line TT$'$ being now parallel to the common X and X' axis, since the two have merged.

In contrast one common scale of length measurement (along the common X and X' axis) applies to both systems of reference.

Now consider an arbitrary event P with co-ordinates (x, t) with respect to the system S. It is projected to the points $(\mathrm{R}', \mathrm{Q}')$ on the $X'-T'$ axis and (R, Q) on the $X-T$ axis. Using the scales as described above one obtains the same values for the co-ordinates (x', t') of P with respect to S' as given in Eq. (3). We get back the GT in space-time as in Sec.3 and Fig. 6(b) becomes similar to Fig.3(a). This is the transformation under the group G_∞.

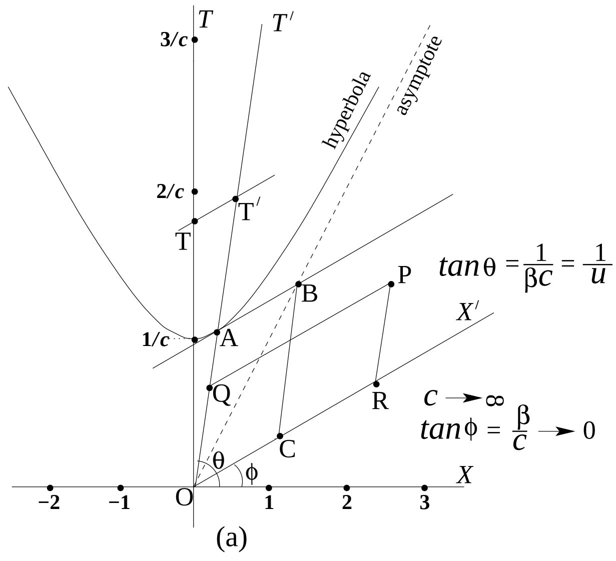

Figure 6: *Metamorphosis* $G_c \to G_\infty$

7 Invariance of the Laws of Nature under G_c

Minkowski observes,

> "The axis of T' may have any upward direction whatever, while X' approaches more and more exactly to X. In view of this it is clear that the group G_c, in the limit when $c = \infty$, that is the group G_∞, becomes no other than that complete group which is appropriate to Newtonian Mechanics. This being so, and since G_c is more intrelligible than than G_∞, it looks as though the thought might have struck some mathematician, fancy-free, that after all, as a matter of fact, natural phenomena do not possess an invariance with the group G_c, but with a group G_c, c being finite and determinate, but in ordinary unit of measure, *extremely great.* Such a premonition would have been an extraordinary triumph for pure mathematics. Well, mathematics, ... with its senses sharpened by an unhampered outlook to far horizona, to grasp forthwith the far-reaching consequences of such a metamorphosis of our concept of nature.
>
> I will state at once what is the value of c with which we shall be finally dealing. *It is the velocity of propagation of light in empty space.*"

Minkowski now touches on the central theme of his paper, viz., invariance of the laws of nature,

> "The existence of the invariance of the natural laws for the relevant group G_c would have to be taken then in this way:-

> From the totality of natural phenomena it is possible, by successive enhanced approximations, to derive more and more exactly a system of reference x, y, z, t, space and time, by means of which these phenomena then present themselves in agreement with definite laws. But when this is done, this system of reference is by no means unequivocally determined by the phenomena. *It is still possible to make any change in the system of reference that is in conformity with the transformations of the group G_c, and leave the laws of nature unaltered.*"

We shall elucidate. There *was* a hypothetical frame of reference, often referred to as the *Absolute frame of reference*, which Newton had in mind when he enunciated the laws of motion. We shall refer to it as the *absolute inertial frame*, or the AIF. However, the basic tenets of Newtonian mechanics hold not only with reference to the AIF but also with reference to any "non rotating frame of reference" whose origin is moving relative to the AIF in a straight line with a constant velocity $\boldsymbol{\beta}c$. Such a frame of reference is called an *inertial frame*, or IF. It had been believed that Maxwell's equations of electrodynamics were valid when referred to the AIF, but not with reference to any other IF. The null result of Michelson-Morley experiments pointed to the fallacy of such a notion. Einstein's formulation of special relativity demonstrated that Maxwell's equations are valid in all IFs, provided one transformed both the co-ordinates and the fields from one IF to another according to Lorentz transformation. What Minkowski proposed in the above statements is an echo of Einstein's relativity postulate, which essentially says that the laws of physics, when formulated with correct mathematical equations, are valid with reference to all IFs. This principle is often referred to as the *Principle of Covariance*. Since different IFs are now connected by Lorentz transformations, we shall use the term Lorentz frames to mean all IFs (as already mentioned.)

Referring to the geometrical construction shown in Fig. 5, in which we have transformed x, t axes to x', t', keeping y, z axes unaltered, Minkowski points out,

> "We may also designate time t', but then must of necessity, in connection therewith, define space by the manifold of the three parameters x', y, z, in which case the physical laws would be expressed in exactly the same way by means of x', y, z, t' as by means of x, y, z, t. We should then have in the *world* no longer space, but an *infinite number of spaces*, analogously as there are in three dimensional space an infinite number of planes. Three dimensional geometry becomes a chapter in four-dimensional physics."

We shall present our interpretation. The term "space" may perhaps be given the following mathematical connotation. It is a *continuum*, being a continuous set $\mathcal{P}$ of points having co-ordinates (x, y, z) given with reference to the Cartesian axes XYZ of a certain IF S, at any instant of time t. $\mathcal{P}$: $\{(x, y, z); -\infty < x, y, z < \infty, t = \text{constant}\}$.

In a sense $\mathcal{P}$ *is the entire set of "simultaneous events"* in S. Similarly the space $\mathcal{P}'$ is the entire set of simultaneous events in S'. However, with respect to S (almost) half of the events comprising $\mathcal{P}'$ occured in the past and (almost) the other half will occur in future, as we know from LT. The "space" $\mathcal{P}'$ is therefore not the same as the space $\mathcal{P}$. Every IF has its own space which, in general, is different from the space of another IF. This point can be further illustrated with reference to Fig. 5a, which, due to sppression of the Y, Z axes, presents the picture of a 1-dimensional universe. The axis X, or any straight line parallel to it, represents the space $\mathcal{P}$. Similarly any straight line parallel to X' represents the space $\mathcal{P}'$. Except one point they have nothing in common.

If we had taken into consideration both the X and the Y axes while constructing the diagram in Fig. 5a, the staright line X axis would be replaced by one $X - Y$ plane. In that case $\mathcal{P}$ would be represented by any plane parallel to the $X - Y$ plane, and $\mathcal{P}'$ by any plane parallel to the $X' - Y'$ plane. There would be an infinite number of such planes representing spaces $\mathcal{P}, \mathcal{P}', \mathcal{P}'', \ldots$, associated with inertial frames $S, S', S'', \ldots$.

Further extension of this picture into the real 3-dimensional universe is difficult to visualize. The spaces $\mathcal{P}, \mathcal{P}', \mathcal{P}'', \ldots$ in this case are 3-dimensional "planes" embedded in the 4-dimensional "world" of Minkowski.

8 The Fundamental Axiom

Sec. II of Minkowski's paper is devoted to two important statements, the first one of which he calls the *fundamental axiom*, and the second one, the *world postulate*.

Let us consider the world lines (WL) of three particles, which we have labelled as #1, #2 and #3 in Fig. 7.

(a) Particle #1 is a stationary particle in the frame $X - T$. Its WL is a staight line parallel to the T axis.

(b) Particle #2 is moving with constant velocity $u = \beta c$ in the X direction. Its WL is a staight line of the form $x = x_0 + ut$, where x_0 is a constant. This WL is therefore parallel to the T' axis (same as the T' axis of Fig. 5.)

(c) Particle #3 is moving with varying speed in the X direction. Its WL is a curved line.

It is then that the *WLs of a stationary or a uniformly moving particle is the same as the time axis of the Lorentz frame in which the particle is at rest.* Minkowski's fundamental postule is an extension of this statement.

Suppose at a world point U the tangent to the WL of #3 is parallel to the straight line OA drawn from the origin to the hyperbola, so that the instantaneous velocity of the particle at the event point U is $u = \beta c$. One can then choose a new set of axes $X'' - T''$ at U, parallel to the $X' - T'$ axes (as in the graphical construction of LT, shown in Fig. 5, in which the X' and the T' axes are given by the lines OR and OA) and the particle will be "at rest" with respect to these axes at the event

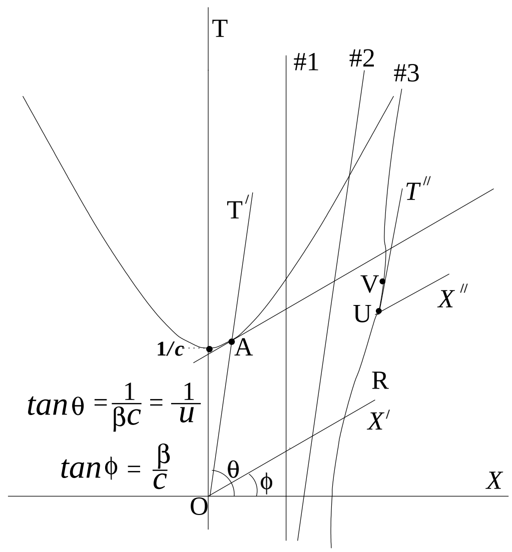

Figure 7: *Rest frame of a moving particle*

U. This, according to Minkowski is a fundamental axiom and states it with emphasis as follows.

Fundamental Axiom: "The substance at any world point may always with the appropriate determination of space-time, be looked upon as at rest."

Minkowski points out an important corollary of this axiom. Let there be two infinitely close events U: (x, y, z, t) and V: $(x + dx, y + dy, z + dz, t + dt)$ on the world line of an arbitrarily moving particle. Let us assume that the instantaneous velocity of the particle at U is $\boldsymbol{\beta}c = (\beta_x,\ \beta_y,\ \beta_z)c$, so that

$$\begin{aligned} dx = \beta_x ct,\ dy = \beta_y ct,\ dz = \beta_z ct;\\ \text{with}\quad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2. \end{aligned} \tag{20}$$

Now, since all transformations under G_c are linear transformations, the co-ordinate differentials (dx, dy, dz, dt) transform linearly in the same way as the coordinates (for example, their transformations may follow Eq. (9) with the differentials $dx, .., dt; dx', ..., dt'$ replacing $x, ...t; x', ..., t'$). All transformations under G_c leave the expression $F(x, y, z, t)$ invariant, as indicated in Eqs. (7), (8). The transformation of the co-ordinate differentials (dx, dy, dz, dt) must then leave the following expression invariant.

$$ds^2 \stackrel{\text{def}}{=} c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{21}$$

This is the same thing as saying

$$\begin{aligned} &c^2dt'^2 - dx'^2 - dy'^2 - dz'^2\\ &= c^2dt^2 - dx^2 - dy^2 - dz^2. \end{aligned} \tag{22}$$

Applying the above equation to the infinitesimal displacement along the WL #3, noting that $dx'' = dy'' = dz'' = 0$, since the particle is instantaneously at rest, and using Eq. (20), we get

$$c^2 dt''^2 = c^2(1-\beta^2)dt^2. \qquad (23)$$

Since the left side is positive definite, it follows that $\beta < 1$, so that $u < c$. This is same as the following corollary, as stated by Minkowsi:

Corollary: "c would stand as the upper limit of all substantial velocities."

The expression written in Eq. (21) is now known as the *Minkowski metric*. Minkowski wrote only the expression on the right side of (21), but not the ds^2 appearing on the left side, which we have inserted following the existing convention, and in recognition of the *major role it played in Einstein's construction of a relativistic theory of gravity* (known as the general theory of relativity).

In retrospect the Minkowski metric contains the seed of a geometrical idea which Einstein absorbed to formulate the new theory of gravity based on differential geometry. The metric brings in geometrical concepts, like curvature, geodesics (shortest line joining two points). Minkowski metric represents "flat" space-time in which gravity is absent, and the geodesics, representing world lines of particles, are straight lines.

In Einstein's theory the metric of *space-time* is more general than the form given in Eq. (21), and gravity is recognized by how much it differed from the Minkowskian one. In Einstein's theory of gravity "the metric is the foundation of all[5]." Distribution of matter, or rather distribution of energy-momentum in space distorts space-time into a curved one, making the geodesics curved, and the world lines of free falling objects, like planets, into curved lines.

9 Group G_c for Optics and G_∞ for Rigid Bodies

Before establishing a justification for the group G_c Minkowski raises the following questions.

> "The question is what are the circumstances which force this changed conception of space and time upon us? Does it actulally contradict experience? And is it advantageous for describing phenomena?"

To the first question he gives the following answer.

> "The impulse and true motive for assuming the group G_c came from the fact that the differential equation for the propagation of light in empty space possesses that group G_c."

We shall present our intepretation of the above statement. By "the differential equation for the propagation of light in empty space" we mean the following homogeneous wave equation:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(x,y,z,t) = 0. \qquad (24)$$

in which ψ represents any one of the 6-components of the propagating electromagnetic field $(\mathbf{E},\mathbf{B}) = (E_x, E_y, E_z, B_x, B_y, B_z)$

in any region of space where there are no sources of electric charge and current. It is known that the wave equation (both the homogeneous form shown above which is valid in empty space, and the inhomogeneous form for the scalar and vector potentials which is valid everywhere) is a direct consequence of Maxwell's equations. Einstein had shown in the first one of his 1905 relativity papers that the transformation (9) leads to transformations of the electromagnetic field $(\mathbf{E}, \mathbf{B}) \to (\mathbf{E}', \mathbf{B}')$, and electric charge-current densities $(\rho, \mathbf{J}) \to (\rho', \mathbf{J}')$, in such a way that Maxwell's equations remain invariant[6]. That would mean that Lorentz transformation will transform $[(x, y, z, t), \psi] \to [(x', y', z', t'), \psi']$, and when this is done the "equation for the propagation of light" given in Eq. (24) will transform into a new equation in which the unprimed quantities will be replaced by primed quantities. In other words Eq. (24) will remain invariant under any transformation under the group G_c.

Referring to the second question Minkowski surmises, "the concept of rigid bodies has meaning only in mechanics satisfying the group G_∞," suggesting thereby that G_c contradicts our concept of rigid bodies.

How? Let the XYZ axes be fixed in a rigid body with its origin at O, and let (x, y, z) be the space co-ordinates of any arbitrary point P in the rigid body. *Mechanics of a rigid body* starts with the *axiom* that the distance $\sqrt{x^2 + y^2 + z^2}$ between O and P is unchanged in any motion of the body. It is easy to show that this assumption is violated by LT.

Proof : For convenience we redesignate the space co-ordinates of any arbitrary point P in the rigid body with respect to its *rest frame* S' as (x', y', z'). The distance $\widehat{OP}$ is $r' = \sqrt{x'^2 + y'^2 + z'^2}$. Let us now imagine that the same rigid body is moving in the X direction with velocity βc. We would like to know the distance $\widehat{OP}$ as measured by a laboratory observer S.

