An Alternate Method for Deriving Relativistic Momentum and Energy of a Particle

T.Shivalingaswamy[1] and B. AKagali[2]

[1]Department of Physics
Government College for Women
Mandya-571401,India.
tssphy@gmail.com

[2]Department of Physics
Bangalore University
Bangalore-560056, India.
bakagali@gmail.com

(*Submitted* 06-01-2013)

Abstract

In this article we discuss an alternate and a simple method employing only the relativistic rules for the addition of velocities to deduce the well-known expressions for the relativistic momentum and energy of a particle, instead of using conventional methods that either involves interaction of more than one particle or require an advanced level of theory of relativity

1. Introduction.

Expressions for relativistic momentum and energy of a material body are derived using several methods. Use of the law of conservation of momentum in the elastic collision of equal masses [1-4], the Lorentz invariant action integral [5] and the four vector acceleration [6,7] are the most common among the methods. While first of the cited methods involves interaction of more than one particle, the other two methods require an 'advanced' level of understanding of the theory of relativity.

In this short article, a fairly simple method employing only the velocity addition rules for a single particle is shown to yield the well-known expressions for momentum and energy.

2. Lorentz Transformation of velocities

Let us consider two identical frames *S*and S^1 such that S^1 is moving with respect to *S* along their common X-axis with velocity *V*. Let us consider a particle of mass *m* that is having a velocity *u* along Y-direction in S^1. Therefore, the particle's velocity components in S^1 are

$$v_x' = 0, \quad v_y' = u \quad \text{and} \quad v_z' = 0 \tag{1}$$

Using Lorentz transformations, The X,Y and Z component so f the particle's velocity in *S*can be found to be[8]

$$v_x = V,\ v_y = u\left(1-\frac{V^2}{c^2}\right)^{\frac{1}{2}} \text{ and } v_z = 0 \qquad (2)$$

3. Relativistic momentum

Using the Newtonian definition as a guide, we can expect the relativistically correct momentum to take the form

$$\vec{P} = m\vec{v}f(\vec{v})$$

Where $f(\vec{v})$ is an unknown function that depend so nylon the magnitude of the particle's velocity in such a way that $f(\vec{v})$ should approach unity in the limit of $\left(\frac{v}{c}\right)$ tending to zero.

Using this ansatz, we get the following components in S for the momentum of the particle being considered.

$$P_x = mVf(|\vec{v}|),\ P_y = mu\sqrt{\left(1-\frac{V^2}{c^2}\right)}$$

$$\text{and } P_z = 0 \qquad (3)$$

Where,

$$|\vec{v}| = \sqrt{\left(v_x^2 + v_y^2 + v_z^2\right)} = \sqrt{V^2 + u^2\left(1-\frac{V^2}{c^2}\right)}$$

But as $V\rightarrow 0$,the particle simplyhasonly$Y$ component of velocity equal to u. Hence, its momentum tends to $mu\,f(u)$. In the limit $V\rightarrow 0$, this May be written as

$$mv_y\ f\left(\sqrt{V^2 + v_y^2}\right) = mu\,f(u) \qquad (4)$$

Expanding in powers of $\frac{V^2}{c^2}$ and retaining the terms with first powers of $\frac{V^2}{c^2}$, we get LHS of equation (4) as

$$\text{LHS} \cong mu\left(1-\frac{1}{2}\frac{V^2}{c^2}\right) f\left(\sqrt{V^2 + u^2\left(1-\frac{V^2}{c^2}\right)}\right)$$

$$\text{LHS} \cong mu\left(1-\frac{1}{2}\frac{V^2}{c^2}\right) f\left(u + \frac{1}{2}\frac{V^2}{u}\left(1-\frac{u^2}{c^2}\right)\right)$$

But

$$f\left(u + \frac{1}{2}\frac{V^2}{c^2}\left(1-\frac{u^2}{c^2}\right)\right) \cong f(u) + \frac{df}{du}\left(\frac{1}{2}\frac{V^2}{u}\left\{1-\frac{u^2}{c^2}\right\}\right)$$

(5)

This should be equal to $\text{RHS} = mu\,f(u)$.

Equating the coefficients of V^2 on both sides of the equation (4), we get the following equation for $f(u)$

$$-\frac{1}{c^2}\,f(u) + \left(\frac{df}{du}\right)\frac{1}{u}\left(1-\frac{u^2}{c^2}\right) = 0 \qquad (6)$$

This is the first order equation for $f(u)$ can be written as

$$\frac{df}{du} = \frac{\frac{u}{c^2}}{1-\frac{u^2}{c^2}}\,f(u) \qquad (7)$$

On integration, we get

$$f(u) = \frac{\text{constant}}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (8)$$

The constant in the above equation can be found by taking recourse to the Newtonian limit: $f(u)\rightarrow 1$ as $\frac{u}{c}\rightarrow 0$. Therefore the constant must be simply equal to unity.

Since the Y direction may be chosen arbitrarily, we may conclude that the relativistically correct momentum for a particle of mass m and velocity $\vec{v}$ is given by

$$\vec{P} = \frac{m\vec{\mathrm{v}}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} \tag{9}$$

This is well-known expression we sought to derive.

4. Relativistic Energy

Whenever there is change in momentum of a body, there will be change in the energy of the body such that

$$\Delta E = \vec{u}.\Delta\vec{P} \tag{10}$$

$$\Delta E = \vec{u}.\Delta\left(\frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\therefore\ \Delta E = \frac{m\vec{u}.\Delta\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{m(\vec{u}.\vec{u})\left(\frac{-2\vec{u}.\Delta\vec{u}}{c^2}\right)}{2\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}$$

This can be simplified to

$$\Delta E = \frac{m(\vec{u}.\Delta\vec{u})}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}} = \Delta\left(\frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\right) \tag{11}$$

Hence, we get the well-known expression for relativistic energy

$$E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \tag{12}$$

5. Discussion

We have shown here that with reasonable assumptions, Lorentz transformation of velocities of a single particle can lead to the correct expression for relativistic momentum and the energy. There is noneed to invoke two-body collisions or Lorentz invariant action. Hence this alternate method will help the students and teachers who take up the derivation while understanding relativistic expressions for energy and momentum.

6. References

[1]. Bergmann P. G. (1960). Introduction to theory of Relativity, Asia Publishing house.

[2]. Rosser W. G. V. (1964), An Introduction to theory of Relativity, Butter worths publishing.

[3]. Robert Resnick (2007), An Introduction to special Relativity, Wiley India Edition.

[4]. Bohm D. (1965). The Special Theory of Relativity, Benjamin.

[5]. Landau L. D. and Lifshitz E. M. (1975). The Classical Theory of Fields, pergamon Press.

[6]. Synge J. L. (1965). Relativity: The Special Theory, North HolandPubl.Co.

[7]. Pauli W. (1958). Theory of Relativity, B. I Publications.

[8]. James B. Hartle. (2003). An Introduction to Einstein's General Relativity, Pearson Education.