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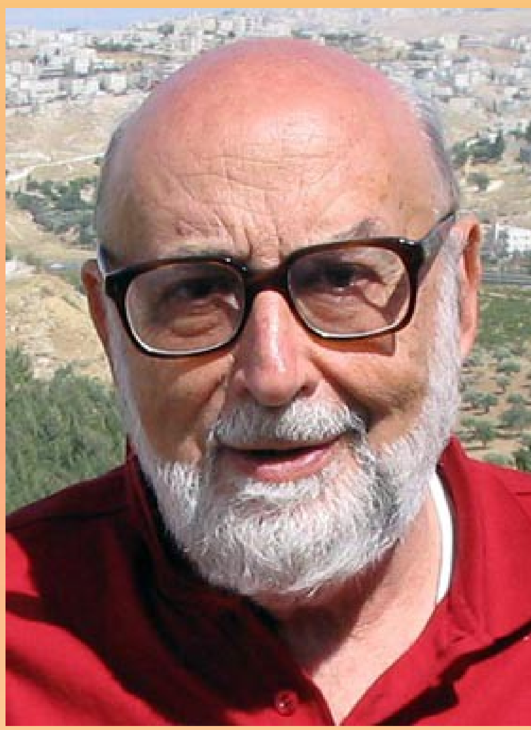

François Englert

Peter W. Higgs

The Nobel Prize Winners in Physics 2013

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Volume 29, Number 4

In this Issue

EDITORIAL

It is a pleasure to publish issue 29.4 of Physics Education. In this issue we announce a new feature, namely a series of pedagogical articles on a special topic / technique influencing recent research activity in physics. Such a series will be guest-edited by a senior specialist concerning the special topic. From the next issue we start a series on the simulation of physical processes and systems and computational methods, guest-edited by prof. D. G. Kanhere, who is with the Center for Modeling and Simulation, University of Pune. Physics Education welcomes new proposals/ suggestions regarding such a series, from specialists in specific techniques/ topics. Many ideas come to mind, namely techniques involved in molecular and atomic spectroscopy, band theory of solids, energy structure and other properties of mesoscopic systems, lowest energy states of many body systems (eg DMRG), correlated electron systems, quantum optics, quantum information and so forth.

This issue contains an article on tachyons, an all time exciting idea, by S. Sahoo, M. Kumar and M. Goswami. Praveen C. Joins M. N., Krishna Kumar Kowshik and Sai Smurti Samantaray present a new application of WKB method. S. Sivakumar makes Cauchy - Riemann conditions for the analyticity of a function more explicit. The article on cosmological special relativity by M. GuhaMajumdar gives an aspect of cosmological relativity, which is not a part of the mainstream, but nevertheless an interesting option. The article by A. Ghorai connects corpuscular theory of light with the observed laws of reflection and refraction of light. Prof. R. Ramchandran's coverage of ideas leading to the 2013 nobel prize makes an interesting reading. The report on the workshop of women in physics by Deepti Sidhaye touches upon some of the core issues.

I wish you a very happy reading experience!

Wish you a Very Happy New Year 2014 !

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Cosmological Special Relativity: Fundamentals and Applications

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Abstract

The theory of Special Relativity has been a centre of interest for the scientific community for more than a century now. In recent times, people have worked on various aspects of the subjects, besides attempting to disprove some of the postulates of Special Relativity and the emergence of ideas such as that of *Doubly Special Relativity*, one effort is especially noteworthy: Dr. Moshe Carmeli's attempt at extending the concept of Special Relativity to the larger picture - our Cosmos. In his famous paper '*Cosmological Special Relativity: The Large-Scale Structure of Space, Time and Velocity*' published in 1997, Dr.Carmeli has laid the framework for Cosmological Special Relativity. This paper is an attempt at understanding the nuances of the subject and trying to work on certain key ideas put forth by the theory. I have focussed on the explanations provided by Cosmological Special Relativity for the accelerated expansion of the universe and the existence of Dark Matter, as per Galaxy Rotation Curves.

Keywords: Cosmological Special Relativity, Galaxy Rotation Curves, Hubble's Law

Introduction

Cosmological Special Relativity (CSR) is an off-shoot of Einstein's Special Relativity, which associates the idea of Hubble Expansion of the Universe with a four-dimensional *space-velocity*, which can be used as a tool for describing the state of our cosmos.[1] Though not a widely followed paradigm, the theory is an interesting theory and a mathematical formulation that tries to explain certain aspects of Cosmology.

To begin with, one needs to define the mathematical space and relevant transformations for CSR. One needs a coordinate for the velocity of the object, which is independent of the spatial coordinates and is analogous to the time coordinate used in special relativity in general, besides the spatial dimensions. This unified space-velocity is used to describe our expanding universe. For every point in space-velocity, one has three spatial coordinates and a velocity coordinate.

The line element in this space is given by

$$ds^2 = \tau^2 dv^2 - (dx^2 + dy^2 + dz^2)$$

τ – Hubble Carmeli Constant

As will be mentioned later, τ is actually the inverse of Hubble's constant in empty space. Physically, it gives us the age of the Universe. If our present time is taken as t=0, then the time, moving backwards, when the Big Bang occurred would be given by τ.

1. Hubble's Law

Since ages, man has been awed by the universe. So magnanimous does it seem to us that the concept of an expanding universe was hard to accept, when it was put forth by Edwin Hubble. Building on George Lemaitre's idea of an expanding universe, Hubble came up with the idea that the universe is expanding at a constant rate: the Hubble's constant (H_O). All celestial objects in intergalactic space are found to have a Doppler Shift in the relative velocity, when observed from the Earth, and also relative to each other. Moreover, this velocity is found to be directly proportional to the distance of the object from the observing point (be it on the Earth or on some other celestial object).

Mathematically,

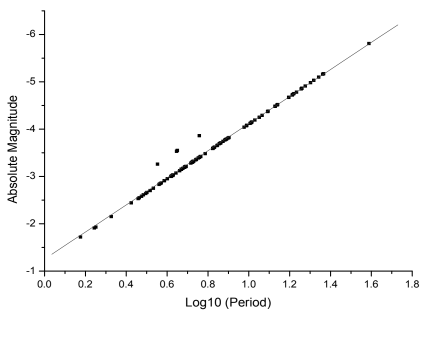
$$v = H_O D$$

where v = velocity of recession of the object, D = Proper Distance, as per *Cosmological Special Relativity*.

Hubble studied the *Radial Velocity vs. Distance* relations, derived with the help of the period-luminosity relation of Cepheid variables, as shown below for a set of 100 Cepheids. For calculating the distance from the PL relationship, one uses the formula

$$m - M = 5 \log \left(\frac{Distance}{10\, pc.}\right)$$

where m is the apparent magnitude and M is the absolute magnitude of the pulsating star.



Graph: *Period-Luminosity Relation* of 100 Galactic Classical Cepheids with absolute magnitudes ranging between -1.72 to -5.81. *Period is in days.*
Courtesy: David Dunlap Observatory

$$Distance = (10\, pc.) \times 10^{\frac{m-M}{5}}$$

One can then plot the radial velocity vs. Distance curve. Given below is the plot for values obtained by Hubble-Humason and published in their 1931 paper

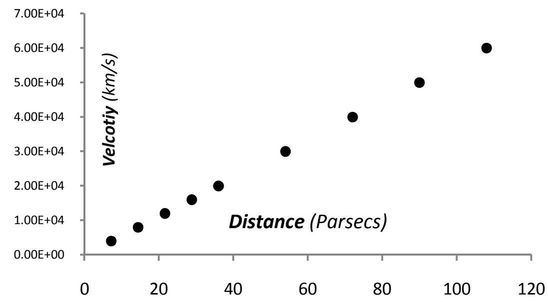


Figure 1: Courtesy: Edwin Hubble, Milton L. Humason. *The Velocity-Distance Relation among Extra-Galactic Nebulae*, Astrophysical Journal, vol. 74, p.43 (1931)

Various Models and Cosmological Special Relativity

Friedmann-Robertson-Walker metric

In the Friedmann-Robertson-Walker metric, one has the line element

$$ds^2 = -c^2 dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$$

Here a(t) serves as a scale factor to describe points in four-dimensional space-time for an expanding universe. In this model, the Hubble's constant is defined as

$$H = \frac{1}{a}\frac{da}{dt}$$

The solution to Einstein's field equations, in Friedmann-Robertson-Walker model is found to be

$$\frac{1}{a}\frac{d^2 a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \Lambda \frac{c^2}{3}$$

In this equation, one finds that the pressure and density terms contribute to the decrease in the expansion rate of the universe, whereas the last term leads to the increase in rate of expansion of the universe. Physically, this term is associated with the negative density energy or the *dark energy.*

Cosmological Special Relativity (CSR) is an off-shoot of Einstein's Special Relativity, which associates the idea of Hubble Expansion of the Universe with a four-dimensional *space-velocity*, which can be used as a tool for describing the state of our cosmos.[1]

To begin with, one needs to define the mathematical space and relevant transformations for CSR. One needs a coordinate for the velocity of the object, which is independent of the spatial coordinates and is analogous to the time coordinate used in special relativity in general, besides the spatial dimensions. This unified space-velocity is used to describe our expanding universe. For every point in space-velocity, one has three spatial coordinates and a velocity coordinate.

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Accelerated Expansion of the Universe

Recent astronomical studies on type Ia Supernovae have shown that the receding velocities of celestial bodies are not linearly dependent on the distance of the object from the Earth. This gives rise to the idea that our Universe may be expanding in an accelerated manner. (Fig. 3)

The Supernova Cosmology Project (SCP) and the High-Z Supernova Search team (HST) showed that the universe is expanding and the results were sensitive to combinations $a\Omega_M + b\Omega_\Lambda$ [2], where $ab < 0, |a| > |b|$. Here Ω_M represents the matter density and Ω_Λ represents the vacuum energy densities. The vacuum energy contributes to the expansion of the Universe whereas matter reduces it. This can be understood by considering the attractive gravitational force of matter, against which the vacuum energy tends to contribute to the expansion. This vacuum energy is also called *dark energy*. Eventually the contribution of the dark energy is more than that of matter, and as is well-established now, we have an accelerated expansion of the universe.

One pertinent question that has been addressed by Kaya[3] is regarding the idea of expansion of the Universe at a speed greater than speed of light. Kaya[3] has mentioned about the idea that the expansion in an inertial frame may not have a physical significance, and may give rise to faster-than-light expansion. Rather when one considers the other objects with respect to the rest frame of one, then using Relativistic Addition of velocities, one can see that one does not have faster-than-light expansion.

However, the idea of faster-than-light expansion is addressed by the CSR model in a different way. One can see that the coordinates themselves are in space-velocity. For two objects moving in relative cosmic time t,

$$\tau^2 v^2 - x^2 - y^2 - z^2 = \tau^2 v'^2 - x'^2 - y'^2 - z'^2$$

Taking one-dimensional case,

$$\tau^2 v^2 - x^2 = \tau^2 v'^2 - x'^2$$

If we take this in one frame of reference, say in the un-primed frame of reference with v = 0 and x = 0,

$$\tau^2 v'^2 = x'^2$$
$$\tau v' = x'$$
... (VX)
$$\tau \frac{dv'}{dt} = \frac{dx'}{dt}$$

$$\tau \times Acceleration = v'_{rel}$$

Now, using observations, one sees that with increase in cosmic time, the acceleration of expansion of the universe is increasing.