Since the body is moving with respect to S, we have to think of two *simultaneous events* $\mathbb{O}$: "$O(0, 0, 0)$ at $t = 0$" and $\mathbb{P}$: "$P(x, y, z)$ at $t = 0$", and find out the distance between their locations in S. Using Lorentz transformation (9) we get $x' = \gamma x$. Hence the distance $\widehat{OP}$, as measured in S, is

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{x'^2/\gamma^2 + y'^2 + z'^2}.$$
$$\text{Thus } r < r'. \tag{25}$$

The body has contracted in the direction of motion by a factor γ (Lorentz contraction).

Q.E.D.

10 Graphical Construction of Length Contraction

Minkowski does not derive the formula (25) using LT. He presents a graphical construction of length contraction. His motivation is to show that

> "the Lorentzian hypothesis (of length contraction) is completely equivalent to

> the new conception of space and time, which, indeed makes the hypothesis much more intelligible."

We shall present Minkowski's construction in a slightly different form to suit our taste. In Fig. 8(a) we have redrawn the hyperbola of Eq. (11) and the $X' - T'$ axes. The parallelogram OA′B′C′ is same as the parallelogram OABC of Fig. 5. The X' axis (the straight line OC′) represents an infinite set of events which are *simultaneous* in S' at $t' = 0$. The straight line A′B′, which is parallel to the X' axis, represents another infinite set of *simultaneous* events (in S') when $t' = \frac{1}{c}$ (see specification of time scale in Sec. 5, following Eq. (11).)

Imagine two frames of reference S and S', characterized by coordinate axes X, T and X', T' respectively, such that S' has boost $c\beta$ in the X direction with respect to S. Let M and M′ be two standard (hence identical) meter sticks. These sticks are lain, respectively, along the X axis of S, and along the X' axis of S' such that their left ends coincide with the respective origins. The right ends of M and M′ will coincide with the points C and C′ on the X axis of S and X' axis of S' respectively (Fig. 8b). Note that the segment OC′, shown with thick line, represents unit length on the X' axis. The world line of its left end will be the T' axis represented by the line OA′t′ and the world line of the right end will be the parallel line C′B′ intersecting the X axis at D.

What is the length ℓ of M′ in the frame S? It is the distance between the points O and D on the x axis which coincide with the left end and the right end of the rod simultaneously (at $t = 0$). In other words $\ell = \widehat{OD}$.

In the last row of table-equation (18) we had obtained the x co-ordinate of the point D to be equal to $\frac{1}{\gamma}$. Hence,

$$\widehat{OD} = \frac{1}{\gamma}. \tag{26}$$

Eq. (26) gives *contracted length* of a 1 meter long meter rod in motion.

Let there be now two objects (which Minkowski calls"images of two equal Lorentzian electrons") each of length ℓ. The first one of them is at rest and the second one moving with uniform velocity βc, when seen from the system S. Then the length of this second object will be $\ell' = r\ell = \frac{\ell}{\gamma}$ in the system S.

There is an alternative graphical construction method suggested by Loedel for obtaining length construction and time dilation[7].

11 The World Postulate

Much of classical mechanics and classical electrodynamics is built on geometrical picturization of physical quantities - e.g., velocity, acceleration, electric and magnetic fields - as vectors, which are *directed line segments* in a 3-dimensional Euclidean space E^3 spanned by the X, Y, Z axes.

We have two fundamental equations of classical mechanics. One of them equates the time rate of change of linear momentum vector $\mathbf{p}$ to force vector $\mathbf{F}$. The other one equates the time rate of change of angular momentum vector $\mathbf{L}$ to torque vector $\mathbf{N}$. Similarly, the

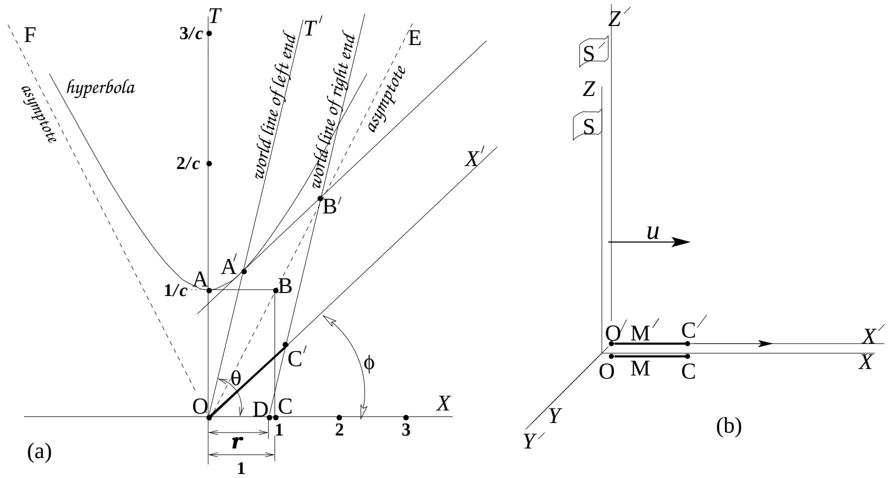

Figure 8: *Geometrical construction of length contraction*

laws of classical electrodynamics are written in the form of four equations involving electric field vector **E** and magnetic field vector **B**. In a sense the important physical quantities of classical physics are vectors, or geometrical objects, in the 3-dimensional Euclidean space, and what we call physical laws are relationships among such 3-dimensional geomterical objects.

Minkowski feels that the term *relativity postulate* (coined by Einstein) is a rather feeble word to emphasize invariance of the laws of the laws of nature under G_c. He prefers the term the *Postulate of the Absolute World*, or, in brief, the *World Postulate* for which he makes the following statement:

> "Only four dimensional world in space and time is given by the phenomena, but that the projection in space and in time may still be undertaken with a certain degree of freedom."

Here we may add that the above statement is similar to what Minkowski says earlier, "*It is still possible to make any change in the system of reference...*", quoted by us in Sec. 7, on page 15.

12 Construction of Time-like and Space-like Vectors

Minkowski now follows up the World Postulate to propose a 4-dimensional equivalent of Newton's 2nd Law of motion, familiar in the form $\mathbf{F} = m\boldsymbol{a}$. As a first step towards this goal he begins by showing how to construct 4-vectors. He begins Sec.III of his paper with the following remarks.

> "The world postulate permits identical treatment of the four co-ordinates x, y, z, t. By this means, as I shall now

> show, the forms in which the laws of physics are displayed gain in intelligibility. In particular the idea of acceleration gains a clear-cut character."

He explains his ideas using a space-time diagram which we have expanded into four subfigures in Fig. 9.

In Fig. 9a we have presented a flat picture of space-time by showing only two axes viz., X and T, whereas in the remaining ones of Fig. 9 we have tried to present a kind of 3-dimensional view by including one extra space axis.

Let O be an arbitrary event point. Taking O as the origin and using the function $F(x, y, z, t) = c^2t^2 - x^2 - y^2 - z^2$, as defined in Eq. (7) we draw the following three surfaces of revolution around the T axis: (a) $F(x, y, z, t) = 0$; (b) $F(x, y, z, t) = 1$; and (c) $F(x, y, z, t) = -1$. To be more explicit, these three surfaces are given by the following three implicit equations.

$$\begin{aligned} (a) \quad \Rightarrow \quad c^2t^2 - x^2 - y^2 - z^2 &= 0. \\ (b) \quad \Rightarrow \quad c^2t^2 - x^2 - y^2 - z^2 &= 1. \\ (c) \quad \Rightarrow \quad c^2t^2 - x^2 - y^2 - z^2 &= -1. \end{aligned} \tag{27}$$

They are described below.

(a) two branches of the cone $F(x, y, z, t) = 0$, corresponding to $t < 0$, called the *front cone*, and $t > 0$, called the *back cone*, as shown in Figs. 9a and 9b. We shall refer to these cones as *light cones*.

(b) two branches of the hyperboloid $F(x, y, z, t) = 1$, corresponding to $t < 0$ and $t > 0$, as shown in Figs. 9a and 9c.

(c) the hyperboloid $F(x, y, z, t) = -1$, as shown in Figs. 9a and 9d.

Minkowski goes further with his description.

> "The territory between the cones is filled by the one-sheeted hyperbolidal figures $-F = k^2$. ... We are specially interested in the hyperbolas with O as centre, lying on the latter figures. The single branches of these hyperbolas may be called briefly the *internal hyperbolas* with centre O. One of these branches, regarded as a world line, would represent a motion which, for $t = -\infty$ and $t = +\infty$, rises asymptotically to the velocity of light."

We have shown one such hyperbolid, corresponding to $k = 1$ in Fig. 9d., and one such internal hyperbola, labelled Ω, inscribed on it. It represents the the world line of a particle that comes from infinity at $t = -\infty$ with the speed of light, approaches the origin at $t = 0$ where it momentarily stops, and then recedes back to infinity at $t = \infty$ with the speed of light.

The territory *within the cones* is also filled with two sheeted hyperboloids $F = k^2$. One of them shown in Figs. 9a and 9b corresponds to $k = 1$. There is however no mention of the family of hyperboloids in Minkowski's statements.

The reader should note that the two dimensional curves $F = 1$ and $F = 0$ appearing in Fig. 9a are the same as the hyperbola and its asymptotes shown in Fig. 5.

> "If we now, on the analogy of vectors in space, call a directed length in the manifold of x, y, z, t a vector, we have

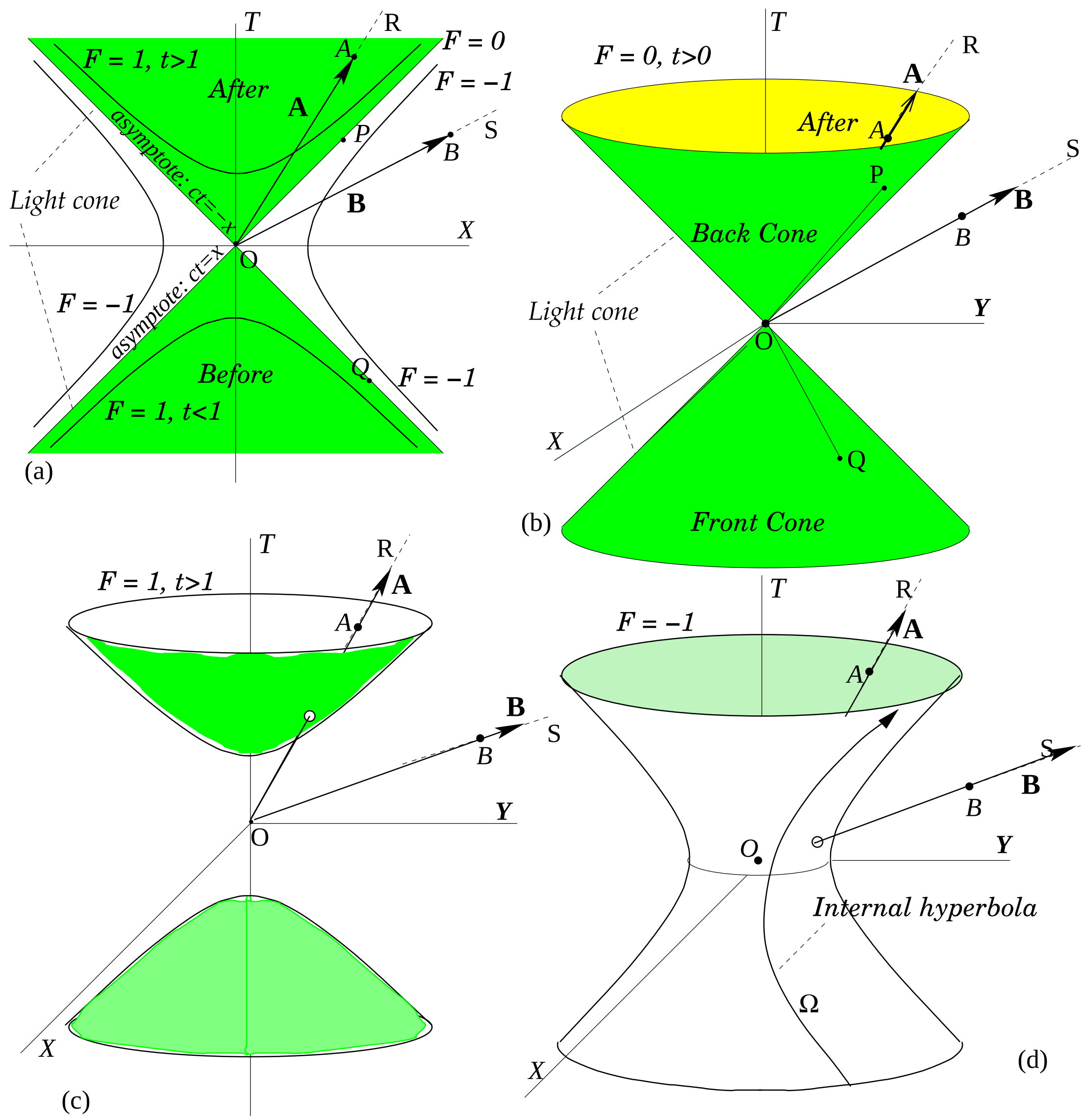

Figure 9: *Time-like and Space-like vectors and their relation to the Light Cone. (a) 2-dimensional (x-t) view of the light cone $F = 0$, the hyperbolids $F = 1$ and $F = -1$, and time-like, space-like vectors* $\mathbf{A}, \mathbf{B}$. *3-dimensional (x-y-t) views of (b) the light cone and* $\mathbf{A}, \mathbf{B}$*; (c) the hyperboloid $F = 1$; (d) the hyperboloid $F = -1$ and an internal hyperbola.*

to distinguish between the time-like vectors with directions from O to the sheet $+F = 1, t > 0$, and the space-like vectors with directions from O to $-F = 1$.

> The time axis may run parallel to any vector of the former kind."

The most elementary example of a 4-vector is a *4-dimensional directed line segment stretching from one event P to another event Q* drawn in Space-time (i.e., Minkowski's World.) Minkowski proposes two types of 4-vectors, viz., *time-like* and *space-like* vectors.

Minkowski proposes the components of these vectors in the following language.

> "We divide any vector we choose, e.g., from O to x, y, z, t into four components x, y, z, t."

We shall make a small departure from Minkowski's convention and take the components of the 4-vector $\overrightarrow{\mathrm{OR}}$ *as* (x, y, z, ct)*, i.e., multiply with c the time component proposed by Minkowski so that all the components of a 4-vector have the same dimension, viz., the dimension of length. This practice will be followed in the rest of this article*

It will be interesting to quote Minkowski's definition of two new types of vector.

> "If we now, on the analogy of vectors in space, call a directed length in the manifold of x, y, z, t a *vector*, we have to distinguish between the *time-like* vectors with directions from O to the sheet $+F = 1, t > 0$ and the *space-like* vectors with directions from O to $-F = 1$."