For
$v'_{rel} > c$, and Age of Universe[4] $= 13.798 \times 10^9$ years.

$$\tau \times Acceleration > c$$

$$Acceleration > \frac{3 \times 10^8 m/s}{13.798 \times 10^9 \times 3.16 \times 10^7 s} = 6.88 \times 10^{-10} m/s^2$$

Thus, using CSR, one can define no upper bound for the velocity v until one can definitely define the acceleration of the universe. Only if the acceleration is above the calculated value, as per CSR, can one see faster-than-light expansion.

One can also see that for celestial bodies to move away from each other at a speed faster than light,

$$\frac{x'}{\tau} = v > c$$

Using (VX)

$$x' > c\tau = 130.8 \times 10^{24} m = 42.39 \times 10^5 kpc$$

$$x' > 42.39 \times 10^2 mpc \approx 4240\ mpc$$

Thus, the bodies need to be at a distance of 4240 mpc to have faster than light expansion. That means light has to travel roughly 13.826 billion years to travel this distance!

2. Postulates of Cosmological Special Relativity

The postulates put forth by Dr.Carmeli, for Cosmological General Relativity[1], in his seminal paper were:

1. *Principle of Constancy of Expansion of the Universe* at all cosmic times.
2. *Principle of Cosmological Relativity*: Principles of Physics are the same at all cosmic times.

One can find an analogy between these postulates and those of Special Relativity in general. I have tried to sum up the analogy in the given table.

Velocity is given fundamental importance in this theory over time since one is more interested with the velocity at which celestial bodies are receding rather than the time of the movement. However, just as in the case of inertial frames in Special Relativity in general, one has a Cosmic Time for any particular event. This is the time an event has taken place, with respect to a particular origin-time. Usually our present time is given the value t=0 and all other events are assigned cosmic time based on this assumption.

The fundamental reason for taking velocity as a more fundamental variable is because of the dimensionality of the most significant constant in the theory: the Hubble-Carmeli Constant. In Special Relativity, c has the dimensions of length/time, and hence one has Space-Time to describe the displacement of an object in terms of the time taken, subject to the constraint of an upper velocity bound.

$$[\tau] = Distance/Velocity$$

Hence we have the idea of Space-Velocity.

Table 1: Analogies	
Special Relativity	Cosmological Special Relativity
Inertial Frames	Cosmic Times
Space-Time	Space-Velocity
Speed of Light	Hubble-Carmeli Constant

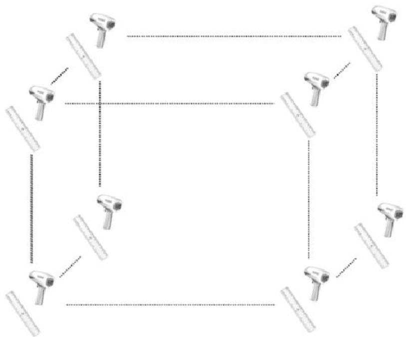


Figure 2: Hypothetical Lattice Unit for describing points in Space-Velocity, as per Cosmological Special Relativity. Just as observers describing an event in Space-Time would require scales and stop-clocks at each point to describe events in Space-Time, one could hypothetically think of scales and velocity trackers at each point in Space-Velocity for doing so in Space-Velocity. One could think of a lattice of points in Space-Velocity, each equipped with a scale and a velocity tracker.

3. Cosmological Transformation

Just as in the case of the Lorentz transformation one uses to convert space-time coordinates of an entity in one coordinate system to the space-time coordinates in another inertial system, one has Cosmological Transformation. Just as the corner-stone of the Lorentz transformation was the constant velocity of light, for Cosmological transformation, the constant expansion described by Hubble serves the purpose. When one is converting space-velocity coordinates of an event from one cosmic-time frame to another, one needs to keep the following quantity constant:

$$\tau^2 v^2 - x^2 - y^2 - z^2$$

For systems where the coordinates in one frame is given by x, y, z, v and in another is given by x', y', z', v' at a relative cosmic time t with respect to the first frame, the transformation is such that it should satisfy

$$\tau^2 v^2 - x^2 - y^2 - z^2 = \tau^2 v'^2 - x'^2 - y'^2 - z'^2$$

Let us take the simplest case of one dimensional motion.

$$\tau^2 v^2 - x^2 = \tau^2 v'^2 - x'^2 \text{... (1)}$$
$$y = y' \text{and} z = z'$$

I have tried to derive the transformation using first principles.

Let us have a transformation of the form,

$$x' = ax + b\tau v$$
$$\tau v' = cx + d\tau v$$

For x' = 0

$$ax = -b\tau v$$

$$\frac{b}{a} = -\frac{x}{v\tau} = -\frac{t}{\tau}$$

$$x' = a\left(x + \frac{b}{a}\tau v\right) = a(x - tv) \text{...(2)}$$

Using (2) in (1),

$$\tau^2 v^2 - x^2 = (cx + d\tau v)^2 - a^2(x - tv)^2$$

$$\tau^2 v^2 - x^2 - (cx)^2 - 2(cx)(d\tau v) - (d\tau v)^2 + a^2 x^2 + a^2 (tv)^2 - 2a^2(x)(tv) = 0$$

Evaluating coefficients of x^2, v^2 and xv

$$a^2 - c^2 = 1, \tau^2(d^2 - 1) = a^2 t^2, cd\tau + a^2 t = 0$$

Simplifying,

$$\frac{t}{\tau} = -\left(\frac{cd}{a^2}\right) = \pm\sqrt{\frac{d^2-1}{a^2}},\ a^2 = 1 + c^2,$$

$$d = \pm\sqrt{\frac{a^2}{a^2 - c^2}} = \pm\sqrt{1 + c^2}$$

$$\frac{t}{\tau} = \pm\sqrt{\frac{c^2}{(1+c^2)}},\ 1 - (\frac{t}{\tau})^2 = \frac{1}{1+c^2}$$

So

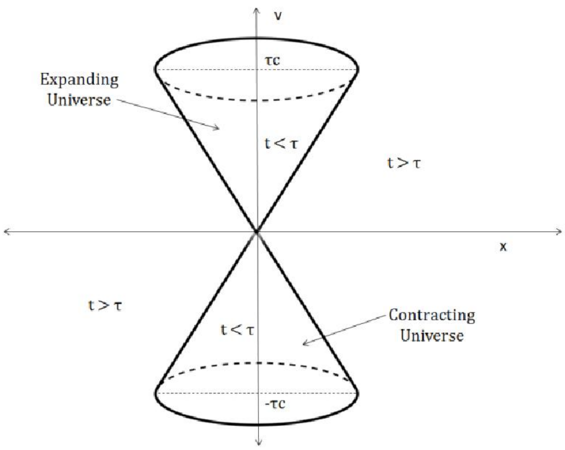
$$d = \pm\frac{1}{\sqrt{1-(t/\tau)^2}},\ c = \pm\sqrt{\frac{(t/\tau)^2}{1-(t/\tau)^2}}$$

$$a = \pm\frac{1}{\sqrt{1-(t/\tau)^2}},\ b = \mp\frac{t}{\tau}\frac{1}{\sqrt{1-(t/\tau)^2}}$$

Going by signs assigned as per the relations

$$x' = \frac{x-tv}{\sqrt{1-(t/\tau)^2}},\ v' = \frac{v-\frac{xt}{\tau^2}}{\sqrt{1-(t/\tau)^2}}$$

$$x = \frac{x'+tv'}{\sqrt{1-(t/\tau)^2}},\ v = \frac{v'+\frac{x't}{\tau^2}}{\sqrt{1-(t/\tau)^2}}$$



This gives us the set of equations that define the Cosmological Transformations. One cannot help but observe that this is similar to the Lorentz Transformation but with $\beta = \frac{t}{\tau}$ and not $\beta = \frac{v}{c}$.

One can define a transformation matrix for transformation between frames separated with a relative cosmic time. The transformation matrix is given by

$$\begin{pmatrix} 1\Big/\sqrt{1-(t/\tau)^2} & -t\Big/\sqrt{1-(t/\tau)^2} & 0 & 0 \\ \left.-1/\tau^2\right/\sqrt{1-(t/\tau)^2} & 1\Big/\sqrt{1-(t/\tau)^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Space-Velocity Diagram

Now we can represent any entity on the space-velocity diagram.

Just as in Space-Time Diagrams for Special Relativity, one can construct Space-Velocity Diagrams, as shown. In such diagrams, one can define what is known as a *Galaxy Cone*. This is the cone defined by the lines t = τ, followed by the galaxies in our universe. Any space-velocity line outside the cone is not found physically in the universe, and I am using the term *vel-like* for such lines, which satisfy $t > \tau$ i.e. $\tau^2\Delta v^2>\Delta r^2$ and there is a relative velocity between two events which allows it to recede in a time period greater than the age of the Universe, with respect to each other. One also has cases for which $\tau^2\Delta v^2<\Delta r^2$.

1. *Length and Velocity Contraction*
   We know that

$$x = \frac{x' + tv'}{\sqrt{1 - (t/\tau)^2}}$$

So,

$$\Delta x = x_1 - x_2 = \frac{x'_1 - x'_2}{\sqrt{1-(t/\tau)^2}} = \frac{\Delta x'}{\sqrt{1-(t/\tau)^2}}$$

Also,

$$v = \frac{v' + \frac{x' t}{\tau^2}}{\sqrt{1-(t/\tau)^2}}$$

For x' = 0

$$v = \frac{v'}{\sqrt{1-(t/\tau)^2}}$$

Hence, in cosmological space-velocity, the length and velocity of an object moving in a cosmic time frame with cosmic time t has lesser value than the values obtained in our cosmic- time frame (t=0).

This raises a fundamental question to the idea of an accelerating universe. What if the universe is not accelerating, but it is only because of velocity contraction that we feel that the Universe is undergoing accelerated expansion?

Law of Addition of Cosmic Times:

$$\frac{x}{v} = \frac{x' + v' t_1}{v' + \frac{x' t_1}{\tau^2}} = \frac{\frac{x'}{v'} + t_1}{1 + \frac{x' t_1}{v' \tau^2}}$$

Using coordinate transformation for frames moving away with a relative cosmic time t_1.

For $t = \frac{x}{v}$, $t_2 = \frac{x\prime}{v\prime} t = \frac{t_1 + t_2}{1 - \frac{t_1 t_2}{\tau^2}}$

One can never reach the valueτ, even with large values of t_1 and t_2, which would have yielded a number larger than τ using simple addition.

2. *Cosmological Red-Shift*

Applying the idea of Doppler Effect for Cosmological Special Relativity, one can write the wavelength of light emitted by a source in the cosmic time frame with relative cosmic time t, with respect to us (t = 0), as

$$\frac{\lambda}{\lambda_O} = \sqrt{\frac{1+\beta}{1-\beta}}, \beta = \frac{t}{\tau}$$

For $\frac{t}{\tau} \ll 1$,

$$z = \frac{\lambda}{\lambda_O} - 1 \approx \frac{t}{\tau} + \frac{1}{2}\left(\frac{t}{\tau}\right)^2$$

Thus, sources further back in time are more red-shifted.Also,

$$1 + z = \sqrt{\frac{1 + t/\tau}{1 - t/\tau}}$$

$$\beta = \frac{t}{\tau} = \frac{(1+z)^2 - 1}{(1+z)^2 + 1}$$

5. Rotation Curves

Rotation curves are the plots between the rotational velocities of components of galactic systems and the distance of the component from the centre of the galaxy.

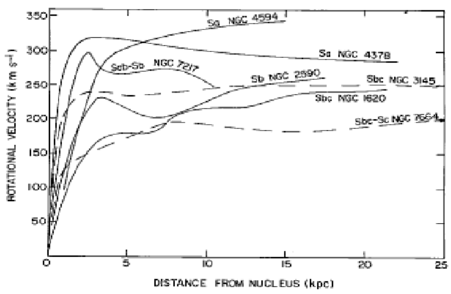
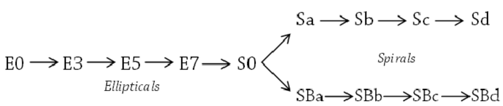


Figure 4: A series of rotation curves for spiral galaxies.
Figure from Rubin, Ford, and Thonnard (1978), Ap. J. Lett., 225, L107

One of the most significant results obtained by the study of the galaxy rotation curves has been the conjecture of the existence of dark matter. Given the luminous matter in galaxies such as the Sc 1 Spiral Galaxies, and considering the mass distribution of the galaxy to be equal to the mass of only luminous matter (*Rubin et al. 1985; Corradi& Capaccioli1990; Persic et al. 1996 have studied the relation between luminous properties and the rotation curves of galaxies*) one would expect the galaxy rotation curves to follow a curve similar to that of celestial bodies orbiting around a star or a planet: to decline at large distances from the central body. However, what one observes is that the velocity tends to become nearly constant and does not decline, as expected. The most commonly accepted idea for this is given by the conjecture that one has non-luminous massive entities that comprise *Dark Matter*, which contributes to the gravitational effect on the orbiting bodies. It is conjectured that the Dark Matter forms a halo around the luminous galactic matter.

To understand the relation between luminous matter and the rotation curves, *R. A. Swaterset al*[5] have used the rotation-curves of low-luminosity galaxies and dwarf-galaxies, as part of the *Westerbork H1 Survey of Spiral and Irregular Galaxies Project* (WHISP). They focussed on galaxies of Hubble type later than *Sd*, and used a refinement technique that involved initial rotation and final rotation curve estimates, using *H1* distribution rather than the conventional *Hα* distribution. They established some interesting characteristic relations for galaxies, with emphasis on Dwarf Galaxies.



HUBBLE SEQUENCE

- *Rubin et al*., *Persic&Salucci*and*Broeils* showed that rotation curves rise more rapidly towards highly luminosities as compared to lower luminosities. To this *R. A. Swaters et al*. added that galaxies with luminosity magnitude range = -20 to -16, the log-slopes of the outer rotation curves *do not* depend on the luminosity. It is found that the correlation between surface brightness and logarithmic slope of the rotation curve is weak and clustered in an intermediate surface brightness range.

- It is also seen that galaxies of the type Sc, Sc-Irregular and Irregular are most observed, with a luminosity magnitude range = -16 to -19.5. Also MR = -17 to -14 are seen to be irregular galaxies.

- For galaxies for which the last measured point lay beyond 3 disk scale lengths and had inclinations between 39^{O} and 80^{O}, the *logarithmic slopes* of the rotation curves were found to be more in the interval 0 to 0.42, for the maximum velocity range of 20 to 140 km/s.

MOND Theory

Another school of thought puts forth the idea that Newtonian Mechanics breaks down on the galactic scale and the law of gravitational attraction becomes proportional to 1/r (as given by Modified Newtonian Dynamics or MOND theory).[6]

In 1983, MordehaiMilgrom hypothesized a modification of Newtonian Dynamics for cases that involved accelerations below the typical centripetal acceleration of a star in a galaxy

$$a_O \approx 10^{-10} m/s^2$$

This theory is appropriate for minor accelerations in a galactic system. For $g >> a_O$, we get the classical Newtonian case ($a_O \rightarrow 0$): $g = g_N$.

For $a_O \rightarrow \infty$ (i.e. for the small acceleration case), we have the modification

$$Small\ Acceleration: g = \sqrt{g_N a_O} \ \dots \text{(SA)}$$

g_N denotes the Newtonian acceleration

Now using the relation for centripetal acceleration of an object carrying out circular motion around a central body (mass: M) at a distance of R,

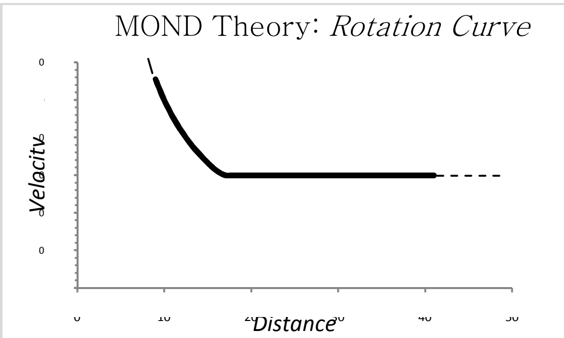
$$\frac{v^2}{R} = g_N = \frac{GM}{R^2}$$

For small values of acceleration, using relation (SA),

$$g = \sqrt{g_N a_o} = \sqrt{\frac{GMa_o}{R^2}}$$

$$\frac{v^2}{R} = \sqrt{\frac{GMa_o}{R^2}} \Rightarrow v^4 = GMa_o$$

As can be seen, this velocity function is independent of distance from the centre, and is directly proportional to the mass of the central object.[7] One can take the simplest normalized case with G = 1, M = 1 and find that the profile of the curve should look something like this far from the centre:



To ensure a smooth transition between the cases for which $g \ll a_o$ and $g \gg a_o$, Milgrom's Law can be written as:

$$\mu\left(\frac{g}{a_o}\right) g = g_N$$

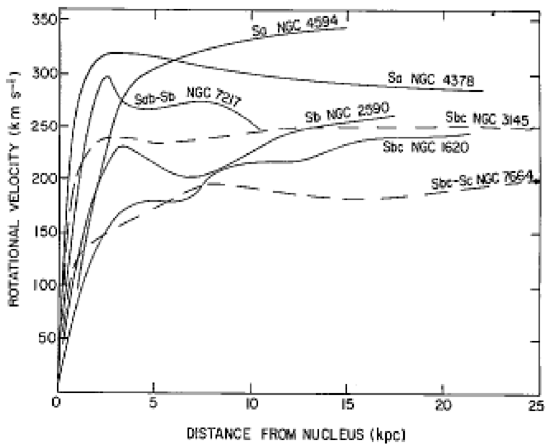
where $\mu(x)$ is an interpolating function with the property:

$$\mu(x) \rightarrow 1, x \gg 1$$

$$\mu(x) \rightarrow x, x \ll 1$$

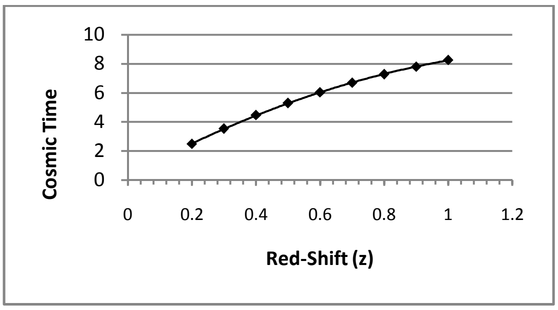
Recently, certain results have brought the attention of the scientific community back to the idea of Dark Matter. One of them is the discovery of ring-like 'Dark Matter' structure in the core of the galaxy cluster C1 0024+17.[8] However, I have tried to focus on CSR and identify how the theory addresses the issues relating to the expansion of the Universe and Galaxy Rotation Curves.

Results



Red-Shift Study and Comparison with Friedmann Model

Experimentally, one of the most useful ways of studying this theory is using Astrometry, especially the study of Red-Shifts of the galaxies. Here, I have tried to compare the plot of *Redshift* vs*Cosmic Time* given by this model, and that given by a popular model: the Friedmann Model.

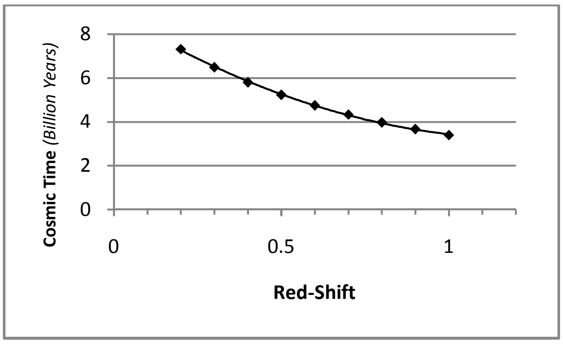


Graph 1: Cosmic Time vs. Red-Shift, as per Cosmological Special Relativistic Model

Now Considering Total Density Parameter (Ω_O) to be ~1.02 for the universe and following the Friedman World Model, we have[9, 10]

$$t = \frac{\Omega_O}{H_O(\Omega_O - 1)^{3/2}}[\sin^{-1} y^{1/2} - y^{1/2}(1-y)^{1/2}]$$

with $y = \frac{\Omega_O - 1}{\Omega_O(1+z)}$ and H_O- Hubble's Constant



Graph 2: Cosmic Time vs. Red-Shift, as per Friedmann Model

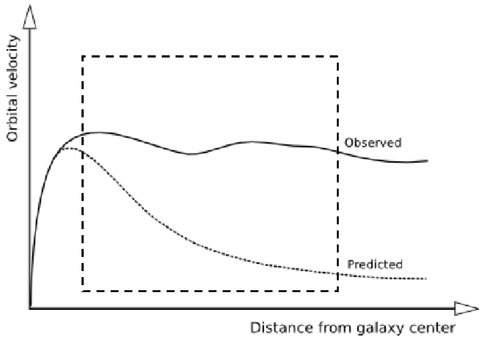
In this case, we have considered zero curvature for the *Friedmann Model*, thereby modelling a flat universe. According to the model, as evident, the model predicts that the universe will expand forever, but at a rate that eventually approaches zero. The zero-curvature assumption is valid for *special* relativity as gravitational effects are neglected.

However, by Cosmological Special Relativity, the acceleration is *increasing*. This is a well-accepted fact today, given the seminal papers by Nobel Laureates Saul Perlmutter, Adam G. Reiss and Brian P. Schmidt [11, 12], based on their observations on Type Ia Supernovae.

Radial Velocity Distributions and Dark Matter

The existence of Dark Matter has been a much debated topic. A primary basis for justifying the existence of Dark Matter has been the discrepancy of theoretically expected values, given assumptions of mass-to-luminosity ratios, and the observed values in Rotation Curves. Rotation curve of a galaxy is the curve that represents the relation between the rotational velocity of the visible entities in the galaxy and their radial distance from the centre of the galaxy.