We shall elucidate. Let $\overrightarrow{\mathrm{OA}}$ be a *directed line segment* marked out on a straight line OR that intersects the upper sheet (i.e., $t > 0$) of the hyperbolid $F(x, y, z, t) = 1$ (Fig. 9c) and let $\overrightarrow{\mathrm{OB}}$ be another directed line segment marked out on another straight line OS that intersects the hyperboloid $F(x, y, z, t) = -1$ (Fig. 9d). Then $\overrightarrow{\mathrm{OA}}$ is a *time-like 4-vector*, and $\overrightarrow{\mathrm{OB}}$ is a *space-like 4-vector.*

Let us denote these vectors as $\overrightarrow{\mathbf{A}}$ and $\overrightarrow{\mathbf{B}}$. Any vector $\overrightarrow{\boldsymbol{\mathcal{A}}}$ which is parallel to $\overrightarrow{\mathbf{A}}$, is a *time-like 4-vector* and any vector $\overrightarrow{\boldsymbol{\mathcal{B}}}$ which is parallel to $\overrightarrow{\mathbf{B}}$, is a *space-like 4-vector* [2]. We shall find a better criterion in the next section for qualifying a vector as time-like or space-like.

In this article we shall denote a 4-vector with a full arrow "$\rightarrow$" on top of the symbol, as in the example $\overrightarrow{\mathbf{A}}$, and the conventional 3-vector with just a bold letter, e.g., **A** without any arrow on top.

Why such peculiar adjectives "time-like" and "space-like"? Minkowski's justification for the first adjective :

> "The time axis may run parallel to any vector of the former kind."

The last sentence is a reminder of the principle adopted in the graphical construction of Lorentz transformation, shown in Fig. 5, and expressed in the paragraph below Eq. (11) on

[2] Author's comment: A line segment that intersects the surface $+F = 1$ will also intersect the family of surfaces $+F = k^2$, since they are all asymptotic to the light cone $F = 0$, and lie inside the cone. Similarly a line segment that intersects the surface $-F = 1$ will also intersect the family of surfaces $\{-F = k^2\}$. In effect then a *time-like* vector is one which has the end point within the light cone, and a *space-like* vector is one having the end point outside it.

page 10: *any radius vector drawn from the origin to any point of the hyperbola $F = 1$ can become the time axis.*

We shall elucidate this further. Let O and A be two events. If it is possible to find a Lorentz frame of referece S', (by transforming from S to S' by a suitable Lorentz transformation) such that these two events accur at the same *spatial* coordinates (i.e., at $x' = x = 0, y' = y = 0, z' = z = 0$), then the radius vector $\overrightarrow{\mathrm{OA}}$ is a time-like vector. In this new frame S' this 4-vector will have a non-zero component $c\tau$ only along the time axis T, its spatial components (i.e., along the X, Y, Z axes) being all equal to zero. This time is the *proper time* between the events as we shall define in the next section.

In general any 4-vector $\overrightarrow{\boldsymbol{\mathcal{A}}}$ is a time-like vector if a suitable Lorentz transformation can reduce its X, Y, Z componets to zero, leaving a non-zero component only for its time component, i.e., along the cT axis.

Mikowski's justification of the second adjective "space-like" can be seen by taking another look at Fig. 5. *Any straight line OX' that does not intersect the hyperbola lies necessarily beyond the asymptotes and can become a space axis.* Analogously, in Fig.9(b) and (c) *any straight line that does not intersect the hyperboloid $F = 1$ lies necessarily beyond the light cone and can become a space axis.* All events are simultaneous along a space axis in a 4-dimensional space-time.

Let us now consider two events O and B. If it is possible to find a Lorentz frame of referece S', in which these two events accur simultaneously, then in this new frame the radius vector $\overrightarrow{\mathrm{OB}}$ has only space components, and no time component. Consequently we call this radius vector $\overrightarrow{\mathrm{OB}}$ a space-like vector.

In general any 4-vector $\overrightarrow{\boldsymbol{\mathcal{B}}}$ is a space-like vector if a suitable Lorentz transformation can reduce its time componet to zero, leaving a non-zero component along the space hyperplane spanned by the X, Y, Z axes.

It will be worthwhile to quote Minkowski's statement, "Any world point between the front and back cones of O can be arranged by means of the system of reference so as to be simultaneous with with O, but also just as well so as to be earlier than O or later than O. Any world point within the front cone of O is necessarily always before O; any world point within the back cone of O necessarily after O."

What about a line segment that intersects neither hyperboloid? Such a line segment must be either $\overrightarrow{\mathrm{QO}}$ lying on the front cone, or $\overrightarrow{\mathrm{OP}}$ lying on the back cone. Minkowski has not given any name for such vectors. In modern textbooks such vectors are called *null vectors*, and the cones they lie on (i.e., the front cone and the back cone) are jointly called the *light cone*. Justification of the first name will become clear as we proceed further. Justification of the second name lies in Minkowski own statement: the front cone of O consists of all the world-points (like Q) which "send light to O" and the back cone of O, of all the world points (like P) which "receive light from O". In brief, *all points lying on the light cone drawn from O are connected to O by light signals.*

13 Orthogonality and Magnitudes of 4-vectors

We shall quote Minkowski's definition of *orthogonality* and magnitudes of 4 vectors.

> "We divide up any vector we choose, e.g., that from O to x, y, z, t into the four components x, y, x, t. If the directions of two vectors are, respectively, that of a radius vector OR from O to one of these two surfaces $\mp F = 1$, and that of a tangent RS at the point R of the same surface, the vectors are said to be normal to each other. Thus the condition that the vectors with components x, y, z, t and x_1, y_1, z_1, t_1 may be normal to each other is
>
> $$c^2 tt_1 - xx_1 - yy_1 - zz_1 = 0. \qquad (28)$$
>
> For the measurement of vectors in different directions the units of measure are to be fixed by assigning to a space-like vector from O to $-F = 1$ always the magnitude 1, and to a time-like vector from O to $+F = 1, t > 0$ always the magnitude $\frac{1}{c}$."

We shall do some thinking to absorb the meaning of the above statement. Let us consider the first part of the statement (i.e., up to Eq. 28). In Fig. 10(a) we have shown the trace of the surfaces $\mp F = 1$ on a 2-dimensional $X - T$ plane. They have been marked by the labels Ω and Γ.

The line OR intersects the hyperboloid $F = 1$, and the line RS is a tangent to it. According to Minkowski's proposition the 4-vector $\overrightarrow{\mathcal{U}} = (u_x, u_y, u_z, u_t)$ which is parallel to the line OR and the 4-vector $\overrightarrow{\mathcal{V}} = (v_x, v_y, v_z, v_t)$ which is parallel to the tangent RS are normal to each other.

Why do we attribute orthogonality between these two vectors? The answer can be found in the graphical construction of Lorentz transformation presented in Sec. 5.

Let us define orthogonal vectors in the 4-dimensional space-time $\mathcal{W}$ by proposing that the unit vectors along the four axes X, Y, Z, T, as defined below,

$$\begin{aligned} \overrightarrow{\mathbf{e}_x} &= (1, 0, 0, 0), \\ \overrightarrow{\mathbf{e}_y} &= (0, 1, 0, 0), \\ \overrightarrow{\mathbf{e}_z} &= (0, 0, 1, 0), \\ \overrightarrow{\mathbf{e}_t} &= (0, 0, 0, 1). \end{aligned} \qquad (29)$$

are the foremost example of a quadruple of mutually orthogonal vectors.

Now, the Lorentz transformation $S \to S'$ can set the time axis along OR and the space axis along the line OX′ which is parallel to the line RS. These directions are then the directions of the transformed unit vectors $\overrightarrow{\mathbf{e}_t}'$ and $\overrightarrow{\mathbf{e}_x}'$, and hence, are orthogonal to each other.

It may require some deeper thinking to appreciate the definition of orthogonality given in Eq. (28). Let us recall how we construct a normal vector on a three dimensional surface. Let $\Phi(x, y, z) = k$ be a surface shown as Σ in Fig.10(b). Let $\mathrm{P}(x, y, z)$ be a point on this surface. The gradient vector $\boldsymbol{\nabla}\Phi$ at

P defined as

$$\boldsymbol{\nabla}\Phi(\mathbf{r}) \stackrel{\text{def}}{=} \mathbf{i}\frac{\partial\Psi(\mathbf{r})}{\partial x} + \mathbf{j}\frac{\partial\Psi(\mathbf{r})}{\partial y} + \mathbf{k}\frac{\partial\Psi(\mathbf{r})}{\partial z}. \quad (30)$$

This vector is necessarily normal to this surface at P. A unit normal vector $\mathbf{n}$ on Σ at P is $\boldsymbol{\nabla}\Phi$ divided by its magnitude, i.e., $\mathbf{n} = \frac{\boldsymbol{\nabla}\Phi}{|\boldsymbol{\nabla}\Phi|}$.

Let Q$(x+dx, y+dy, z+dz)$ be a point *on the surface* Σ infinitesimally close to P, at a displacement vector $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ from P, so that $\Phi(x+dx, y+dy, z+dz) = k$. Hence,

$$\begin{aligned} &\Phi(x+dx, y+dy, z+dz) - \Phi(x,y,z) = 0, && (a) \\ \text{Or,}\quad &\frac{\partial\Phi(\mathbf{r})}{\partial x}dx + \frac{\partial\Phi(\mathbf{r})}{\partial y}dy + \frac{\partial\Phi(\mathbf{r})}{\partial z}dz = 0. && (b) \\ \text{Or,}\quad &\boldsymbol{\nabla}\Phi(\mathbf{r})\cdot d\mathbf{r} = 0. && (c) \end{aligned} \quad (31)$$

Now, $d\mathbf{r}$ is a tangent vector on the surface Σ, passing through the point P. Let $\mathbf{t}$ be a unit tangent vector in the direction of $d\mathbf{r}$. Then the last relation in Eq. (31) is same as the trivial statement $\mathbf{n}\cdot\mathbf{t} = 0$, i.e., $\mathbf{n}$ and $\mathbf{t}$ are orthogonal.

Now we come back to space-time $\mathcal{W}$ and think of the hypersurface Σ shown in Fig. 10(a) of which the equation is

$$F(x,y,z,ct) = c^2t^2 - x^2 - y^2 - z^2 = k^2. \quad (32)$$

We shall obtain the 4-dimensional version of Eq. (31), using the 4-dimensional gradient operator $\overrightarrow{\Box} \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial ct}\right)$.

$$\begin{aligned} \overrightarrow{\Box}F(x,y,z,ct) &= \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial ct}\right) \\ &= 2(-x, -y, -z, ct) \end{aligned} \quad (33)$$

Let $d\overrightarrow{\mathbf{r}} = (dx, dy, dz, dct)$ be a 4-displacement vector on the hypersurface S. Then analogous to Eq. (31b) we have here

$$\begin{aligned} \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz + + \frac{\partial F}{\partial(ct)}d(ct) &= 0. && (a) \\ \text{Or,}\quad -x\,dx - y\,dy - z\,dz + ct\,d(ct) &= 0. && (b) \end{aligned} \quad (34)$$

Here R(x,y,z,ct) is a point on the hypersurface given in (32), so that $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathrm{OR}} = (x,y,z,ct)$ is the 4-radius vector from the origin to the hyperboloid, as shown in Fig.10(a).

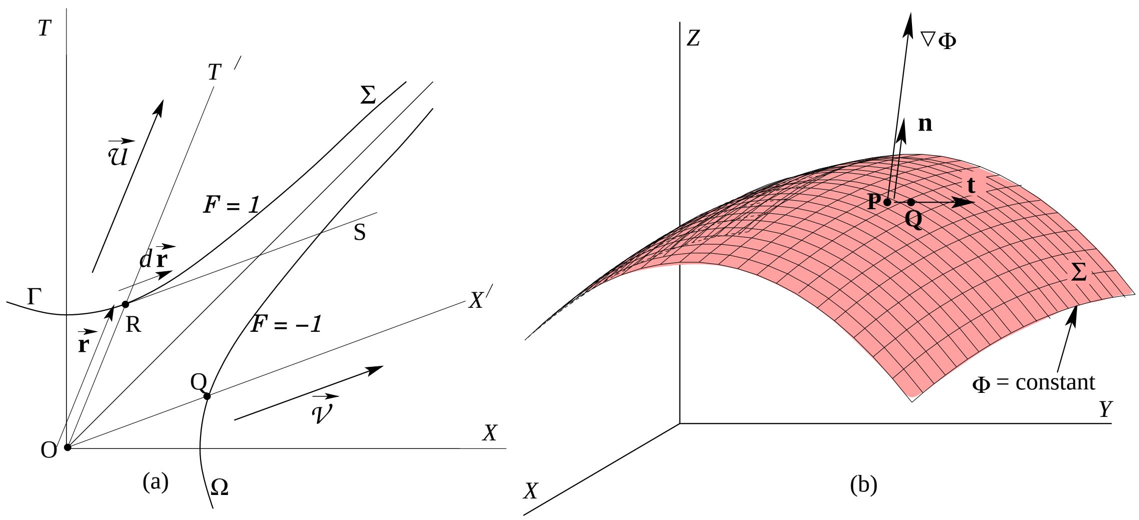

Figure 10: *Orthogonal 4-vectors*

By assumption $\overrightarrow{d\mathbf{r}} = (dx, dy, dz, dct)$ is tangent to this surface at P. Its direction is same as the direction of the tangent RS shown in Fig.10(a). Therefore they are mutually orthogonal.

Let us now think of the 4-vector $\overrightarrow{\mathcal{U}} = (u_x, u_y, u_z, u_t)$ which is parallel to $\overrightarrow{\mathbf{r}} = (x, y, z, ct)$ and the 4-vector $\overrightarrow{\mathcal{V}} = (v_x, v_y, v_z, v_t)$ which is parallel to $\overrightarrow{d\mathbf{r}} = (dx, dy, dz, dct)$. These vectors are orthogonal, as already said. Eq. (34b) would now imply

$$-u_x v_x - u_y v_y - u_z v_z + u_t v_t = 0. \qquad (35)$$

Eq. (35) matches Eq. (28), and is therefore the criterion of orthogonality of any two 4-vectors as laid out by Minkowski.

We shall now interpret the second part of Minkowski's statement (regarding "magnitude") using Fig. 11.

Consider the "unit vectors", in the 3-dimensional Euclidean space, as shown in Fig.(a). These vectors are shown as the line segments $\overrightarrow{\mathrm{OA}}, \overrightarrow{\mathrm{OB}}, \overrightarrow{\mathrm{OC}}, \cdots \overrightarrow{\mathrm{OE}}$. (a) The tips of these vectors lie on the surface of a sphere with O as origin, and given by the equation $x^2 + y^2 + z^2 = 1$. (b) Any vector $\overrightarrow{\mathrm{OA}}$ can be transformed into any other vector $\overrightarrow{\mathrm{OB}}$ by an orthogonal transformation mentioned in Sec. 3 and represented by Eq. (2). (c) One of these vectors, namely $\overrightarrow{\mathrm{OE}}$, intercepts the Z axis at $z = 1$.