Vera Cooper Rubin was one of the pioneers of the Gravity Rotation problem. She worked on the Rotation Curves of the Sc I and Sc II galaxies.



A series of rotation curves for spiral galaxies.
Edited on Figure from www.citizendium.org

Now, the velocity function for a galaxy, in terms of the distance from its centre is given by

$$v(r) = \sqrt{\frac{G_O M}{r}}\left(\frac{r}{r_c + r}\right)^{\left(\frac{3}{2}\right)\beta} \times \sqrt{1 + \sqrt{\frac{M_O}{M}}\left(1 - e^{-\frac{r}{r_O}}\left(1 + \frac{r}{r_O}\right)\right)}$$

Let us concentrate on $r_c << r$ and $r < r_O$, and for HSB Galaxies, β = 1. For a given mass M,

$$v(r) \propto \sqrt{\frac{1}{r}}\sqrt{1+\sqrt{\frac{M_O}{M}}\left(1-e^{-\frac{r}{r_O}}\left(1+\frac{r}{r_O}\right)\right)}$$

where k is a constant dependent on the mass of the celestial body, r_c- the radius of inner core and parameter r_O.[13]

Taking the Milky Way as a test example,

$$M = 5.8\times10^{11}M_{\odot}, M_O = 9.60\times10^{11}M_{\odot}$$

$$r_O = 13.92\ kpc$$

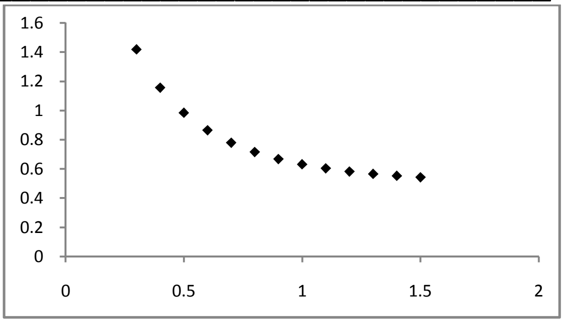
We can now introduce r in *kpc* and with constant K having dimensions of [length$^{3/2}$/time].

$$v(r) = K\sqrt{\frac{1}{r}}\sqrt{2.29-1.29\times\left(e^{-\frac{r}{13.92}}\left(1+\frac{r}{13.92}\right)\right)}$$

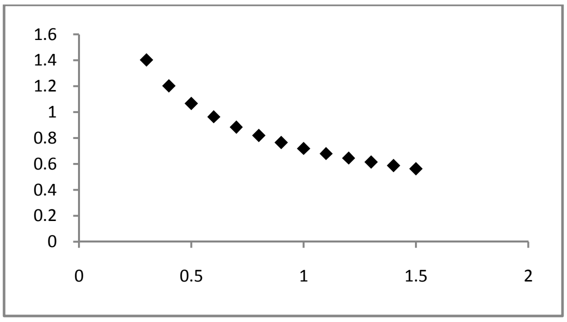
$$v(r) \approx K\times 1.513 \times\sqrt{\frac{1}{r}}\left[1-0.282\left(e^{-\frac{r}{13.92}}\left(1+\frac{r}{13.92}\right)\right)\right]$$

Now, we know that the curve for the function

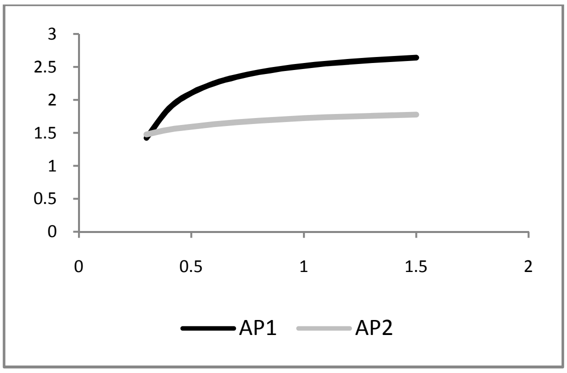
$$\frac{1}{\sqrt{r}}+\sqrt{r}e^{-r}$$



As an exercise, I tried to scale down the variable r in the equation using division of r by a constant (10). And I obtained:

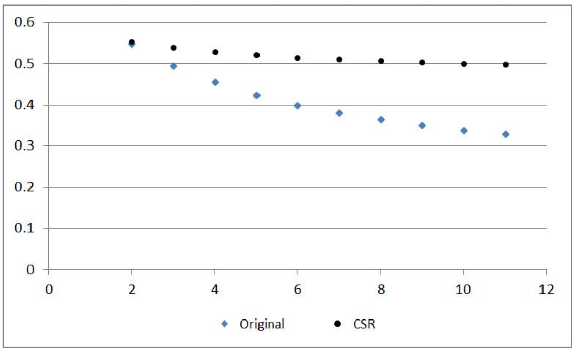


However, the most interesting result was obtained when each point of the data-set was evaluated after r was divided by arithmetic progressions for successive values of r (A.P.$_1$ with a = 1 and r = 1; and A.P.$_2$ with a = 0.1 and r = 0.1):



This, along with my previous conjecture that the discrepancy could possibly be because of varying divisors for the distance variable in the function for v(r), took me to test if I can use the idea of Cosmological Transformation and check the changes brought about.

Now, I have considered a hypothetical case of a galaxy. I have considered 10 data-points at distances 4 to 13 kpc. Also, I have considered the AP case with a = 1.2 and r = 0.2. This means, in terms of Cosmological Transformation, γ is taking values from 1.2 to 0.2. In other words, we have t ~ 7.5 Billion Years to ~13 Billion Years. The radius of the galaxy is taken to be r = 13.92 kpc.



The data-sets have been normalized to meet at the left end-point

However, I would like to point out a few points here, for the given hypothetical case:

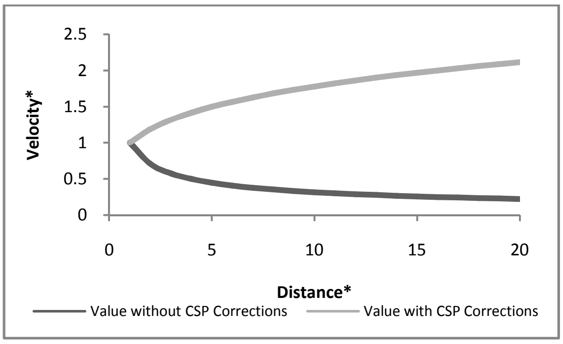
- Firstly, the progression I have taken begins from a large value of t. t~7.6 Billion Years is about 55% of the age of the universe. Thus, we are talking of distant galaxies that are at a distance of about 2.33 mpc or more.
- I have limited the data-points to 13 kpc to limit it within the radius of the galaxy.
- Here one has taken Milky-Way-like configuration with a negligible core radius and a uniform mass distribution.

Use of Newtonian Dynamics

One can use the formula

$$\omega = \frac{v}{R} = \frac{1}{r}\sqrt{\frac{GM}{r}}$$

And try and see if a system with CSP Corrections would give expected results. For Distance from Earth to Galactic Centre = 900 kpc, and uniform mass distribution (with unit mass and unit G, for convenience),



**Variables have been normalized (hypothetical case) with M=1 and G=1*

One final line of thought that can be studied and analyzed is that of trying to explain the galaxy rotation curves using a combination of the MOND theory and Cosmological Special Relativity. It is seen that some galaxies, such as NGC 2841, have a discrepancy in MOND fits and observed values. One can try to see if CSR corrections can be used to make the fitted curve closer to the experimentally obtained values.

Short-Term Targets

For near Galaxies, as one can see, the Cosmological factor γ is negligible and hence the correction is minimal. Hence, by the velocity function given by *Brownstein et al*[12], one cannot find significant corrections. I would like to work on the Mass-corrections if any, after extending the present work to Cosmological General Relativity for various galaxies, as worked on by *J. Hartnett*.

I would also like to extend the present study to more galaxies, with application to particular systems, with emphasis on mass distribution.

Conclusion

I have tried work on the fundamentals and applications of Cosmological Special Relativity, as modelled primarily by Dr. Moshe Carmeli, and develop the theory further. I have worked on proving the validity of the model, as compared to the Friedmann Model, in describing the accelerated expansion of the Universe. I have also studied red-shifts and shown that the apparent disparity between theoretically expected and observed data-points for the *Rotational Velocity vs. Distance* graph can be explained by Cosmological Special Relativity.

Though Cosmological Special Relativity is not a standard paradigm in Cosmology today, it is an interesting theory put forth by Moshe Carmeli, which has been studied in the present paper.

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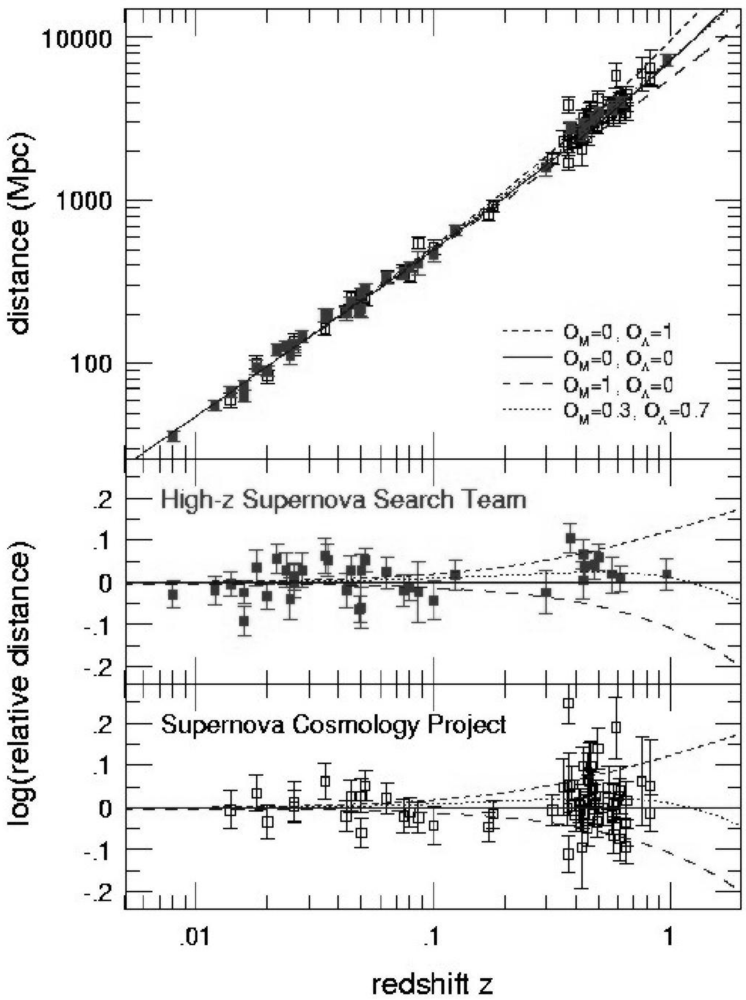