There exists an analogous situation in space-time illustrated in Fig.(b), in which we have shown 4-vectors $\overrightarrow{\mathrm{OA}}, \overrightarrow{\mathrm{OB}}, \overrightarrow{\mathrm{OC}}, \cdots \overrightarrow{\mathrm{OE}}$ drawn against the $X - T$ axes (suppressing the Y, Z axes.) (a) The tips of these vectors lie on the hyperboloid $F(x, y, z, ct) \equiv c^2t^2 - x^2 - y^2 - z^2 = 1$. (b) Any vector $\overrightarrow{\mathrm{OA}}$ can be transformed into any other vector $\overrightarrow{\mathrm{OB}}$

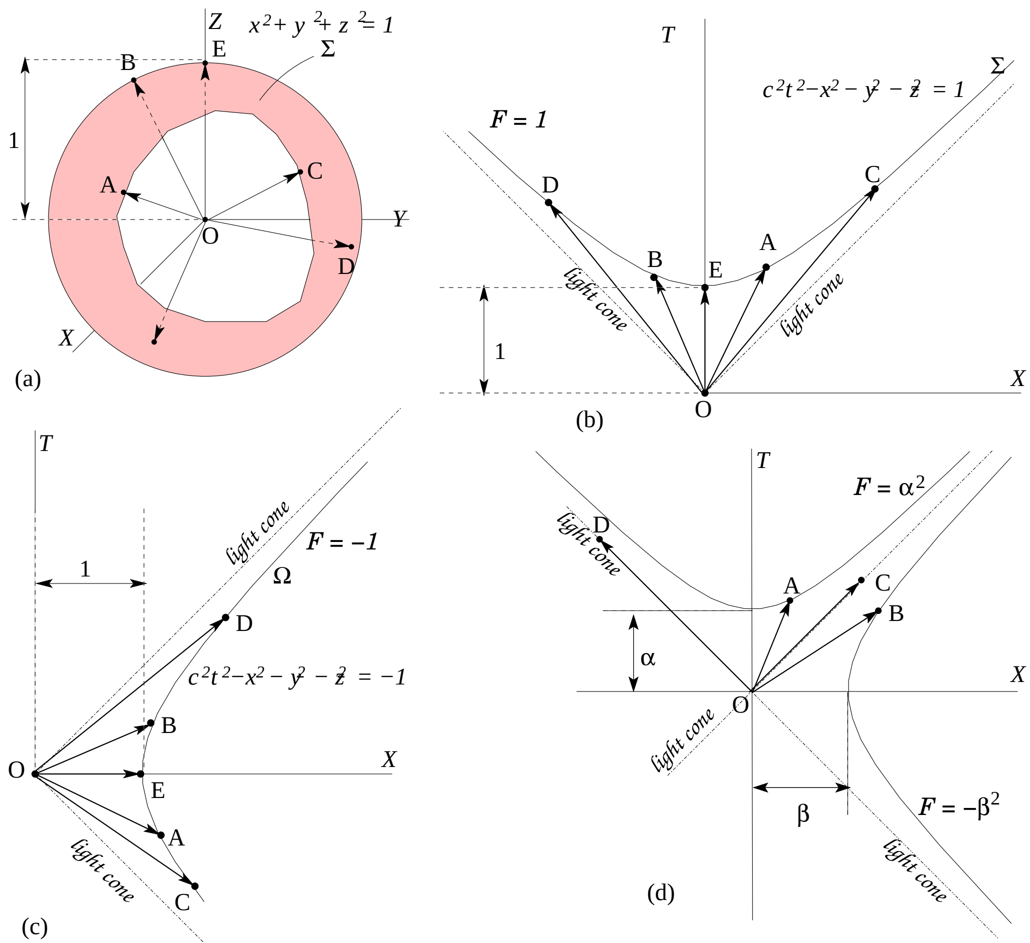

Figure 11: *Magnitude of a 4-vector*

by the Lorentz transformation mentioned in Sec. 4 and represented by Eq. (8). (c) One of these vectors, namely $\overrightarrow{\text{OE}}$ intercepts the T axis at $ct = 1$. Therefore, *we assign to all these vectors unit magnitude.*

The class of unit vectors mentioned in the previous para are all *time-like* vectors. In Fig (c) we have shown shown *space-like* vectors originating from the origin O and terminationg on the surface of the hyperboloid $F(x, y, z, ct) \equiv c^2t^2 - x^2 - y^2 - z^2 = -1$. As in the case of time-like vectors all the space-like 4-vectors shown in the diagram, namely, $\overrightarrow{\text{OA}}, \overrightarrow{\text{OB}}, \overrightarrow{\text{OC}}, \cdots \overrightarrow{\text{OE}}$ have the same length. Since $x = 1, ct = 0$ for the point E, all these space-like 4-vectors have *unit magnitude.*

How does one arrive at the magnitude of any arbitrary 4-vector using Minkowski's proposition? We shall find the answer in

Fig.(d). Here $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ are two arbitray vectors, time-like and space-like respectively. The region within the front cone is filled with a family of hyperboloid hypersurfaces $F = k^2$ corresponding to every positive value of k. One of these hypersurfaces, having value $k = \alpha$ will pass through the point A. Then the magnitude of $\overrightarrow{\mathrm{OA}}$ is α.

Similarly, the space between the front cone and the back cone is filled with another family of hypersurfaces $F = -k^2$ corresponding to every positive value of k. One of these hypersurfaces, having value $k = \beta$ will pass through the point B. Then the magnitude of $\overrightarrow{\mathrm{OB}}$ is β.

Whatever Minkowski has said in the present context can be simplified by defining a *scalar product* of two vectors.

Let us think of two arbitrary vectors $\overrightarrow{\mathbf{A}} = (a_x, a_y, a_z, a_t)$ and $\overrightarrow{\mathbf{B}} = (b_x, b_y, b_z, b_t)$. We define their scalar product as

$$\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}} \stackrel{\text{def}}{=} a_t b_t - a_x b_x - a_y b_y - a_z b_z. \quad (36)$$

Minkowski has not defined scalar product in his article. However, it is an important concept. In particular *Lorentz invariance of scalar product* is useful in working out many relativistic formulas involving energy and momentum. By Lorentz invariance we mean that even though the components of the 4-vectors $\overrightarrow{\mathbf{A}}$ and $\overrightarrow{\mathbf{B}}$ will change under a Lorentz transformation, the scalar product will not change at all. That is, it is same in all Lorentz frames.

We can now define orthogonality and magnitude in a slightly different way. Instead of magnitude we shall use the term "norm" of a vector, which is, in a sense, square of magnitude.

(a) The 4-vectors $\overrightarrow{\mathbf{A}}$ and $\overrightarrow{\mathbf{B}}$ are *orthogonal* if $\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}} = 0$.

(b) The *norm* of a 4-vector $\overrightarrow{\mathbf{A}}$ is defined as $A^2 \stackrel{\text{def}}{=} \overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{A}} = a_t^2 - a_x^2 - a_y^2 - a_z^2$.

(c) $\overrightarrow{\mathbf{A}}$ is (i) *time-like* if $A^2 > 0$, (ii) *space-like* if $A^2 < 0$, (iii) is a *null vector* if $A^2 = 0$.

A null vector lies along the light cone. In Fig. (d) the displacement vectors $\overrightarrow{\mathrm{OC}}$ and $\overrightarrow{\mathrm{OD}}$ are null vectors. A null displacement vector represents propagation of a light signal (i.e., a photon.)

The norm, being a scalar product, is an invariant quantity. If we accept Minkowski's definitions of magnitude, and denote the magnitude of a 4-vector $\overrightarrow{\mathbf{V}}$ as $\check{V}$, then the magnitudes $\check{A}$ and $\check{B}$ of a time-like vector $\overrightarrow{\mathbf{A}}$ and a space-like vector $\overrightarrow{\mathbf{B}}$ are given as

$$\check{A} = \sqrt{(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{A}})} = \sqrt{a_t^2 - a_x^2 - a_y^2 - a_z^2},$$
$$\check{B} = \sqrt{-(\overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{B}})} = \sqrt{b_x^2 + b_y^2 + b_z^2 - b_t^2}. \quad (37)$$

and these magnitudes are same in all Lorentz frames.

14 Proper Time, Proper Velocity, Proper Acceleration

In Fig. 12 we have shown the world line of a material particle and marked it as Ω. The particle is at two infinitesially close world points Q and R at times t and $t + dt$. The 4-

displacement vectors of these two points are: $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathrm{OR}} = (x, y, z, ct)$ and $\overrightarrow{\mathbf{r}} + d\overrightarrow{\mathbf{r}} = \overrightarrow{\mathrm{OQ}} = (x + dx, y + dy, z + dz, ct + d\,ct)$.

We are interested in the infinitesimal 4-displacement $d\overrightarrow{\mathbf{r}} = \overrightarrow{\mathrm{RQ}} = (dx, dy, dz, d\,ct)$ taking place in the infinitesimal time interval dt. The norm of this 4-vector is called the *metric* of space-time. It is given as

$$ds^2 = d\overrightarrow{\mathbf{r}} \cdot d\overrightarrow{\mathbf{r}} = c^2dt^2 - dx^2 - dy^2 - dz^2. \quad (38)$$

The vector $d\overrightarrow{\mathbf{r}}$ *is necessarily a time-like vector*, i.e., $ds^2 > 0$. To realize its time-like nature let assume that the particle has velocity $\mathbf{v} = (v_x, v_y, v_z)$ at R. Then

$$\begin{aligned} ds^2 &= c^2dt^2 - v_x^2dt^2 - v_y^2dt^2 - v_z^2dt^2 \\ &= \left(c^2 - v^2\right) dt^2. \end{aligned} \quad (39)$$

Since $c > v$, the right side is positive.

The time-like property of $d\overrightarrow{\mathbf{r}}$ ensures that it always lies within the light cone. This also implies that the at every world point the world line of the particle lies within the light cone, as we have shown in the figure. The angle that a tangent to the world line will make with the cT axis must be always less than 45^0.

Since ds^2 is positive, we can take its square root and get a positive real number $c\,d\tau$.

$$d\tau \stackrel{\text{def}}{=} \frac{\sqrt{ds^2}}{c} = \frac{1}{c}\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} \quad (40)$$

We integrate this infinitesimal between from the event $\mathrm{P_o}$ to the event P and get

$$\begin{aligned} \tau(\mathrm{P_o} \to \mathrm{P}) &= \frac{1}{c}\int_{t_0}^{t} \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} \\ &= \int_{t_0}^{t} \sqrt{c^2 - v^2}\, dt. \end{aligned} \quad (41)$$

Minkowski calls the integral $\tau(\mathrm{P_o} \to \mathrm{P})$ "the *proper time* of the substantial point at P". In keeping with this nomenclature we shall call the infinitesimal $d\tau$ defined in Eq. (40) the infinitesimal *proper time* of the particle from the event R to the event Q.

It should be remarked in passing that proper time is the time measured by an observer comoving with the particle (whose motion we are monitoring.)

The proper time $d\tau$ is Lorentz invariant, because it is the square root of a norm (which is invariant) divided by the speed of light which is also invariant.

In non-relativistic physics velocity is displacement per unit coordinate time: $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, i.e., infinitesimal displacement $d\mathbf{r}$ divided by infinitesimal coordinate time dt which is invariant under Galilean transformation. By analogy, Minkowski defines 4-velocity $\overrightarrow{\mathbf{v}}$ of the particle as

$$\begin{aligned} \overrightarrow{\mathbf{v}} &\stackrel{\text{def}}{=} \frac{d\overrightarrow{\mathbf{r}}}{d\tau} = \frac{1}{d\tau}(dx, dy, dz, d\,ct) \\ &= \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, c\frac{dt}{d\tau}\right). \end{aligned} \quad (42)$$

Minkowski adopts the convention that a dot $(\cdot)$ over a symbol representing a variable will represent derivative of the vari-

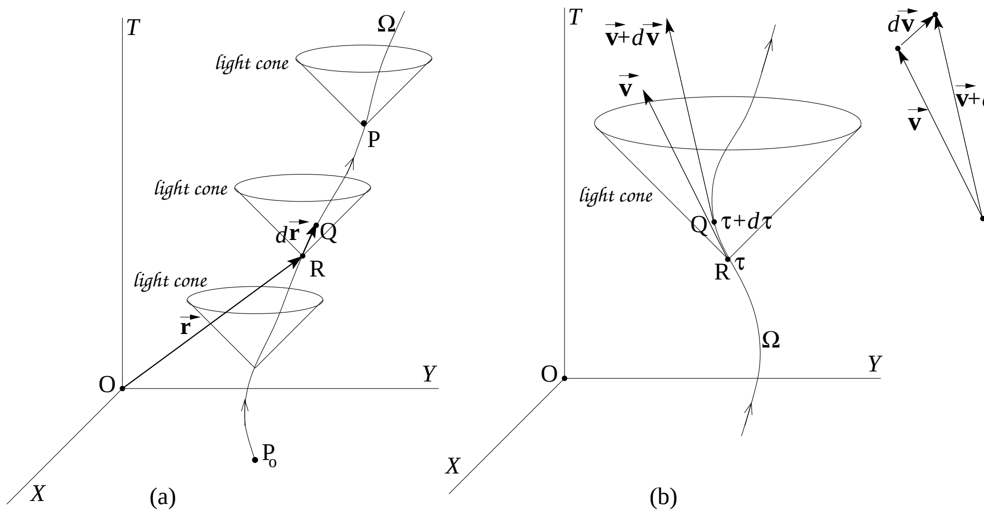

Figure 12: *Construction of velocity and acceleration 4-vectors*

able with respect to proper time τ. Compare this with the convention adopted in non-relativistic mechanics: a dot represents derivative with respect to ordinary time t. Using Minkowski's convention we rewrite the above equation as

$$\overrightarrow{\mathbf{v}} = \dot{\overrightarrow{\mathbf{r}}} = (\dot{x}, \dot{y}, \dot{z}, c\dot{t}). \tag{43}$$

In a similar vein Minkowski defines acceleration 4-vector $\overrightarrow{\boldsymbol{a}}$ as

$$\begin{aligned}\overrightarrow{\boldsymbol{a}} &\stackrel{\text{def}}{=} \dot{\overrightarrow{\mathbf{v}}} = \frac{d\overrightarrow{\mathbf{v}}}{d\tau} \\ &= \left(\frac{d^2x}{d\tau^2}, \frac{d^2x}{d\tau^2}, \frac{d^2x}{d\tau^2}, c\frac{d^2t}{d\tau^2}\right) \\ &= (\ddot{x}, \ddot{y}, \ddot{z}, c\ddot{t}).\end{aligned} \tag{44}$$

At this point Minkowski points out,

> the velocity vector is the time-like vector of unit magnitude in the direction of the world line at P, and the acceleration vector at P is normal to the velocity vector at P, and is therefore, in any case a space-like vector.

We shall prove different parts of this statement. As a prelude to this we define Lorentz factor Γ associated with the particle velocity $\mathbf{v}$ as

$$\Gamma \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{45}$$

It is then seen from (39) that

$$\begin{aligned} &c^2 d\tau^2 = ds^2 = \frac{c^2 dt^2}{\Gamma^2}, \Rightarrow \quad d\tau = \frac{dt}{\Gamma}; \\ &\text{Or,}\ \ \dot{t} = \frac{dt}{d\tau} = \Gamma. \end{aligned} \tag{46}$$

In the sequel we shall find it convenient to convert $\frac{d}{d\tau} \to \frac{d}{dt}$, using the prescription

$$\frac{d}{d\tau} = \Gamma \frac{d}{dt}. \tag{47}$$

(a) *The velocity vector* **v** *is time-like.*
Proof: The norm of $\overrightarrow{\mathbf{v}}$ is given as

$$\begin{aligned}\overrightarrow{\mathbf{v}}\cdot\overrightarrow{\mathbf{v}} &= \mathit{\Gamma}^2c^2-\mathit{\Gamma}^2v^2\\ &= \mathit{\Gamma}^2(c^2-v^2)=c^2.\end{aligned} \tag{48}$$

Since $\overrightarrow{\mathbf{v}}\cdot\overrightarrow{\mathbf{v}} = c^2 > 0$, the 4-vector $\overrightarrow{\mathbf{v}}$ is time-like.

Q.E.D.

The "magnitude" of 4-velocity, according to *our* definition of magnitude given in Eq. (37), is the *same* for all velocities (for a particle at rest as well as for a particle moving with relativistic speed, close to that of light) and equals c. However, the same magnitude is considered to be unit magnitude by Minkowski, who proposes at the end of Sec.IV of his paper to set $c = 1$.

(b) $\overrightarrow{\mathbf{v}}$ *and* $\overrightarrow{\boldsymbol{a}}$ *are mutually orthogonal.*
Proof: Let us first rewrite Eq. (38) as an expression for $c^2d\tau^2$, take derivative with respect to τ.

$$\begin{array}{rrcl} & c^2d\tau^2 &=& c^2dt^2-dx^2-dy^2-dz^2.\\ \text{Dividing both sides with } d\tau^2 & c^2 &=& c^2\dot{t}^2-\dot{x}^2-\dot{y}^2-\dot{z}^2.\\ \text{Differentiating with respect to } \tau & 0 &=& 2\left[c^2\dot{t}\ddot{t}-\dot{x}\ddot{x}-\dot{y}\ddot{y}-\dot{z}\ddot{z}\right],\\ \text{which implies} & 0 &=& v_ta_t-v_xa_x-v_ya_y-v_za_z,\end{array} \tag{49}$$

thereby establishing orthogonality between $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\boldsymbol{a}}$.

Q.E.D.

(c) *The acceleration vector* $\overrightarrow{\boldsymbol{a}}$ *is space-like.*
Proof: Let us first establish a theorem:

Theorem 2 *Let* $\overrightarrow{\boldsymbol{\mathcal{A}}}$ *be a time-like vector which is orthogonal to* $\overrightarrow{\boldsymbol{\mathcal{B}}}$. *Then* $\overrightarrow{\boldsymbol{\mathcal{B}}}$ *is spce-like.*

Proof of the theorem: As shown in Sec.12, a time-like vector $\overrightarrow{\boldsymbol{\mathcal{A}}}$ can be oriented along the time axis by a proper choice of Lorentz frame. Let S be that frame so that the vector $\overrightarrow{\boldsymbol{\mathcal{A}}}$ will have only time-component in S, i.e., $\overrightarrow{\boldsymbol{\mathcal{A}}} = (0, 0, 0, ca_t)$. Let $\overrightarrow{\boldsymbol{\mathcal{B}}}$ have components (b_x, b_y, b_z, cb_t) in this frame S. By our assumption of orthogonality

$$\overrightarrow{\boldsymbol{\mathcal{A}}}\cdot\overrightarrow{\boldsymbol{\mathcal{B}}} = 0\times b_x+0\times b_y+0\times b_z-ca_t\times cb_t = 0,$$

implying that $b_t = 0$, which establishes the space-like property of $\overrightarrow{\boldsymbol{\mathcal{B}}}$.

Q.E.D.

In this particular case $\overrightarrow{\mathbf{v}}$ is time like, and orthogonal to $\overrightarrow{\boldsymbol{a}}$. Therefore by the above theorem we conclude that $\overrightarrow{\boldsymbol{a}}$ is space-like.

Q.E.D.

For future use we shall write the 4-vectors $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\boldsymbol{a}}$ in a different form, identifiable by a bold letter without an arrow on top, often referred to as the 3+1 form.

Therefore, from (43)

$$\overrightarrow{\mathbf{v}} = \Gamma\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, c\frac{dt}{dt}\right) = \Gamma(v_x, v_y, v_z, c) \\ = \Gamma(\mathbf{v}, 1) = (\Gamma\mathbf{v}, \Gamma c). \tag{50}$$

In this case the block $\mathbf{v}$ stands for the non-relativistc 3-velocity

$$\mathbf{v} = (v_x, v_y, v_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \\ = \left(\frac{\dot{x}}{\dot{t}}, \frac{\dot{y}}{\dot{t}}, \frac{\dot{z}}{\dot{t}}\right). \tag{51}$$

We can write the 3+1 form of $\overrightarrow{\boldsymbol{a}}$ as follows.

$$\overrightarrow{\boldsymbol{a}} = \dot{\overrightarrow{\mathbf{v}}} = \Gamma\frac{d\overrightarrow{\mathbf{v}}}{dt} = \left(\Gamma\frac{d(\Gamma\mathbf{v})}{dt}, c\,\Gamma\frac{d\Gamma}{dt}\right). \tag{52}$$

By analogy with centripetal acceleration of a particle moving in a circle Minkowski proposes a formula for the acceleration of a particle in space-time in the following language.

> "Now, as is readily seen, there is a definite hyperebola which has three infinitely approximate points in common with the world line at P, and whose asymptotes are generators of a "front cone" and a "back cone". Let this hyperbola be called hyperbola of curvature at P. If M is the centre of this hyperbola, we here have to do with an internal hyperbola with centre M. Let ρ be the magnitude of the vector MP; then we recognize the acceleration vector at P as vector in the direction MP of magnitude c^2/ρ.
>
> If $(\ddot{x}, \ddot{y}, \ddot{z}, c\ddot{t})$ are all zero, the hyperbola of curvature reduces to a straight line touching the world line at in P, and we must put $\rho = \infty$."

We shall try to explain the statement. However, before proceeding further we shall have a brief review of what is often referred to as "a hyperbolic motion"[8, 9].

Consider a particle moving along the X axis under a constant acceleration a, as measured in its instantaneous rest frame[3]. Let us assume that at $t = 0$ this particle is instantaneously at rest, and located at the origin, in a certain frame S. Then the (x, ct) coordinates of this particle are given in this frame, as functions of the proper time τ, as

$$x = \frac{c^2}{a}\left[\cosh\frac{a\tau}{c} - 1\right];\; ct = \frac{c^2}{a}\left[\sinh\frac{a\tau}{c}\right]. \tag{53}$$

We have shown the world line of this particle in Fig.13(a). By shifting the origin, as shown in Fig.13(b), the same world line can be written as

$$x = \frac{c^2}{a}\left[\cosh\frac{a\tau}{c}\right];\; ct = \frac{c^2}{a}\left[\sinh\frac{a\tau}{c}\right]. \tag{54}$$

Then the above parametric equations of the world line transform into the familiar

[3] An example of this is the motion of a particle of charge e moving under a uniform electric field in the X direction. A detailed analysis for this case for a relativistic particle is not trivial.

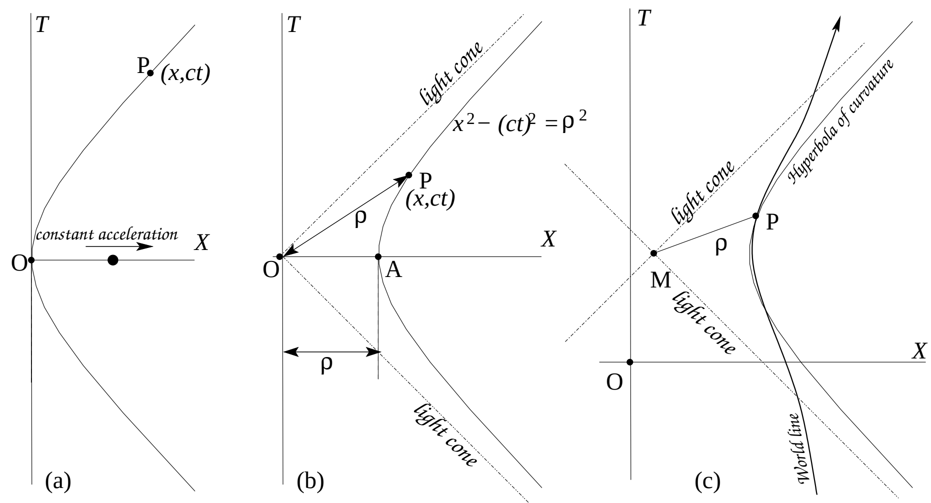

Figure 13: *Hyperbolic trajectory*

equation of a hyperbola, involving only the space and time coordinates.

$$x^2 - (ct)^2 = \rho^2; \quad \text{where } \rho = \frac{c^2}{a}. \qquad (55)$$

We shall differentiate the coordinates (x, ct), as given in (54), with respect to proper time τ to obtain 4-velocity and 4-acceleration.

$$\begin{aligned} &\dot{x} = c \sinh \tfrac{a\tau}{c};\ c\dot{t} = c \cosh \tfrac{a\tau}{c}; \\ &\ddot{x} = a \cosh \tfrac{a\tau}{c};\ c\ddot{t} = a \sinh \tfrac{a\tau}{c}; \\ &\overrightarrow{\boldsymbol{a}} = (\ddot{x}, c\ddot{t}) = a(\cosh \tfrac{a\tau}{c}, \sinh \tfrac{a\tau}{c}). \end{aligned} \qquad (56)$$

The 4-acceleration $\overrightarrow{\boldsymbol{a}}$ is a space-like 4-vector. According to Minkowski's prescription, shown in (37), its magnitude is is

$$\breve{a} = \sqrt{a_x^2 - a_t^2} = a = \frac{c^2}{\rho}. \qquad (57)$$

One can make a formal identity between the above hyperbolic motion and a "circular motion" by adopting an imaginary time coordinate: $u = ict$. Set $\omega = i\frac{a}{c}$ so that $\omega\tau = i\frac{a\tau}{c}$. Then from Eq. (54),

$$\begin{aligned} x &= \rho \left[\cosh \tfrac{a\tau}{c}\right] \\ &= \rho \left[\cos \tfrac{ia\tau}{c}\right] = \rho \cos \omega\tau. \quad (a) \\ u = ict &= \rho \left[i \sinh \tfrac{a\tau}{c}\right] \\ &= \rho \left[\sin \tfrac{ia\tau}{c}\right] = \rho \sin \omega\tau. \quad (b) \\ x^2 + u^2 &= \rho^2. \quad (c) \end{aligned} \qquad (58)$$

The hyperbolic path in space-time looks formally like a circular path of radius ρ followed by a particle with angular velocity ω, except that now this angular velocity is an imaginary quantity.

An infinitesimal segment of any curve (in E^3) at any point P can be considered to have three infinitesimally adjacent points passing through a circle. This circle is referred to as the *circle of curvature* at P and the radius ρ of this circle is called the *radius of curvature*. A particle moving along this curve and having velocity v at P has a normal acceleration equal to $a_n = \frac{v^2}{\rho}$, and directed towards the centre of curvature.

By analogy, any infinitesimal segment of a world line at any event point P can be considered to have three infinitesimally adjacent points passing through a hyperbola. Minkowski calls this hyperbola the *hyperbola of curvature* at P. The parameter ρ of this hyperbola appearing in (55) is analogous to the radius of curvature. What we have just seen in (57) is that the magnitude of the acceleration 4-vector is equal to $\breve{a} = \frac{c^2}{\rho}$.

If we take u to be a real variable in Eq. (58c), then the origin is the centre of the circle. By analogy, Minkowski defines the same origin to be the *centre of the hyperbola of curvature* when u is pure imaginary. How to identify this centre for the general case when the centre is no longer the origin of coordinates (for example, when equation of the hyperbola is transformed by shifting the origin of the coordinate system?) By finding out where the two asymptotes of the hyperbola intersect, as we have shown in Fig. 13(b). These asymptotes are also the traces of the light cones passing through the centre.

In the figure we have denoted the centre as M. Any straight line segment joining M to any point P on the hyperbola of curvature gives the "magnitude" ρ of the displacement 4-vector (see Sec. 13 and Fig. 11) and hence the radius of the hyperbola of curvature.

Fig. 13(c) illustrates the construction of the the hyperbola of curvature at the event point P of a of a particle moving with a variable acceleration. The world line of this particle is shown with a thicker line, and the light cones with broken lines.

15 The Four Dimensional Law of Motion

Section IV of Minkowski's paper is of a greater historical significance. He lays out the foundation and plan for *Mechanics* in the new relativistic order. While unravelling this plan he combines 3-dimensional (Newtonian) momentum with kinetic energy to form a new 4-dimenaional entity which he simply calls *Momentum*. He combines 3-dimensional (Newtonian) force with the rate of work done into another new 4-dimensional entity which he calls *Motive Force Vector*. Many people today refer to Minkowski's motive force as *Minkowski force*.

Minkowski begins with a sort of preamble,

> "To show that the assumption of the group G_c for the laws of physics never leads to a contradiction, it is unavoidable to undertake a revision of the whole of physics on the basis of this assumption. ... For the last branch of physics (i.e., mechanics) it is of prime importance to raise the question - When a force with components X, Y, Z parallel to the axes of space acts at a world point P (x, y, z), where the velocity vector is $\dot{x}, \dot{y}, \dot{z}, \dot{t}$ what must we take this force to be when the system of reference is in any way changed?"

He answers the self-posed question with the following proposition,

> "When the system of reference is changed, the force in question transforms into a force in the new space co-ordinates in such a way that the appropriate vector with the components $\dot{t}X, \dot{t}Y, \dot{t}Z, \dot{t}T$, where
>
> $$T = \frac{1}{c^2}\left(\frac{\dot{x}}{\dot{t}}X + \frac{\dot{y}}{\dot{t}}Y + \frac{\dot{y}}{\dot{t}}Z\right) \qquad (59)$$
>
> is the rate at which work is done by the force at the world point divided by c, remains unchanged. This vector is always normal to the velocity vector at P. A force vector of this kind, corresponding to a force at P, is to be called a "motive force vector" at P."

We shall elucidate. We shall adhere to our convention of multiplying the time component proposed by Minkowski with c to get all components having the same dimension. We shall denote Minkowski's proposed motive force as $\overrightarrow{\boldsymbol{\mathcal{F}}}$ and, using Eq. (46), write its expression as

$$\begin{aligned} \overrightarrow{\boldsymbol{\mathcal{F}}} &= (\mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_z, \mathcal{F}_t) \\ &= (\dot{t}X, \dot{t}Y, \dot{t}Z, \dot{t}cT) \\ &= \Gamma(X, Y, Z, cT) \\ &= \Gamma(\mathbf{F}, cT), \qquad (a) \\ \mathbf{F} &= X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}. \qquad (b) \end{aligned} \qquad (60)$$

Here $\mathbf{F}$ is the 3-dimensional force, as defined in Newton's second law of motion, acting on a particle at a world point P.