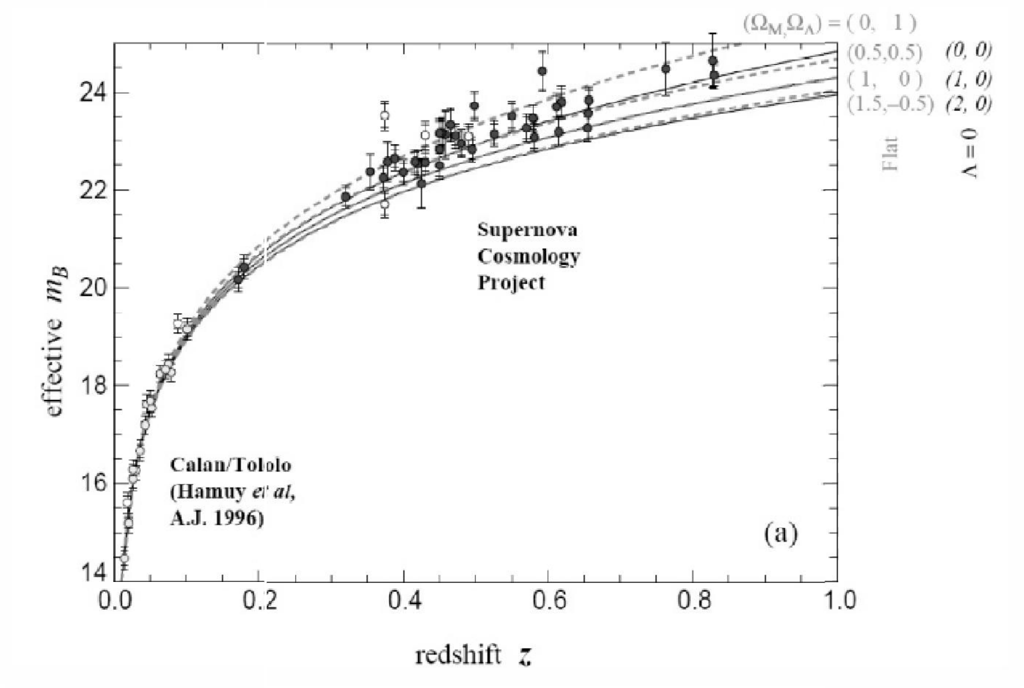

Figure 6

The Hubble diagram for 42 high redshift type Ia supernovae from SCP and 18 low redshift supernovae from the Calan/Tololo Supernova Survey. The dashed curves show a range of "flat" models where$\Omega M + \Omega\Lambda = 1$.

Courtesy: www.nobelprize.org

Analyticity and Cauchy-Riemann Equations

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Abstract

Cauchy-Riemann equations relate the real and imaginary parts of a complex-analytic function. These equations originate from the requirement for the uniqueness of the derivative, independent of the line or direction along which the limit is taken. This note attempts to make a presentation of the derivation in which the "direction of approach" is made explicit. Some interesting implications of the notion of analyticity are given.

What is the analytic function whose real part is x? It is $x+iy+c$, where c is a constant. Is there an analytic function whose real part is x^2? There is none. If we choose two functions $u(x,y)$ and $v(x,y)$ randomly, it is likely that $f(x,y) = u+iv$ is not an analytic function. Complex analyticity is about defining the derivative of a complex-valued function of the complex variable $z = x+iy$. Formally, the complex-valued function f is said to be analytic at a point $z \in C$ if

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}, \qquad (1)$$

exists and is unique irrespective of the direction along which δz approaches zero[1]. This requirement for uniqueness gives rise to the well known Cauchy-Riemann (CR) equations relating the real and imaginary parts of $f(z)$. To ensure uniqueness, two independent directions in the complex $x-y$ plane are chosen and the equality of the derivatives along the respective directions is demanded.

We recall that to define the derivative of a function of a single real variable, the left-sided and right-sided derivatives are to be equal[2]. In the complex plane, the number of directions are unlimited. The conventional choices for the two directions are along the x and y axes respectively. In fact, any two axes which are inclined at a nonzero angle are fine to establish the need for the CR equations. It would be better if this "direction of approach" is made to appear explicitly in the derivation of the CR conditions and in this note we attempt that.

With $f(z) = u(x, y) + iv(x, y)$, u and v being the real and imaginary parts respectively, and $\delta z = \delta x + i\delta y$,

$$f(z + \delta z) = f(x + \delta x + i(y + \delta y)) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y). \tag{2}$$

Expanding $u(x + \delta x, y + \delta y)$ and $v(x + \delta x, y + \delta y)$ about (x, y) and rearranging[3], we get

$$f(x+\delta x, y+\delta y)-f(x, y) = \frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\epsilon_1\delta x+\epsilon_2\delta y+i\left[\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \epsilon_3\delta x + \epsilon_4\delta y\right]. \tag{3}$$

The factors ϵ_1, ϵ_2, ϵ_3 and ϵ_4 vanish as δz approaches zero. These factors are included to represent the contribution from the higher order terms in δx and δy to the expansion of the function.

To bring out the dependence of the derivative on the direction of approach, it is advantageous to adapt the polar representation[4] of complex numbers to write $\delta z = r\exp(i\theta)$, so that $\delta x = r\cos\theta$ and $\delta y = r\sin\theta$. Here θ represents the angle δz makes with x-axis and this angle can be interpreted as the direction of approaching z as $\delta z \to 0$. In Fig. 1, three of the infinitely many possible directions are indicated. Approaching along the x-axis and y-axis correspond to choosing $\theta = 0$ and $\pi/2$ respectively. Using the polar representation of δz in the definition of the derivative of $f(z)$ yields

$$f'(z) = \lim_{r\exp(i\theta)\to 0} \frac{r\left[\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} + i\cos\theta\frac{\partial v}{\partial x} + i\sin\theta\frac{\partial v}{\partial y}\right]}{r\exp(i\theta)},. \tag{4}$$

Since $\exp(-i\theta)$ does not vanish for any real θ, the limit corresponds to $r \to 0$. Therefore,

$$f'(z) = \frac{\cos\theta\frac{\partial u}{\partial x} + i\sin\theta\frac{\partial v}{\partial y} + \sin\theta\frac{\partial u}{\partial y} + i\cos\theta\frac{\partial v}{\partial x}}{\exp(i\theta)}. \tag{5}$$

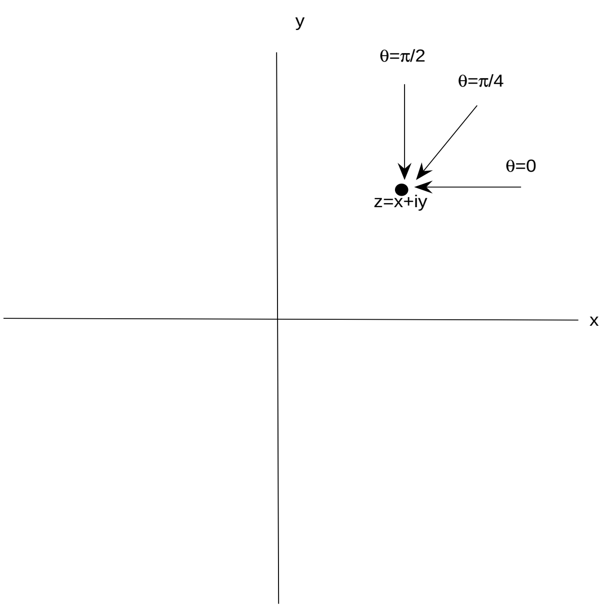

Figure 1: Some possible directions for approaching z as $\delta z \to 0$.

This ratio is independent of r. However, it depends on θ, the direction of approaching z. For the uniqueness of the derivative, the limit should be independent of θ. That is possible only if the factor $\exp(i\theta)$ in the denominator is cancelled by an identical term in the numerator. It is readily recognized that if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{6}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{7}$$

the required cancellation occurs in Eq. 5 and the limit is independent of θ. Consequently, the derivative $f'(z)$ is uniquely defined. The general form, given in Eq. 5, itself is simple enough to identify the conditions for the uniqueness of the derivative. The relations given in Eq. 6 and Eq. 7 are the CR equations. We note that the various ϵ factors in Eq. 3 do not appear in the expression for $f'(z)$ since they become zero if δz approaches zero.

Now, to see why x^2 cannot be the real part of an analytic function, let us assume that $f = x^2 + iv(x, y)$ is an analytic function. We need to find the imaginary part v. The CR conditions imply

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x, \quad (8)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0. \quad (9)$$

The second equation implies that v is a function of y alone, so that its partial derivative with respect to x vanishes. So, the partial derivative of $v(y)$ with respect ot y is also a function of y alone. But the first equation implies that the partial derivative of $v(y)$ depends on x, which is inconsistent. So, the CR conditions cannot be satisfied if the real part of an analytic function is x^2. We, therefore, conclude that an analytic function cannot have x^2 as its real part. In fact, it is straightforward to extend the argument to establish that there is no analytic function, other than $x + iy$, whose real part is solely a function of x.

An exercise to illustrate the severe restriction imposed by the analyticity is to consider those functions whose real part is a sum of the form $f(x) + g(y)$. Though this form appears to be rather nonrestrictive, the CR conditions give the real part to be $a(x^2 - y^2) + bx + cy$ and the corresponding imaginary part is $2axy - cx + by$, where a, b and c are real constants. It is equally possible to choose a specific form for the imaginary part of an analytic function, and that would determine its real part.

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A New and Simpler Method of Calculating Approximate Energies of Bound-State Potentials using the WKB Approximation

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Abstract

In this article, we discuss a new and simpler method employed to determine the approximate energies of certain bound-state potentials using the WKB approximation. In this method, the potentials are represented by a finite number of equally spaced rectangular step functions. The energy for each step is calculated using the Bohr-Sommerfeld quantization condition $\int pdx = \left(n + \frac{1}{2}\right)\hbar\pi$, where 'p' is the classical momentum and 'n' is a quantum number. The, the total energy would be the sum of the energies determined for all steps in the potential. Specifically, we have determined the approximate energies for the Simple Harmonic Potential, The Double-Ramp Potential and the Quartic Potential. However, this method can be extended, in general, to any bound-state potential.

Keywords: Schrodinger equation, WKB approximation, Bohr-Sommerfeld quantization, Bound-State potential.

1. Introduction

All problems in Non-Relativistic Quantum Mechanics cannot be solved exactly using the Schrodinger equation and therefore one has to resort to approximate methods of solving the same. Some of the well known and commonly used methods are the Perturbation theory, the WKB approximation and the Rayleigh-Ritz Variational Method. We focus on the WKB approximation in this article as it is particularly useful for calculating allowed energies of bound-state potentials. In particular, from the solutions of the WKB wave function, one can arrive at the Bohr-Sommerfeld quantization rule. This is a powerful result as it enables us to determine the approximate allowed energies without having to solve the Schrodinger equation. However, for complex potentials, the integrals become hard to solve. In this article, we present a simpler method in which the integrals encountered in the bound-state potentials are easily solved. We consider three such potentials, namely, the Simple Harmonic potential, the Double-Ramp potential and the Quartic potential. For the sake of convenience, we assume the potentials in this article to be a function of position only. Each of them is represented by a finite number of equally spaced rectangular step functions. The simple harmonic potential and the double-ramp potential have been represented by three, five and seven steps and the quartic potential has been represented by three and five steps. In general, the energy for each step has been determined and the total energy is the sum of the energies determined for each step.