Now, the rate at which the force $\mathbf{F}$ is doing work on the particle per unit "co-ordinate time" t is

$$\begin{aligned} \frac{dW}{dt} &= \mathbf{F}\cdot\mathbf{v} = Xv_x + Yv_y + Zv_z \\ &= X\frac{\dot{x}}{\dot{t}} + Y\frac{\dot{y}}{\dot{t}} + Z\frac{\dot{z}}{\dot{t}}. \end{aligned} \qquad (61)$$

We have used Eq. (51) to get the second equality. Comparing Eq. (61) with (59) we find that

$$cT = \frac{1}{c}\frac{dW}{dt}, \qquad (62)$$

as pointed out by Minkowski. We therefore rewrite the expression for $\overrightarrow{\boldsymbol{\mathcal{F}}}$ as

$$\overrightarrow{\boldsymbol{\mathcal{F}}} = \Gamma\left(\mathbf{F}, \frac{1}{c}\mathbf{F}\cdot\mathbf{v}\right) = \Gamma(\mathbf{F}, \mathbf{F}\cdot\boldsymbol{\beta}), \qquad (63)$$

where we have set $\boldsymbol{\beta} = \frac{\mathbf{v}}{c}$.

It has now to be shown that the motive force $\overrightarrow{\boldsymbol{\mathcal{F}}}$ is normal to the velocity 4-vector $\overrightarrow{\mathbf{v}}$. The proof is easy and straight forward.

Proof:

$$\begin{aligned} \overrightarrow{\boldsymbol{\mathcal{F}}}\cdot\overrightarrow{\mathbf{v}} &= \Gamma\mathbf{F}\cdot\Gamma\mathbf{v} - \Gamma cT \times \Gamma c \\ &= \Gamma^2\left(\frac{dW}{dt} - c^2T\right) = 0. \end{aligned}$$

Q.E.D.

He proceeds further,

> "I shall now describe the world-line of a substantial point with constant mechanical mass $\bar{m}$, passing through P. Let the velocity vector at P, multiplied by $\bar{m}$, be called the "momentum vector" at P, and the "acceleration vector" at P, multiplied by $\bar{m}$ be called the "force vector" of the motion at P. With these definitions the

law of motion of a point of mass with given motive force vector runs thus:- *The Force Vector of Motion is equal to the Motive Force Vector.*"

It should be remarked here that the term "constant mechanical mass $\bar{m}$" used in the above statement is same as what is generally referred to as "rest mass" *(for which Minkowski has used the symbol m.)* In this article we shall use the symbol m_0 to mean the same rest mass, so that we can reseve m for "relativistic mass" which will be defined in Eq (65) below. We can now represent Minkowski's "momentum vector" by the symbol $\overrightarrow{\boldsymbol{\mathcal{P}}}$, and his "force vector" by the symbol $\overrightarrow{\mathbf{F}}$. We shall refer to the first one as *4-momentum.* Using Eq. (43)

$$\begin{aligned} \overrightarrow{\boldsymbol{\mathcal{P}}} &\stackrel{\text{def}}{=} m_0 \overrightarrow{\mathbf{v}} = \mathit{\Gamma} m_0(\mathbf{v}, c) \\ &= (\mathbf{p}, mc), \qquad (a) \\ \overrightarrow{\mathbf{F}} &\stackrel{\text{def}}{=} m_0 \overrightarrow{\boldsymbol{a}} \qquad (b) \end{aligned} \tag{64}$$

At this point the reader should note that we have defined two quantities while writing (64), namely, the relativistic mass m and the relativistic 3-momentum $\mathbf{p}$.

$$\begin{aligned} \text{Relativistic mass} \quad m &\stackrel{\text{def}}{=} \mathit{\Gamma} m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad (a) \\ \text{Relativistic 3-momentum} \quad \mathbf{p} &\stackrel{\text{def}}{=} m\mathbf{v} = \mathit{\Gamma} m_0 \mathbf{v} = \frac{m_0 \mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad (b) \end{aligned} \tag{65}$$

Minkowski's proposition of "the law of motion of a point of mass" runs as follows:

$$\overrightarrow{\mathbf{F}} = \overrightarrow{\boldsymbol{\mathcal{F}}}. \tag{66}$$

In view of Eq. (64b) the law of motion looks like the 4-dimensional version of Newton's law of motion:

$$\boxed{m_0 \overrightarrow{\boldsymbol{a}} = \overrightarrow{\boldsymbol{\mathcal{F}}}.} \tag{67}$$

It may be noted, from the definitions of momentum in Eq. (64a), and acceleration in (44), that the above equation of motion can be written in the following alternative form:

$$\boxed{\frac{d\overrightarrow{\boldsymbol{\mathcal{P}}}}{d\tau} = \overrightarrow{\boldsymbol{\mathcal{F}}}.} \tag{68}$$

We shall find the form (68) to be more convenient than (67).
Minkowski continues:

"This assertion comprises four equations for the components corresponding to the four axes, and since both vectors are a priori normal to the velocity vector, the fourth equation may be looked upon as a consequence of the other three. In accordance with the above significance of T, the fourth equation undoubtedly represents the law of energy. Therefore the component of the momentum vector along the axis of t, multiplied by c, is to be defined as kinetic energy of the point mass. The expression for this is

$$= \bar{m}c^2 \frac{dt}{d\tau} = \frac{\bar{m}c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad (69)$$

i.e., after removal of the additive constant $\bar{m}c^2$, the expression $\frac{1}{2}\bar{m}v^2$ of Newtonian mechanics down to the magnitudes of the order $\frac{1}{c^2}$."

To explain the above statemant we shall resolve the equation of motion (68) into four components, converting $\frac{d}{d\tau} \to \frac{d}{dt}$ using (47) and (60).

$$\begin{array}{lcllr} \frac{d\mathcal{P}_x}{d\tau} & = & \mathcal{F}_x, & \Rightarrow \frac{dp_x}{dt} = X. & (a) \\ \frac{d\mathcal{P}_y}{d\tau} & = & \mathcal{F}_y, & \Rightarrow \frac{dp_y}{dt} = Y. & (b) \\ \frac{d\mathcal{P}_z}{d\tau} & = & \mathcal{F}_z, & \Rightarrow \frac{dp_z}{dt} = Z. & (c) \\ \frac{d\mathcal{P}_t}{d\tau} & = & \mathcal{F}_t, & \Rightarrow \frac{d(mc)}{dt} = cT. & (d) \end{array} \qquad (70)$$

In view of Eq. (62) the line (d) of the above equations implies

$$\frac{d(mc^2)}{dt} = c^2 T = \frac{dW}{dt}. \qquad (71)$$

Since the work done results in changing the kinetic energy K, i.e., $\frac{dW}{dt} = \frac{dK}{dt}$, Minkowski sets the kinetic energy to be equal to the expression given in (69). We shall however follow the current usage and call the expression *total energy E*.

$$E = mc^2 = \Gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad (72)$$

which is the famous energy equation attributed to Einstein.

The total energy E of the particle consists of a kinetic energy part K and a "rest energy" part V_o.

$$K = E - m_0c^2,\ V_o = m_0c^2. \qquad (73)$$

To appreciate this we shall expand the right side of (72) in powers of $\frac{v}{c}$.

$$\begin{aligned} E \approx m_0c^2\left[1 + \frac{1}{2}\frac{v^2}{c^2}\right] &= m_0c^2 + \frac{1}{2}m_0v^2 \\ = V_o + K. & \end{aligned} \qquad (74)$$

In non-relativistic physics, $\frac{1}{2}m_0v^2$ is the kinetic energy of the particle. Therefore we have represented it by K, the symbol for kinetic energy. The balance part V_o can be considered to be potential energy, in view of the fact that matter can be converted entirely to energy (as in the cases like electron-positron annihilation resulting into a pair of gamma rays.)

We can now see the significance of the last sentence in Minowski's statement following Eq. (69).

We shall now rewrite the 4-dimensional equation of motion (70) compactly in two lines, breaking it up into a 3+1 form. For this we note that the external force $\mathbf{F}$ acting on the particle is as given in (60b). Also from (61) and (62), $c^2T = \frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v}$. Hence, from (70) and (72),

$$\begin{array}{lcll} \frac{d\mathbf{p}}{dt} & = & \mathbf{F}. & (a) \\ \frac{dE}{dt} & = & \mathbf{F}\cdot\mathbf{v}. & (b) \end{array} \qquad (75)$$

The 4-dimensional equation of motion breaks up into Newton's second law of motion (except that $\mathbf{p}$ has a extra factor Γ), as shown in line (a), and an energy equation, as shown in line (b). This is because E/c is the time component of the 4-momentum, as seen from Eq. (64a), which we should now write in the following preferable form.

$$\vec{\boldsymbol{\mathcal{P}}} = \left(\mathbf{p}, \frac{E}{c}\right). \tag{76}$$

Before concluding Sec. III Minkowski introduces "natural unit of velocity" and imaginary coordinate for time.

> "We can determine the ratio of the units of length and time in such a way that the natural unit of velocity becomes $c = 1$. If we then introduce, further, $\sqrt{-1}\,t = s$ in place of t, the quadratic expression
>
> $$d\tau^2 = -dx^2 - dy^2 - dz^2 - ds^2 \tag{77}$$
>
> thus becomes perfdectly symmetrical in x, y, z, s; and this symmetry is communicated to any law which does not contradict the world-postulate. Thus the essence of the postulate may be clothed mathematically in a very pregnant manner in the mystic formula $3 \cdot 10^5\,\text{km} = \sqrt{-1}$ secs."

16 Lienard-Weichert 4-Potential - the Minkowski Way

Section V of the paper begins with the following statement.

> "The advantages afforded by the world-postulates will perhaps be most strikingly exemplified by indicating the effects proceeding from a point charge in any kind of motion according to the Maxwell-Lorentz theory. Let us imagine the world-line of such a point electron with charge e, and introduce upon it the proper time τ from any initial point. In order to find the field caused by the electron at any world point P_1, we construct the front cone belonging to P_1. The cone evidently meets the world line, since the directions of the line are everywhere those of time-like vectors, at the single point P, We draw the tangent to the world-line at P and construct through P_1 the normal P_1Q to this tangent. Let the length of P_1Q be r. Then by the definition of a front cone, the length of PQ must be r/c. Now the vector in the direction PQ of magnitude e/r represents by its components along the axes of x, y, z, the vector potential multiplied by c, and by the component along the axis of t, the scalar potential of the field excited by e at the world-point P. Herein lie the elementary laws formulated by A.Lienard and E.Wiechert. "

This is a crucial statement which we shall try to interpret using our understanding of the *Lienard-Weichert potentials* $(\mathbf{A}, \Phi)$ of a point charge e moving arbitrarily along an arbitrary path and the Lorentz transformation of these potentials from a frame S to another frame S', or vice versa.

The word "electron" used by Minkowski will mean any charged particle in our article. Fig. 14(a) shows the particle of charge e moving along the trajectory Λ. At any instant of time t' it is located at some point P′, having coordinates $\mathbf{r}'(t') = (x'(t'), y'(t'), z'(t'))$ where its velocity is $\mathbf{v}(t') = c\boldsymbol{\beta}(t')$. Due to its charge it generates an electromagnetic field $(\mathbf{E}, \mathbf{B})$ at all times, which propagates with the speed of light, reaching all points in space. D is one such point, located at the radius vector $\mathbf{r} = (x, y, z)$, fixed in space, and equipped with a detector. We may like to call it a "field point". ,

Let t be any arbitrary time. Then "the field is detected at D at time t" is a certain "event" having space-time coordinates (x, y, x, ct), which we shall designate as the "field event" P_1. The radius vector $\mathbf{R}(t') = \mathbf{r} - \mathbf{r}'(t')$ gives the displacement of D with respect to P′ at time t'. Let $R(t')$ be the distance between P′ and D, and $\mathbf{n}(t') = \frac{\mathbf{R}(t')}{R(t')}$ a unit vector directed from P′ to D. Then the the e.m. field at the event P_1 is given by the following expressions (in gaussian units)[10]:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\boldsymbol{\nabla}\Phi(\mathbf{r}, t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, && (a) \\ \mathbf{B}(\mathbf{r}, t) &= \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t), && (b) \\ \text{where} \quad \mathbf{A}(\mathbf{r}, t) &= \left[\frac{e\boldsymbol{\beta}(t')}{(1-\mathbf{n}\cdot\boldsymbol{\beta}(t'))R(t')}\right]_{t'=t_r}, && (c) \\ \text{and} \quad \Phi(\mathbf{r}, t) &= \left[\frac{e}{(1-\mathbf{n}\cdot\boldsymbol{\beta}(t'))R(t')}\right]_{t'=t_r}. && (d) \end{aligned} \tag{78}$$

Here $\Phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t)$ are the scalar and vector potentials at $\mathbf{r}$ at time t, and are known as Lienard-Weichert potentials. The expressions for these potentials given in lines (c) and (d) require some explanation.

Let us consider t as the "present time" (when the field is detected). The charge is located at $\mathrm{P_o}$ at the present time. However, the field that reaches D at t originated somewhere in the past, at time t_r, called the *retarded time* (corresponding to the present time t) when the particle was located at the *retarded point* $\mathrm{P_r}$. The retarded time t_r is determined by solving the equation

$$R(t_r) \equiv |\mathbf{r} - \mathbf{r}'(t')|_{t'=t_r} = c(t - t_r), \tag{79}$$

because the field propagates with the speed of light. The expressions given within square brackets in Eqs.(78c,d) are to be evaluated at $t' = t_r$ after solving Eq. (79).

The scalar potential Φ and the vector potential $\mathbf{A}$ together form a 4-vector $\overrightarrow{\mathbf{A}} = (\mathbf{A}, \Phi)$ which we shall call *4-vector potential.* For the simple boost along the X axis explained in Sec. 4 the the components of the 4-potential will undergo the same Lorentz transformation from S to S' as given in Eq. (9) with (x, y, z, ct) replaced by (A_x, A_y, A_z, Φ).

We shall first try to interpret the statement for the simpler *special case* in which a point charge e charge is moving with a constant velocity $\mathbf{v}$ along a straight line which we take as the X axis. We have shown this path at the bottom of Fig. 14(b) as the straight line OX, coinciding with the X axis, and superimposed above this line the construction of the potentials using the hint given by Minkowski.