This paper is organized as follows: in Sections 2, 3 and 4 we discuss the allowed energies for the simple harmonic potential, the double-ramp potential and the quartic potential respectively, Section 5 is dedicated to the discussion of results and in Section 6 we express our acknowledgements.

2. Simple Harmonic Potential

We choose to solve the simple harmonic potential first as it is frequently encountered in the literature. Using the standard WKB method, the exact allowed energies for this potential can be obtained. The potential of a simple harmonic oscillator is given by

$$V = \frac{1}{2} m\omega^2 x^2$$

where 'm' is the mass, 'ω' is the angular frequency and 'x' is the position. At the turning points $V = E$ and let $x = A$,

$$V = E = \frac{1}{2} m\omega^2 A^2$$

Solving for

$$A = \left[\frac{2E}{m\omega^2}\right]^{\frac{1}{2}}$$

The potential is now approximated by a finite number of rectangular steps. Consider the potential to be approximated by three steps as shown below.

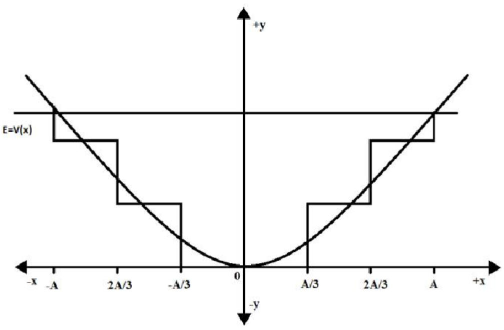

Figure 1: Simple Harmonic Potential approximated by three steps

From equation (2), the potentials at different steps are given by

$$V_1 = \frac{1}{2} m\omega^2 \left(\frac{A}{6}\right)^2,$$

$$V_2 = \frac{1}{2} m\omega^2 \left(\frac{A}{2}\right)^2,$$

$$V_3 = \frac{1}{2} m\omega^2 \left(\frac{5A}{6}\right)^2$$

As $p(x) = [2m\{E - V(x)\}]^{1/2}$ the Bohr-Sommerfeld quantization condition can be written as $\int [2m\{E - V(x)\}]^{1/2}\, dx = \left(n + \frac{1}{2}\right) \hbar\pi$(3)

For the simple harmonic potential as approximated above, we write equation (3) as

$$\int_{-A}^{-5A/6} \sqrt{2m(E - V_3(x))}\, dx + \int_{-5A/6}^{-2A/3} \sqrt{2m(E - V_3(x))} dx + \int_{-2A/3}^{-A/2} \sqrt{2m(E - V_2(x))} dx + \int_{-A/2}^{-A/3} \sqrt{2m(E - V_2(x))} dx + \int_{-A/3}^{-A/6} \sqrt{2m(E - V_1(x))} dx + \int_{-A/6}^{0} \sqrt{2m(E - V_1(x))} dx + \int_{0}^{A/6} \sqrt{2m(E - V_1(x))} dx + \int_{A/6}^{A/3} \sqrt{2m(E - V_1(x))} dx + \int_{A/3}^{A/2} \sqrt{2m(E - V_2(x))} dx + \int_{A/2}^{2A/3} \sqrt{2m(E - V_2(x))} dx + \int_{2A/3}^{5A/6} \sqrt{2m(E - V_3(x))} dx + \int_{5A/6}^{A} \sqrt{2m(E - V_3(x))} dx = \left(n + \frac{1}{2}\right) \hbar\pi$$

Solving the integrals we get,

$$\sqrt{(2m(E-V_3)}\left[\frac{-5A}{6}+A\right]+\sqrt{2m(E-V_3)}\left[\frac{-2A}{3}+\frac{5A}{6}\right]+\sqrt{2m(E-V_2)}\left[\frac{-A}{2}+\frac{2A}{3}\right]$$
$$+\sqrt{2m(E-V_2)}\left[\frac{-A}{3}+\frac{A}{2}\right]+\sqrt{2m(E-V_1)}\left[\frac{-A}{6}+\frac{A}{3}\right]+\sqrt{2m(E-V_1)}\left[\frac{A}{6}\right]$$
$$+\sqrt{2m(E-V_1)}\left[\frac{A}{6}\right]+\sqrt{2m(E-V_1)}\left[\frac{A}{3}-\frac{A}{6}\right]+\sqrt{2m(E-V_2)}\left[\frac{A}{2}-\frac{A}{3}\right]$$
$$+\sqrt{2m(E-V_2)}\left[\frac{2A}{3}-\frac{A}{2}\right]+\sqrt{2m(E-V_3)}\left[\frac{5A}{6}-\frac{2A}{3}\right]+\sqrt{2m(E-V_3)}\left[A-\frac{5A}{6}\right]$$
$$=\left(n+\frac{1}{2}\right)\hbar\pi$$

Simplifying the above we get,

$$\frac{2A}{3}\sqrt{2m(E-V_3)}+\frac{2A}{3}\sqrt{2m(E-V_2)}+\frac{2A}{3}\sqrt{2m(E-V_1)}=\left(n+\frac{1}{2}\right)\hbar\pi$$

$$\frac{2A}{3}(2mE)^{1/2}\left[\left(1-\frac{V_3}{E}\right)^{1/2}+\left(1-\frac{V_2}{E}\right)^{1/2}+\left(1-\frac{V_1}{E}\right)^{1/2}\right]=\left(n+\frac{1}{2}\right)\hbar\pi$$

Using the forms of V_1, V_2 and V_3 in the above equation, we get

$$\frac{2A}{3}(2mE)^{\frac{1}{2}}\left[\left(1-\frac{25}{36}\right)^{\frac{1}{2}}+\left(1-\frac{1}{4}\right)^{\frac{1}{2}}+\left(1-\frac{1}{36}\right)^{\frac{1}{2}}\right]=\left(n+\frac{1}{2}\right)\hbar\pi$$

Substituting for A from equation (2) in the above equation, we get,

$$\frac{4E}{3\omega}(3.206045)=\left(n+\frac{1}{2}\right)\hbar\pi$$

Simplifying the above equation for E, we get

$$E=0.7349\left(n+\tfrac{1}{2}\right)\hbar\omega \qquad (4)$$

Now let us consider the potential to be approximated by five steps. The Bohr-Sommerfeld quantization rule would be now be written as

$$\int_{-A}^{-9A/10} \sqrt{2m(E-V_5(x))}\,dx + \int_{-9A/10}^{-4A/5} \sqrt{2m(E-V_5(x))}\,dx + \int_{-4A/5}^{-7A/10} \sqrt{2m(E-V_4(x))}\,dx$$
$$+ \int_{-7A/10}^{-3A/5} \sqrt{2m(E-V_4(x))}\,dx + \int_{-3A/5}^{-5A/10} \sqrt{2m(E-V_3(x))}\,dx + \int_{-5A/10}^{-2A/5} \sqrt{2m(E-V_3(x))}\,dx$$
$$+ \int_{-2A/5}^{-3A/10} \sqrt{2m(E-V_2(x))}\,dx + \int_{-3A/10}^{-A/5} \sqrt{2m(E-V_2(x))}\,dx + \int_{-A/5}^{-A/10} \sqrt{2m(E-V_1(x))}\,dx$$
$$+ \int_{-A/10}^{0} \sqrt{2m(E-V_1(x))}\,dx + \int_{0}^{A/10} \sqrt{2m(E-V_1(x))}\,dx + \int_{A/10}^{A/5} \sqrt{2m(E-V_1(x))}\,dx$$
$$+ \int_{A/5}^{3A/10} \sqrt{2m(E-V_2(x))}\,dx + \int_{3A/10}^{2A/5} \sqrt{2m(E-V_2(x))}\,dx + \int_{2A/5}^{5A/10} \sqrt{2m(E-V_3(x))}\,dx$$
$$+ \int_{5A/10}^{3A/5} \sqrt{2m(E-V_3(x))}\,dx + \int_{3A/5}^{7A/10} \sqrt{2m(E-V_4(x))}\,dx + \int_{7A/10}^{4A/5} \sqrt{2m(E-V_4(x))}\,dx$$
$$+ \int_{4A/5}^{9A/10} \sqrt{2m(E-V_5(x))}\,dx + \int_{9A/10}^{A} \sqrt{2m(E-V_5(x))}\,dx = \left(n+\frac{1}{2}\right)\pi\hbar$$

where

$$V_1 = \frac{1}{2}m\omega^2\left(\frac{A}{10}\right)^2, V_2 = \frac{1}{2}m\omega^2\left(\frac{3A}{10}\right)^2,$$

$$V_3 = \frac{1}{2}m\omega^2\left(\frac{5A}{10}\right)^2, \qquad V_4 = \frac{1}{2}m\omega^2\left(\frac{7A}{10}\right)^2\ V_5 = \frac{1}{2}m\omega^2\left(\frac{9A}{10}\right)^2$$

Solving the integrals we get,

$$\sqrt{2mE}\left[\left(1-\frac{V_1}{E}\right)\left(\frac{A}{10}+\frac{A}{5}-\frac{A}{10}+\frac{A}{10}-\frac{A}{10}+\frac{A}{5}\right)+\left(1-\frac{V_2}{E}\right)\left(\frac{3A}{10}-\frac{A}{5}+\frac{2A}{5}-\frac{3A}{10}-\frac{A}{5}+\frac{3A}{10}-\frac{3A}{10}+\frac{2A}{5}\right)\right.$$
$$+\left(1-\frac{V_3}{E}\right)\left(\frac{5A}{10}-\frac{2A}{5}+\frac{3A}{5}-\frac{5A}{10}-\frac{5A}{10}+\frac{3A}{5}-\frac{2A}{5}+\frac{5A}{10}\right)$$
$$+\left(1-\frac{V_4}{E}\right)\left(\frac{7A}{10}-\frac{3A}{5}+\frac{4A}{5}-\frac{7A}{10}-\frac{3A}{5}+\frac{7A}{10}-\frac{7A}{10}+\frac{4A}{5}\right)$$
$$\left.+\left(1-\frac{V_5}{E}\right)\left(\frac{9A}{10}-\frac{4A}{5}+A-\frac{9A}{10}-\frac{9A}{10}+A-\frac{4A}{5}+\frac{9A}{10}\right)\right]=\left(n+\frac{1}{2}\right)\hbar\pi$$

Simplifying the above we get,

$$\sqrt{2mE}\left[\left(1-\frac{V_1}{E}\right)\frac{2A}{5}+\left(1-\frac{V_2}{E}\right)\frac{2A}{5}+\left(1-\frac{V_3}{E}\right)\frac{2A}{5}+\left(1-\frac{V_4}{E}\right)\frac{2A}{5}+\left(1-\frac{V_5}{E}\right)\frac{2A}{5}\right]=\left(n+\frac{1}{2}\right)\hbar\pi$$