The world line OPQ represents the uniform motion of the source charge e moving along the X axis. The observer is the detector D,

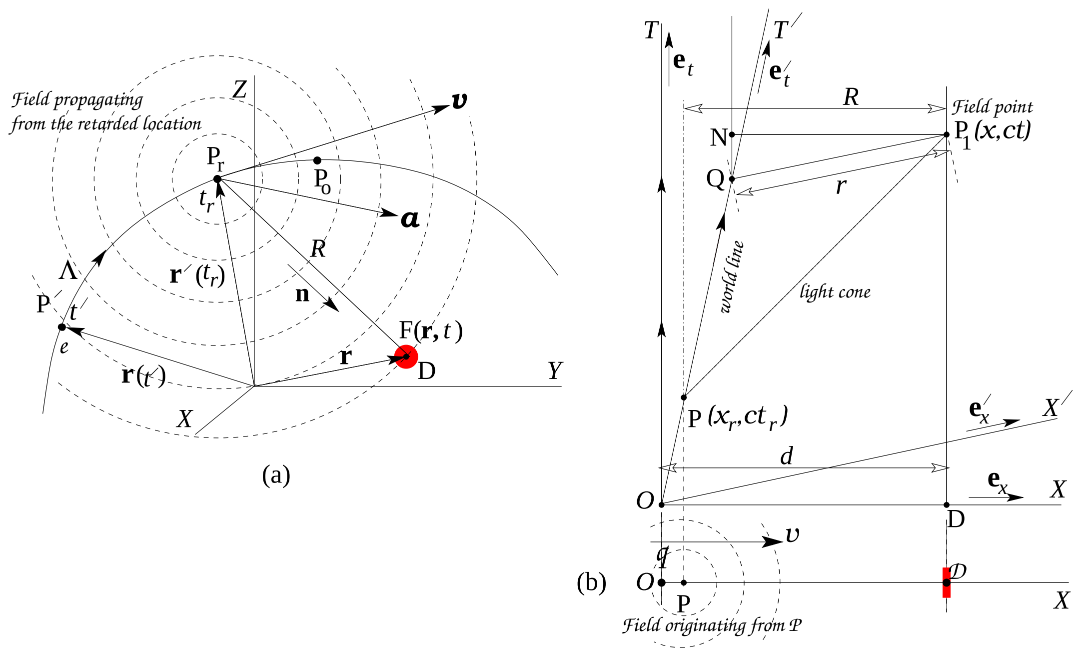

Figure 14: *Lienard Weichert 4-vector Potential. (a) Field originating at the retarded time t_r from an arbitrarily moving charge e; (b) Field originating from a charge e in uniform motion seen in space-time.*

located at a fixed distance d from the origin. The world lines OP of the source and DP_1 of the detector will intersect at some event point (not shown in the diagram) indicating the the two will collide somewhere. We are however considering an event P_1 before such a collision occurs.

We identify the laboratory frame S with the space-time axes $X-T$ and the rest frame S' of e with the axes $X'-T'$. The axis cT' coincides with the world line of e. (See statement in italics at the beginning of Sec. 8.)

From the event P_1 drop a "normal" (in Minkowskian sense) $\mathrm{P}_1\mathrm{Q}$ to the world line OPQ, meeting the world line at Q. Draw the *front light cone* (i.e., a straight line at 45^o with the X axis) meeting the world line at P. The event P, located at coordinates (x_r, ct_r), is the *source* of the event P_1 located at (x, ct), i.e., the field detected at P_1 originated from P at the retarded time t_r. It should be rememberd that the 4-vector $\overrightarrow{\mathrm{PP}_1}$ is a null vector that is, the events P and P_1 are connected by a light ray.

The 4-vectors $\overrightarrow{\mathrm{P}_1\mathrm{Q}}$ and $\overrightarrow{\mathrm{OP}}$ are normal to each other (in the Minkowski sense), and $\overrightarrow{\mathrm{OP}}$ is parallel to the T' axis. Therefore $\overrightarrow{\mathrm{P}_1\mathrm{Q}}$ is parallel to the X' axis. Let the intercept $\widehat{\mathrm{P}_1\mathrm{Q}}$ (measured along the X' axis) be r. According to Minkowski's statement,

$$\overrightarrow{\mathbf{A}} = \frac{e}{r}\overrightarrow{\mathbf{e}_t}', \tag{80}$$

where $\overrightarrow{\mathbf{e}_t}'$ is a unit 4-vector along the T' axis.

Resolving either side of (80) along the time and space axes of the frame S' we get the vector and scalar potentials $(A'_x, A'_y, A'_z, \Phi')$ in S'.

$$A'_x = A'_y = A'_z = 0;\ \Phi' = \frac{e}{r}. \tag{81}$$

The equations are familiar. In the frame S' the point charge e is at rest, and the detector which is moving towards it is at a distance r from from it at the detection time t. The detector sees only a static Coulomb electric field derivable from the scalar potential Φ' as shown, but no magnetic field at all, so that $\mathbf{A}' = \mathbf{0}$.

In order to get the potentials in the in the Lab frame S we have to perform Lorentz transformation from S' to S.

$$\begin{aligned} \Phi &= \gamma(\Phi' + \beta A'_x) = \tfrac{\gamma e}{r}. \\ A_x &= \gamma(A'_x + \beta\Phi') = \tfrac{\gamma\beta e}{r}. \\ A_y &= A'_y = 0;\ A_z = A'_z = 0. \end{aligned} \tag{82}$$

The above equations show the potentials in the Lab frame S as functions of the distance r between e and D, measured in the rest frame S' of e, *at the time of detection*. They are now to be expressed as functions of the distance R, between D and e, measured *at the retarded time* t_r in the Lab frame S. The answer can be found in the following lemma.

Lemma: 2 *Consider an emitter E moving along the X axis with velocity $v = c\beta$ and a detector D placed on the same axis at a distance R from the origin. The emitter E emits a sharp light pulse at time $t = 0$ when it is at the origin O (Event $\mathcal{E}$). This pulse is received by the detector D at time t (event $\mathcal{D}$). Let r be the distance between D and E, as measured in S', when the pulse is detected. Then*

$$r = \gamma(1-\beta)R. \tag{83}$$

Proof of the lemma. In Fig. 15(a) we have shown the events in space-time against X, T axes as seen from S. In Fig. 15(b) we have shown the same events against X', T' axes, as seen from S'. We set the coordinates of $\mathcal{E}$ to be $(x = 0, ct = 0)$ in S and $(x' = 0, ct' = 0)$ in S'. Since the event $\mathcal{D}$ is connected to $\mathcal{E}$ by a light signal, the coordinates of $\mathcal{D}$ are $(x = R, ct = R)$ in S and $(x' = r, ct' = r)$ in S'. Applying Lorentz transformation we get

$$x' = \gamma(x - \beta ct), \quad \text{Or,} \quad r = \gamma(1-\beta)R.$$

Q.E.D.

The last line gives the desired relation connecting the unknown variable r to the distance R between the source point and the field point at the retarded time. Now going back to Eqs. (82), and using (83) we get the Lienard-Weichert potentials for the simple case of point charge in uniform motion:

$$\Phi = \frac{e}{(1-\beta)R};\ \mathbf{A} = \frac{\beta e}{(1-\beta)R}\mathbf{i} = \frac{\boldsymbol{\beta} e}{(1-\beta)R}. \tag{84}$$

In the second equation $\mathbf{i}$ is a unit 3-vector in the direction of the X axis.

We shall now come back to the *general case* in which the particle e is moving with arbitrary velocity and acceleration. In order to

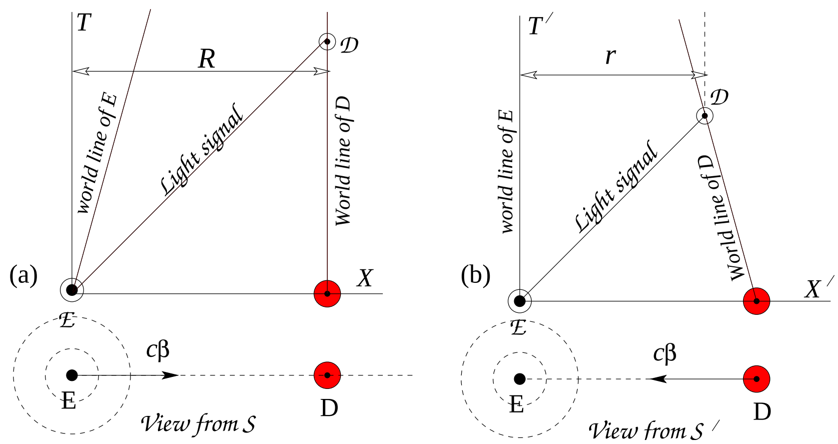

Figure 15: *Emission and detection of a light pulse seen from two frames seen in space-time.*

show the picture on a sheet of paper we shall imagine that the trajectory of the particle is confined to a plane which we shall take as the XY plane, and that the detector D is located at (x, y) on this plane. The detector D receives the field at time t, so that the field event is P_1 (as before), but now having coordinates (x, y, ct). This field originates from the event point P having coordinates (x_r, y_r, ct_r). We have illustrated this case in Fig. 16 in which we have shown the XY plane with a light colour to mark it out from space-time.

We have shown the path of e on the XY plane (with a thick line) as Λ, and its world line (with a broken thin line) as Ω. The points 1,2,P,3,4 on Λ have their corresponding images 1,2,P,3,4 on Ω.

As in the special case, we identify the laboratory frame S with the space-time axes $X - T$. To make the work easier we orient the axes such that the X axis is in the direction of the instantaneous velocity at the source event P, so that $\mathbf{v} = \boldsymbol{\beta} c = \beta c \mathbf{i}$ is the velocity of e at the event P. We shall now consider the instantaneous rest frame of e at the event P to be the S' frame. The time axis T' of S' is parallel to the 4-velocity vector $\overrightarrow{\mathbf{v}}$ at the event P (making angle α with the T axis), as we have shown in the figure.

From the event P_1 drop a "normal" P_1Q to the axis T' meeting it at Q. The *front light cone* drawn from the event P_1 meets the T' axis at P, which is the source event for the field event P_1.

The rest of the arguments, including Eqs. (82)-(84) are the same as for the "special case" and will not be repeated. The lemma 2, valid for the special case, is to be replaced by the following lemma which is valid for the general case.

Lemma: 3 *Consider an emitter E moving arbirarily and emitting a sharp light pulse at time $t = 0$ when it is at a point P (Event $\mathcal{E}$). This pulse is received by a stationary detector D at time t (event $\mathcal{D}$). Let S be the Lab*

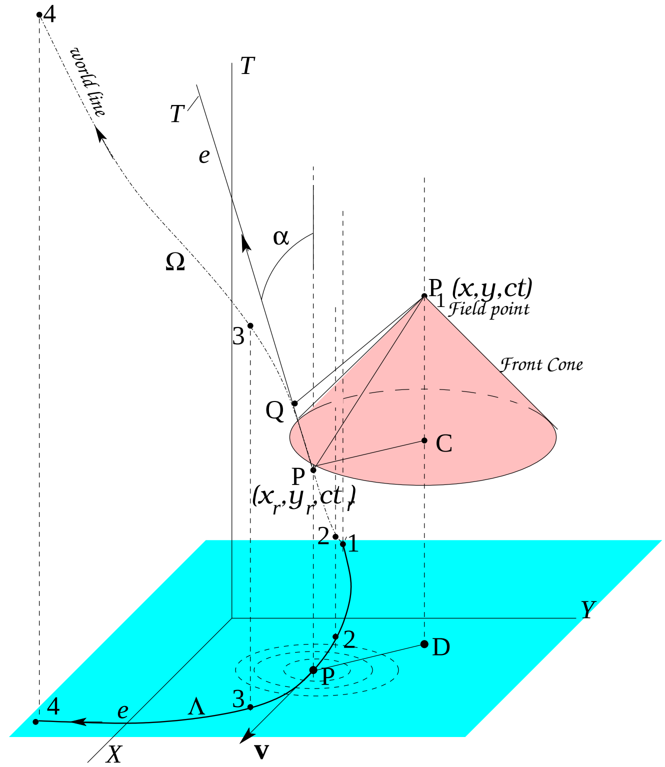

Figure 16: *Construction of Lienard Weichert 4-vector Potential*

frame (i.e., the frame of the detector), and S' the instantaneous rest frame of E at the event $\mathcal{E}$. If R is the (fixed) distance between D and P, as measured in S, and r the distance between D and E, as measured in S', when the pulse is detected, then

$$r = \gamma(1 - \boldsymbol{\beta} \cdot \mathbf{n})R, \tag{85}$$

where $c\boldsymbol{\beta}$ is the instantaneous velocity of E at the event $\mathcal{E}$, and $\mathbf{n}$ is a unit vector directed from P to D.

Proof of the lemma. We shall consider only (x, y, ct) coordinates, and suppress the z coordinate. We take the origins of the Lorentz frames S and S' to be the event $\mathcal{E}$.

Let us take the plane polar coordinates of the detector D as (R, θ) so that $x = R\cos\theta$; $y = R\sin\theta$. Here $\cos\theta = \boldsymbol{\beta} \cdot \mathbf{n}$. The event $\mathcal{D}$ has coordinates $(R\cos\theta, R\sin\theta, ct)$ in S and $(r\cos\phi, r\sin\phi, ct')$ in S'. Since the signal propagtes with the speed of light, $ct = R$, $ct' = r$. Therefore, performing Lorentz transformation (9) from S to S',

$$ct' = \gamma(ct - \beta x).$$

$$\text{Or, } r = \gamma(R - \beta R\cos\theta) = \gamma R(1 - \boldsymbol{\beta} \cdot \mathbf{n}). \tag{86}$$

Q.E.D.

Now we go back to Eqs. (82) and (85) to get the Lienard-Weichert potentials for this general case.

$$\begin{aligned} \Phi &= \frac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}; \\ c\mathbf{A} &= \frac{\beta e\,\mathbf{i}}{(1-\beta)R} = \frac{\boldsymbol{\beta}\, e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}. \end{aligned} \quad (87)$$

Note that $\mathbf{i}$ is a unit 3-vector in the direction of the X axis which is same as the direction of $\boldsymbol{\beta}$ at the retarded time $t = 0$. Therefore we have set $\beta\mathbf{i} = \boldsymbol{\beta}$ in the second line.

The potentials written in Eq. (87) can be found in Jackson[10].

17 Force between two charged particles moving in arbitrary trajectories

In Fig. 17(a) we have shown two particles of charges e and e_1 moving along arbitrary trajectories Λ and Σ respectively. The shaded plane represents the space E^3 with the Z axis suppressed. The particle e_1 is acted on by the e.m. field originating from e. The 4-force $\overrightarrow{\boldsymbol{\mathcal{F}}}$ that is exerted on e_1 at the event point P_1 can be computed by first finding out the electromagnetic field emanating from e at the retarded point P and traveling along the back light cone to reach P_1 (as already explained in the previous section,) and then using the Lorentz force equation (see Eq. (91) below.) Minkowski gives a formula for this force in the following language.