Using the forms of V_1, V_2, V_3, V_4 and V_5 in the above equation, we get

$$\frac{2A}{5}\sqrt{2mE}\left[\left(1-\frac{1}{100}\right)+\left(1-\frac{9}{100}\right)+\left(1-\frac{25}{100}\right)+\left(1-\frac{49}{100}\right)+\left(1-\frac{81}{100}\right)\right]=\left(n+\frac{1}{2}\right)\hbar\pi$$

or

$$\frac{4E}{5\omega}\left[\left(\frac{99}{100}\right)^{1/2}+\left(\frac{91}{100}\right)^{1/2}+\left(\frac{75}{100}\right)^{1/2}+\left(\frac{51}{100}\right)^{1/2}\left(\frac{19}{100}\right)^{1/2}\right]=\left(n+\frac{1}{2}\right)\hbar\pi$$

Simplifying the above equation for E, we get

$$E=0.99066\left(n+\tfrac{1}{2}\right)\hbar\omega \qquad (5)$$

Now let us consider the potential to be approximated by seven steps. The Bohr-Sommerfeld quantization rule would be now be written as

$$\int_{-A}^{-13A/14} \sqrt{2m(E-V_7(x))}\,dx + \int_{-13A/14}^{-6A/7} \sqrt{2m(E-V_7(x))}\,dx + \int_{-6A/7}^{-11A/14} \sqrt{2m(E-V_6(x))}\,dx$$

$$+ \int_{-11A/14}^{-5A/7} \sqrt{2m(E-V_6(x))}\,dx + \int_{-5A/7}^{-9A/14} \sqrt{2m(E-V_5(x))}\,dx + \int_{-9A/14}^{-4A/7} \sqrt{2m(E-V_5(x))}\,dx$$

$$+ \int_{-4A/7}^{-A/2} \sqrt{2m(E-V_4(x))}\,dx + \int_{-A/2}^{-3A/7} \sqrt{2m(E-V_4(x))}\,dx + \int_{-3A/7}^{-5A/14} \sqrt{2m(E-V_3(x))}\,dx$$

$$+ \int_{-5A/14}^{-2A/7} \sqrt{2m(E-V_3(x))}\,dx + \int_{-2A/7}^{-3A/14} \sqrt{2m(E-V_2(x))}\,dx + \int_{-3A/14}^{-A/7} \sqrt{2m(E-V_2(x))}\,dx$$

$$+ \int_{-A/7}^{-A/14} \sqrt{2m(E-V_1(x))}\,dx + \int_{-A/14}^{0} \sqrt{2m(E-V_1(x))}\,dx + \int_{0}^{A/14} \sqrt{2m(E-V_1(x))}\,dx$$

$$+ \int_{A/14}^{A/7} \sqrt{2m(E-V_1(x))}\,dx + \int_{A/7}^{3A/14} \sqrt{2m(E-V_2(x))}\,dx + \int_{3A/14}^{2A/7} \sqrt{2m(E-V_2(x))}\,dx$$

$$+ \int_{2A/7}^{5A/14} \sqrt{2m(E-V_3(x))}\,dx + \int_{5A/14}^{3A/7} \sqrt{2m(E-V_3(x))}\,dx + \int_{3A/7}^{A/2} \sqrt{2m(E-V_4(x))}\,dx$$

$$+ \int_{A/2}^{4A/7} \sqrt{2m(E-V_4(x))}\,dx + \int_{4A/7}^{9A/14} \sqrt{2m(E-V_5(x))}\,dx + \int_{9A/14}^{5A/7} \sqrt{2m(E-V_5(x))}\,dx$$

$$+ \int_{5A/7}^{11A/14} \sqrt{2m(E-V_6(x))}\,dx + \int_{11A/14}^{6A/7} \sqrt{2m(E-V_6(x))}\,dx + \int_{6A/7}^{13A/14} \sqrt{2m(E-V_7(x))}$$

$$+ \int_{13A/14}^{A} \sqrt{2m(E-V_7(x))} = \left(n+\frac{1}{2}\right)\hbar\pi$$

where $V_1 = \frac{1}{2}m\omega^2\left(\frac{A}{14}\right)^2, V_2 = \frac{1}{2}m\omega^2\left(\frac{3A}{14}\right)^2, V_3 = \frac{1}{2}m\omega^2\left(\frac{5A}{14}\right)^2, V_4 = \frac{1}{2}m\omega^2\left(\frac{A}{2}\right)^2$

$V_5 = \frac{1}{2} m\omega^2 \left(\frac{9A}{14}\right)^2, V_6 = \frac{1}{2} m\omega^2 \left(\frac{11A}{14}\right)^2, V_7 = \frac{1}{2} m\omega^2 \left(\frac{13A}{14}\right)^2$

Simplifying in a similar way as done in the three steps and five steps, we obtain

$$E = 0.9941 \left(n + \frac{1}{2}\right) \hbar\omega$$

3.The Double-Ramp Potential

The Double-Ramp potential or also called the Gravitational Potential is given by

$$V = mg|x|$$

At turning points; $E = V$

$$E = mg|x|$$

$$x = \frac{E}{mg} \qquad (7)$$

Consider the potential to be approximated by three steps as shown below.

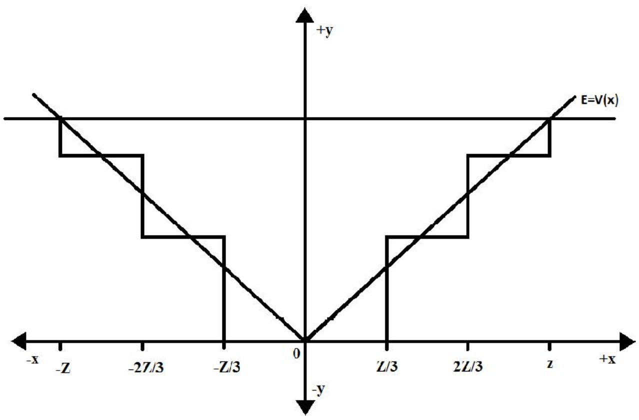

Figure 2: The Double-Ramp Potential approximated by three steps.

From equation (7), the potentials at different steps are given by

$$V_1 = mg\left|\frac{x}{6}\right|;\ V_2 = mg\left|\frac{x}{2}\right|; V_3 = mg\left|\frac{5x}{6}\right|$$

The Bohr-Sommerfeld quantization condition is written as,

$$\int_{-Z}^{-5Z/6} \sqrt{2m(E - V_3(x))}\,dx + \int_{-5Z/6}^{-2Z/3} \sqrt{2m(E - V_3(x))}\,dx + \int_{-2Z/3}^{-Z/2} \sqrt{2m(E - V_2(x))}\,dx$$
$$+ \int_{-Z/2}^{-Z/3} \sqrt{2m(E - V_2(x))}\,dx + \int_{-Z/3}^{-Z/6} \sqrt{2m(E - V_1(x))}\,dx + \int_{-Z/6}^{0} \sqrt{2m(E - V_1(x))}\,dx$$
$$+ \int_{0}^{Z/6} \sqrt{2m(E - V_1(x))}\,dx + \int_{Z/6}^{Z/3} \sqrt{2m(E - V_1(x))}\,dx + \int_{Z/3}^{Z/2} \sqrt{2m(E - V_2(x))}\,dx$$
$$+ \int_{Z/2}^{2Z/3} \sqrt{2m(E - V_2(x))}\,dx + \int_{2z/3}^{5Z/6} \sqrt{2m(E - V_3(x))}\,dx + \int_{5z/6}^{z} \sqrt{2m(E - V_3(x))}\,dx$$
$$= \left(n + \frac{1}{2}\right)\hbar\pi$$

Simplifying in the same way as in Section 2, we get:

$$E = 0.649\left(n + \tfrac{1}{2}\right)^{2/3} (\pi^2\hbar^2 mg^2)^{1/3} \qquad (8)$$

Similarly, when the potential is approximated by 5 steps, we get:

$$E = 0.652\left(n + \tfrac{1}{2}\right)^{2/3} (\pi^2\hbar^2 mg^2)^{1/3} (9)$$

And when approximated by 7 steps, we get:

$$E = 0.653\left(n + \tfrac{1}{2}\right)^{2/3} (\pi^2\hbar^2 mg^2)^{1/3} (10)$$

4.The Quartic Potential.

The potential of a quartic oscillator is given by:
$V = \lambda x^4$

At turning points: $E = V$ and let $x = A$

$$E = \lambda A^4$$
$$A = \left(\frac{E}{\lambda}\right)^{1/4} \qquad (11)$$

Consider the potential to be approximated by three steps as shown below.

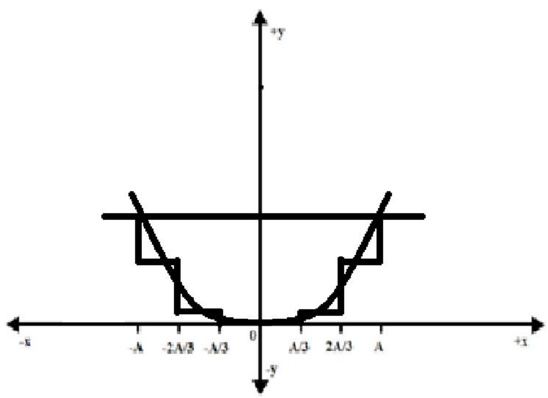

Figure 3: The Quartic Potential approximated by three steps.

From equation (11), the potentials at different steps are given by

$$V_1 = \lambda \left(\frac{A}{6}\right)^4 \; ; \; V_2 = \lambda \left(\frac{A}{2}\right)^4 \; ; \; V_3 = \lambda \left(\frac{5A}{6}\right)^4$$

The Bohr-Sommerfeld quantization condition is written as,

$$\int_{-A}^{-5A/6} \sqrt{2m(E - V_3(x))}\, dx + \int_{-5A/6}^{-2A/3} \sqrt{2m(E - V_3(x))}\, dx + \int_{-2A/3}^{-A/2} \sqrt{2m(E - V_2(x))}\, dx$$
$$+ \int_{-A/2}^{-A/3} \sqrt{2m(E - V_2(x))}\, dx + \int_{-A/3}^{-A/6} \sqrt{2m(E - V_1(x))}\, dx + \int_{-A/6}^{0} \sqrt{2m(E - V_1(x))}\, dx$$
$$+ \int_{0}^{A/6} \sqrt{2m(E - V_1(x))}\, dx + \int_{A/6}^{A/3} \sqrt{2m(E - V_1(x))}\, dx + \int_{A/3}^{A/2} \sqrt{2m(E - V_2(x))}\, dx$$
$$+ \int_{A/2}^{2A/3} \sqrt{2m(E - V_2(x))}\, dx + \int_{2A/3}^{5A/6} \sqrt{2m(E - V_3(x))}\, dx + \int_{5A/6}^{A} \sqrt{2m(E - V_3(x))}\, dx$$
$$= \left(n + \frac{1}{2}\right) \hbar \pi$$

Simplifying in the same way as in Sections 2 and 3, we get:

$$E = 0.289 \left(n + \tfrac{1}{2}\right)^{4/3} \left(\frac{\hbar^4 \pi^4 \lambda}{m^2}\right)^{1/3} \qquad (12)$$

Similarly, when the potential is approximated by 5 steps, we get:

$$E = 0.29 \left(n + \tfrac{1}{2}\right)^{4/3} \left(\frac{\hbar^4 \pi^4 \lambda}{m^2}\right)^{1/3} \qquad (13)$$

4.Results and Discussion.