> "I will now describe the ponderomotive action of a moving point charge on another moving point charge. Let us imagine the world line of second point electron of the charge e_1, passing through the world-point P_1. We define P, Q, r as before, then construct the centre M of the hyperbola of curvature at P, and finally the normal MN from M to a straight line imagined through P parallel to QP_1. With P as starting point we now determine a system of reference as follows:- The axis of t in the direction PQ, the axis of x in direction QP_1, the axis of y in direction MN, whereby finally the axis of z is also defined as normal to the axes of t, x, y. Let the acceleration vector at P be $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{t}$, the velocity vector at P_1 be $\dot{x}_1, \dot{y}_1, \dot{z}_1, \dot{t}_1$. The motive force vector exerted at P_1 by the first moving electron e on the second moving electron e_1 now takes the form
>
> $$-ee_1\left(\dot{t}_1 - \frac{\dot{x}_1}{c}\right)\overrightarrow{\boldsymbol{\mathcal{R}}}, \quad (88)$$
>
> where the components $\mathcal{R}_x, \mathcal{R}_y, \mathcal{R}_z, \mathcal{R}_t$ of the vector $\overrightarrow{\boldsymbol{\mathcal{R}}}$ satisfy the three relations
>
> $$c\mathcal{R}_t - \mathcal{R}_x = \frac{1}{r^2},\ \mathcal{R}_y = \frac{\ddot{y}}{c^2 r},\ \mathcal{R}_z = 0, \quad (89)$$
>
> and where, fourthly, this vector is normal to the velocity vector at P_1, and through this circumstance alone stands in dependence on the latter velocity vector."

We shall make our work simpler by assuming that (a) the charge e is moving along a straight line which is also identified with the Y axis, (b) its acceleration is constant, equal

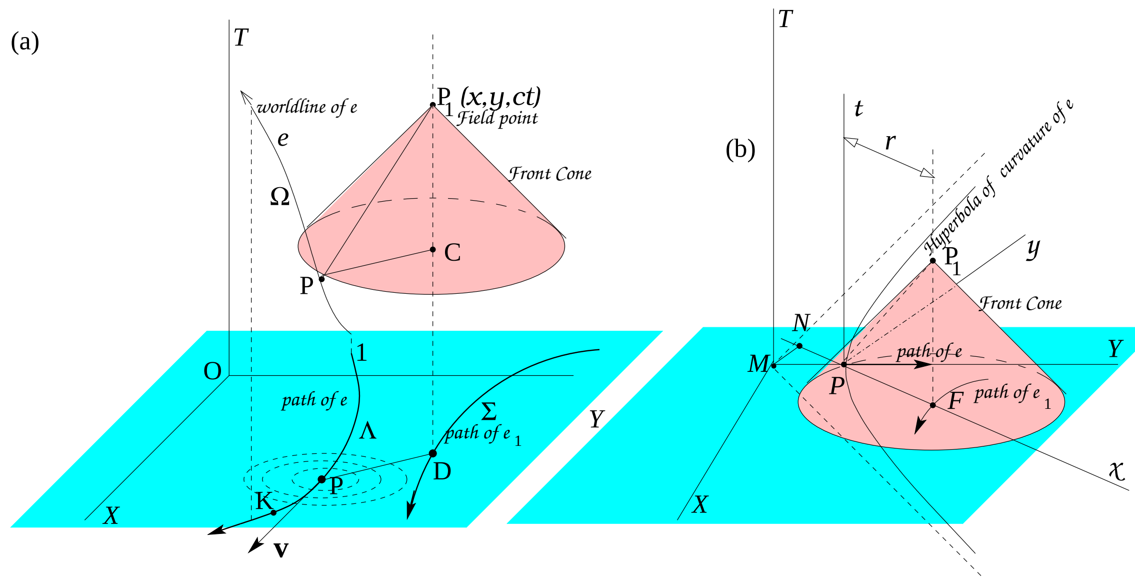

Figure 17: *Charged particles on arbitrary trajectories*

to a, so that the path is a hyperbola, and that (c) the second particle e_1 is also moving on the XY plane. We shall take the time axis in the direction of the instantaneous 4-velocity $\overrightarrow{\mathbf{v}}$ of e at the event P. We have shown this configuration in Fig. 17(b). The front cone with apex at P_1 intersects the XY plane in a circle of radius R. In the present case $r = R$. The retarded event P lies on this circle.

M is the centre of the hyperbola of curvature, which, for this special one dimensional constant acceleration motion is same as the hyperbola we have already drawn, and lies on the Y axis, at the intersection of the asymptotes.

We shall now set up a new frame of reference S' according to Minkowski's prescription. The axes of this new frame will be designated as x, y, z, t, the z axis being suppressed. The new t axis is taken parallel to the T axis. From the field event P_1 we drop a "normal" P_1Q to the t axis. Then $\overrightarrow{QP_1}$ lies parallel to the XY plane, and equals $\overrightarrow{PF}$, and has length $r = R$. From M we drop a normal MN to the straight line FP, both these lines being on the XY plane. The new x axis is along PF, the new y axis is parallel to $\overrightarrow{MN}$, so that the $x-y$ axes lie on the XY plane.

We shall now obtain an expression for $\overrightarrow{\boldsymbol{\mathcal{F}}}$. The scalar and vector potentials for the field created by the moving charge e was written in Eqs. (87). The $(\mathbf{E}, \mathbf{B})$ fields are now obtained by performing the differentiations indicated in Eqs. (78a,b). These differentiations are tricky and involved because the potentials involve "retarded" variables $[R(t'), t']_{t'=t_r}$, whereas the differentiations are to be carried out with respect to the field variables x, y, z, t. These two sets of variables are connected through Eqs. (79).

We shall not perform these differentiations. Instead we shall quote the resulting $(\mathbf{E}, \mathbf{B})$ fields from Jackson[11].

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\Gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{\text{ret}} \\ &\quad + \frac{e}{c}\left[\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\boldsymbol{\beta}'\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{\text{ret}}. \\ \mathbf{B}(\mathbf{r}, t) &= [\mathbf{n}\times\mathbf{E}(\mathbf{r}, t)]_{\text{ret}}. \end{aligned} \tag{90}$$

Here we have used the prime symbol $'$ to mean derivative with respect to coordinate time t, i.e., $\boldsymbol{\beta}' = \frac{d\boldsymbol{\beta}}{dt} = \boldsymbol{a}/c$, the subscript "$_{\text{ret}}$" indicates that the quantities within the square brackets are to be evaluated at the "retarded event" P, and $\mathbf{n}$ is a unit vector in E^3 from the source point P to the field point F.

The motive force $\overrightarrow{\boldsymbol{\mathcal{F}}}$ is now given by Eq. (60a), where $\mathbf{F}$ is the Lorentz force on the point charge e_1, given as

$$\mathbf{F} = e_1[\mathbf{E} + \boldsymbol{\beta}_1 \times \mathbf{B}], \tag{91}$$

$(\mathbf{E}, \mathbf{B})$ being the electromagnetic field experienced by the point charge e_1, and $\mathbf{v}_1 = c\boldsymbol{\beta}_1$ is its velocity, all at the field event P_1.

In the present case the new frame of reference is an instantaneous rest frame of the source particle, so that $\boldsymbol{\beta} = \mathbf{0}$. Also $\mathbf{n} = \mathbf{i}, \Gamma = 1, R = r$. Hence from (90),

$$\mathbf{E}(P_1) = \frac{e}{r^2}\mathbf{i} + \frac{e}{cr}\left[\mathbf{i}\times(\mathbf{i}\times\boldsymbol{\beta}')\right]. \tag{92}$$

Since $\Gamma = 1$, it follows from Eq. (50) that

$$c\boldsymbol{\beta}' = \boldsymbol{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}}{d\tau^2} = \ddot{\mathbf{r}}. \tag{93}$$

Therefore,

$$\mathbf{i}\times\boldsymbol{\beta}' = \frac{1}{c}[\mathbf{i}\times(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j})] = \frac{\ddot{y}}{c}\mathbf{k}. \tag{94}$$

Hence,

$$\begin{aligned} \mathbf{E}(P_1) &= \frac{e}{r^2}\mathbf{i} - \frac{e\ddot{y}}{c^2 r}\mathbf{j}, \\ \mathbf{B}(P_1) &= \mathbf{i}\times\mathbf{E}(P_1) = -\frac{e\ddot{y}}{c^2 r}\mathbf{k}. \end{aligned} \tag{95}$$

We shall divide the force on e_1, given in (91) into an electric part $\mathbf{F}_e = e_1\mathbf{E}$ and a magnetic part $\mathbf{F}_m = e_1\boldsymbol{\beta}_1\times\mathbf{B}$, for easier calculation. From (50),

$$\dot{\overrightarrow{\mathbf{r}}}_1 = \frac{d\overrightarrow{\mathbf{r}}_1}{d\tau} = c\Gamma_1\boldsymbol{\beta}_1 = c\dot{t}_1\boldsymbol{\beta}_1. \tag{96}$$

Hence,

$$\begin{aligned} \mathbf{F}_e &= ee_1\left[\frac{\mathbf{i}}{r^2} - \frac{\ddot{y}}{c^2 r}\mathbf{j}\right]. \ \mathbf{F}_m = \frac{ee_1\ddot{y}}{c^3 r\dot{t}_1}\left[\dot{x}_1\mathbf{j} - \dot{y}_1\mathbf{i}\right]. \\ \mathbf{F} &= \mathbf{F}_e + \mathbf{F}_m = ee_1\left[\left\{\frac{1}{r^2} - \frac{\ddot{y}\dot{y}_1}{c^3\dot{t}_1 r}\right\}\mathbf{i} + \left\{-\frac{\ddot{y}}{c^2 r} + \frac{\ddot{y}\dot{x}_1}{c^3 r\dot{t}_1}\right\}\mathbf{j}\right]. \end{aligned} \tag{97}$$

We shall use Eq. (63), set $\Gamma_1 = \dot{t_1}$, and write the motive force 4-vector on the particle e_1 as

$$\overrightarrow{\boldsymbol{\mathcal{F}}} = \Gamma_1(\mathbf{F},\ \boldsymbol{\beta}_1 \cdot \mathbf{F}) = \Gamma_1(\mathbf{F},\ \boldsymbol{\beta}_1 \cdot \mathbf{F}_e). \quad (98)$$

We shall now obtain the time component of $\overrightarrow{\mathbf{F}}$:

$$\mathcal{F}_t = \Gamma_1 \boldsymbol{\beta}_1 \cdot \mathbf{F}) = \Gamma_1 \boldsymbol{\beta}_1 \cdot \mathbf{F}_e = \Gamma_1 \frac{\dot{\mathbf{r}_1}}{\dot{t_1}} \cdot \mathbf{F}_e = \frac{ee_1}{c^2}\left\{\frac{\dot{x_1}}{r^2} - \frac{\dot{y_1}\ddot{y}}{c^2 r}\right\}, \quad (99)$$

and its space components:

$$\mathcal{F}_x = ee_1\left(\frac{1}{r^2} - \frac{\ddot{y}\dot{y_1}}{c^3 t_1 r}\right); \mathcal{F}_y = ee_1\left(-\frac{\ddot{y}}{c^2 r} + \frac{\ddot{y}\dot{x_1}}{c^3 r \dot{t_1}}\right). \quad (100)$$

It is now seen that

$$\mathcal{F}_t - \mathcal{F}_x = -\frac{ee_1}{r^2}\left(\dot{t_1} - \frac{\dot{x_1}}{c}\right). \quad (101)$$

Therefore, if we write

$$\overrightarrow{\boldsymbol{\mathcal{F}}} = -ee_1\left(\dot{t_1} - \frac{\dot{x_1}}{c}\right)\overrightarrow{\boldsymbol{\mathcal{R}}}, \quad (102)$$

then

$$\mathcal{R}_t - \mathcal{R}_x = \frac{1}{r^2};\ \mathcal{R}_y = \frac{\ddot{y}}{c^2 r}. \quad (103)$$

We now have to establish that $\overrightarrow{\boldsymbol{\mathcal{R}}}$ is a 4-vector.

Proof: $\mathbf{v}_1(\mathrm{P}_1) = (\dot{x_1}, \dot{y_1}, c\dot{t_1})$; $\mathbf{r}_1(\mathrm{P}_1) = (r, 0, 0, r)$. Hence, $\mathbf{v}_1(\mathrm{P}_1) \cdot \mathbf{r}_1(\mathrm{P}_1) = r(\dot{x_1} - c\dot{t_1})$.
It follows that $\overrightarrow{\boldsymbol{\mathcal{F}}} = \frac{ee_1}{cr}[\mathbf{v}_1(\mathrm{P}_1) \cdot \mathbf{r}_1(\mathrm{P}_1)]\overrightarrow{\boldsymbol{\mathcal{R}}}$. The distance r is not a coordinate diatance between two events. It is a distance defined in the rest frame of e (like the proper time which is defined in the rest frame of a particle.) Hence, $\overrightarrow{\boldsymbol{\mathcal{F}}}$ = scalar × $\overrightarrow{\boldsymbol{\mathcal{R}}}$. The left hand side is a 4-vector. Therefore $\overrightarrow{\boldsymbol{\mathcal{R}}}$ is a 4-vector.

Q.E.D.

References

[1] A. Einstein, *"On the Electrodynamics of Moving Bodies"*, Translated from "*Zur Elektrodynamik bewegter Körper*, Annalen der Physik, **17**, (1905)", and
A. Einstein, *"Does the Inertia of a Body Depend on its Energy Content?"*, Translated from "*Ist die Trägheit eines Körpers von seinem Energiegehalt abhängig ?*, Annalen der Physik, **17**, (1905),"
reproduced in *The Principle of Relativity, Collection of Original Papers in Relativity*, H. A. Lorentz, A. Einstein, H.

Minkowski and H. Weyl; , tr. by W. Perret and G. B. Jeffrey, Dover, New York (1952). See pp. 37-71.

[2] A. Einstein, *"On the Influence of Gravitation on Propagation of Light"*, Translated from "*Über den Einfluss der Schwercraft auf die Ausbreiung des Lichtes*, Annalen der Physik, **35**, (1911)", and A. Einstein, *"The Foundatiion of the General Theory of Relativity"*, Translated from "*Die Grundlage der allgemienen Relativitätstheorie", Annalen der Physik*, **49**, (1916), reproduced in pp 99-194 of " *The Principle of Relativity ...* " cited in Ref. 1.

[3] H. Minkowski, "*Space and Time*, A Translation of an Address delivered at the 80th Assembly of German Natural Scientists and Physicians, at Cologne, 21 September, 1908 ", reproduced in pp 75-96 of " *The Principle of Relativity ...* " cited in Ref. 1.

[4] S. Datta, *1905 Relativity Papers of Einstein*, Physics Education, Vol. 22, No.1, Indian Association of Physics Teachers (April-June 2005). Pages 39-68.

[5] C.W.Misner, K.S.Thorne, J.A.Wheeler, *Gravitation*, W.H.Freeman & Co., San Francisco (1972). see p.304.

[6] S. Datta, *op cit*, pp. 53-55.

[7] R.Ramachandran, *Lodel Diagrams and Relativistic Mechanics*, Physics Education(IAPT), Vol. 22, No.1, (April-June 2005). Pages 69-73.

[8] C.Moller, *The Theory of Relativity*, 2nd Ed., Oxford University Press, New Delhi (1972). See 74.

[9] S. Datta, *Introduction to Special Theory of Relativity*, Allied Publishers Ltd, New Delhi (1988/2001). See p. 98.

[10] J.D.Jackson, *Classical Electrodynamics*, 2nd Ed., Wiley Eastern Ltd., New Delhi (1978). See p. 656.

[11] J.D.Jackson, *op cit*, See p.657.