At the very first glance, we see that the accuracy of the calculations increases as we increase the number of rectangular steps in each of the potentials. The best possible values of the approximate energies so obtained are summarized in the table below.

Simple harmonic potential	$E = 0.99066 \left(n + \frac{1}{2}\right) \hbar \pi$

Quartic potential	$E = 0.29\left(n+\frac{1}{2}\right)^{\frac{4}{3}}\left(\frac{\hbar^4\pi^4\lambda}{m^2}\right)^{\frac{1}{3}}$
Gravitational potential	$E = 0.653\left(n+\frac{1}{2}\right)^{\frac{2}{3}}(\pi^2\hbar^2 mg^2)^{\frac{1}{3}}$

The potentials that we have discussed in this article are fairly simple. The method can also be extended to complex potentials of physical interest such as the Morse potential in Molecular Spectroscopy or the Wood-Saxon potential in Nuclear Physics and so on. The spacing between the rectangular steps and the number of steps can be appropriately varied, depending on the nature of the potential, to achieve the desired convergence of energy eigenvalues.

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TACHYONS: FASTER THAN LIGHT PARTICLES

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Abstract

A tachyon is a hypothetical particle that always moves with a velocity greater than the velocity of light in vacuum, *c* = 3×10^8 m/s. It has several peculiar properties such as velocity greater than that of light and can be arbitrarily large even velocity v = ∞ is allowed, energy of a tachyon decreases with increasing velocity, its energy can be negative, even though it has spin zero it is a fermion. Tachyon particles are appeared theoretically in a variety of theories but there is no experimental evidence for the existence of tachyon particles yet. However, the study of its very reason of non-existence can be a useful tool to solve the mysteries of the universe.

1. Introduction

According to their velocities, particles can be classified into three categories [1–3]: (i) particles which travel at velocities smaller than the velocity of light (bradyons) are classified as *time-like* particles (ii) particles moving with the velocity of light (luxons) are classified as *light-like* particles e.g. photons and (iii) particles which travel with velocities greater than the velocity of light (tachyons) are classified as *space-like* particles. Now consider the third category of particles which travel with velocities greater than the velocity of light. If we try to ascribe a proper (rest) mass to such particles according special relativity it will be imaginary.

In 1962, Ennackal Chandy George Sudarshan (an Indian-American Physicist) and his collaborators [2] introduced the idea of faster-than-light particles, which were subsequently named tachyons by Gerald Feinberg [4] in 1967. The name comes from the Greek word $\tau\alpha\chi\omega$ (tachys) meaning "swift, quick, fast, rapid". Tachyons have several wondeful properties [5]: (a) Velocity of tachyons is always greater than the velocity of light and can be arbitrarily large, even velocity v = ∞ is allowed (b) Energy of a tachyon increases with decreasing velocity, which means that tachyons accelerate as they lose energy. (c) Energy of a free tachyon can be negative. (d) Tachyons are fermions, even though they have spin zero [4]. (e) Tachyons obey the fundamental conservation laws of energy, momentum and electric charge [6]. (f) Tachyons can interact with ordinary *i.e.* bradyonic particles. In order to obey the conservation laws, they must carry real energy and real momentum when they interact with bradyons. Special relativity (SR) is used for particles which travel at velocities smaller than the velocity of light (bradyons). But tachyons always move faster than light. Hence, extended theory of relativity is required for faster than light phenomena, particles and reference frames. Recami [7] called this new domain as extended relativity (ER). Thus, ER is the theoretical framework which describes the motion and interactions of tachyons. Such formulation does not change SR and gives similar results. A 'bradyonic observer' travels at a velocity less than *c*, while a 'tachyonic observer' travels at a velocity greater than *c*. In this article, some properties of tachyon particles are discussed. Most of the time we have made comparisons between ordinary particles (bradyons) and tachyon particles.

2. Mass, Momentum and Energy of a Tachyon Particle

A tachyon is a particle with space-like four-momenta. The possibility of particles whose four-momenta are always space-like and whose velocities are therefore always greater *c* is not in contradiction with special relativity [4]. There are two equivalent approaches to discuss the kinematics of tachyon particles:

(a) According to theory of special relativity [8], the energy of a particle moving with velocity v is given by

$$E = \frac{m c^2}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}, \tag{1}$$

where *m* is the *proper mass* or *rest mass* of the particle. We can see that, in the case of v > c, the denominator becomes imaginary. Since the energy must be a real quantity this implies that the proper mass of tachyon particle must be imaginary (since a pure imaginary number divided by another pure imaginary number is a real number). The term *proper mass* [9] is appropriate for all categories of particles including luxons whose proper mass is zero and tachyons whose proper mass is imaginary. The word *rest mass* is not suitable for a tachyon because it never comes to rest.

(b) Describing tachyons with real masses: Let us consider the proper mass $m = i m_*$ (where $i^2 = -1$ and m_* is real). With this approach the energy equation (1) becomes

$$E = \frac{m_* c^2}{\sqrt{\frac{\mathrm{v}^2}{c^2} - 1}} \tag{2}$$

Now the denominator is real and therefore the same can be said for the mass. Here, m_* is known as *meta-mass* that is the absolute value of the proper mass of a tachyon ($m = i m_*$). Both approaches are equivalent mathematically. Generally imaginary proper mass has no physical significance and Eq. (1) is valid for ordinary particles (bradyons) that is when v < c. That is why most of the physicists are using Eq. (2) as the energy equation for a tachyon particle.

From the theory of special relativity we know that the energy of a time-like particle is related to its mass and momentum through the expression [8]

$$E^2 = m^2 c^4 + c^2 p^2 \tag{3}$$

This quadratic equation has two solutions: one is the positive root of the equation and the other is the negative root (i.e. for a given mass *m*, this equation describes the surface of a two-sheet hyperboloid). In other words, we can say that Eq. (4) represents a two-sheeted hyperboloid of revolution around the *E* axis. A three-dimensional model of such an (E,p) surface is shown in Fig. 1(a) [2]. The slope of the hyperbola at a given point gives the velocity of the particle in the corresponding reference frame,

$$\mathrm{v} = \frac{dE}{dp} \qquad (4)$$

In the theory of relativity, the state of positive energy is associated with forward particle that means a forward particle always has a positive energy [1] and a backward particle has a negative energy. Particles can be classified according to their 'direction of movement' in time: A particle which moves to the future is called as a *forward particle* and a particle moving to the past is called a *backward particle*. A particle which has an infinite velocity exists only in the very present instant is called a *momentary particle*. According to the *switching principle* [1,9,10] a backward particle (whose energy is negative) should always be physically observed as a typical forward particle (whose energy is positive). This is also known as *reinterpretation principle*. The interpretation is such that negative energy tachyons propagating backward in time are equivalent to positive energy tachyons propagating forward in time.

The energy of a light-like particle (mass m = 0) is related to its momentum through the expression

$$E^2 = c^2 p^2 \qquad (5)$$

The three-dimensional (E,p) surface becomes a cone of revolution about the *E* axis, as shown in Fig. 1(b). In this case, slope of the curve at any given point gives the velocity $\mathrm{v} = dE/dp = c$. This represents that the particle (e.g. photon) moves always with the velocity c. There is no point at which the particle would be at rest. Here, a Lorentz transformation can take a point on the upper cone only into another point on the upper cone (Fig. 1(b)). Any transformation into a point on the lower cone appears to introduce a particle traveling with negative energy i.e. traveling backward in time.

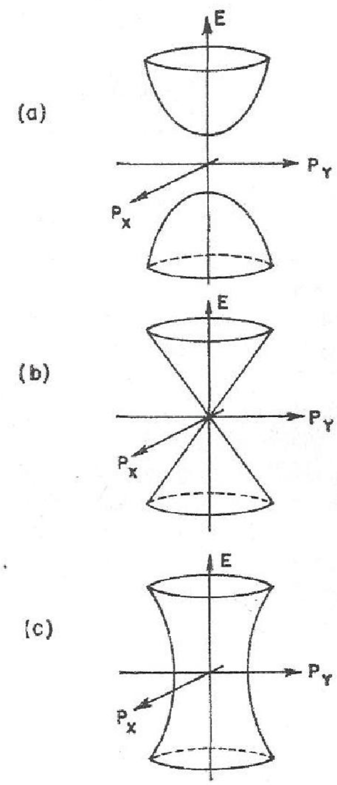

Fig. 1: Three-dimensional models of the (E,p) surfaces (a) for time-like particles, (b) for light-like particles, and (c) for space-like particles [2]. Slope of a curve at any given point gives the velocity of the particle whose energy and momentum are furnished by the coordinates of the point.

The energy of a space-like particle (tachyon) is related to its mass and momentum through the expression

$$E^2 = c^2 p^2 - m^2 c^4 \qquad (6)$$

This equation implies that the (E,p) surface is a single-sheeted hyperboloid of revolution around the E axis, as shown in Fig. 1(c) [2]. This represents that tachyons can change its status of a forward particle to the status of a backward one (and vice versa) [1]. Slope of the curve is everywhere greater than c. This represents that if particles with an imaginary proper mass do exist, their velocity would never be less than c and they can not be at rest.

The momentum of a tachyon particle is given by

$$p = \frac{m_* \mathrm{v}}{\sqrt{\frac{\mathrm{v}^2}{c^2} - 1}}, \qquad (7)$$

whereas the momentum of an ordinary particle (time-like particle) is given by

$$p = \frac{m \mathrm{v}}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} \qquad (8)$$

From Eqs. (2) and (7), it is clear that both the energy and momentum are monotonic decreasing functions of the velocity i.e. the energy of a tachyon decreases with increasing velocity. In other words, if it is required to slow down a tachyon particle, energy must be supplied to it instead of taking away from it. If the tachyon particle is to be decelerated to its lowest possible velocity, which is the velocity of light, an infinite amount of energy would have to be supplied [9].

Furthermore, the range of the energy and momentum, of a tachyon particle, are given by [4]:

$$0 < \mathrm{E} < \infty \quad , \qquad m_* \mathrm{c} < |\mathrm{p}| < \infty \qquad (9)$$

The value v = ∞ is allowed for tachyons, and at v = ∞ we have

$$\mathrm{E} = 0 \quad , \qquad |\mathrm{p}| = m_* c \qquad (10)$$

So tachyons at infinite velocity carry momentum but no energy. Infinite velocity tachyon having finite momentum and zero energy is known as a *transcendent tachyon* [9]. Another interesting property of tachyon: we know that the velocity of light is invariant from observer to observer. It does not depend upon its energy. But tachyons velocity varies from observer to observer and it depends upon its energy.

3. Rods and Clocks

The study of tachyons depends on the same postulates as in special relativity for bradyons [6]: (i) The laws of physics are the same in all inertial systems. (ii) The speed of light in free space has the same value c in all inertial systems. According to the first postulate the laws of physics that are treated as being the same in all inertial frames are mainly the conservation of energy, momentum and electric charge. Thus, we can say that tachyons conserve energy, momentum and charge in a given reference frame like bradyons. The second postulate means that electromagnetic waves travel at the same speed regardless of whether they are generated by a charged tachyon or a charged bradyon.

The tachyonic transformations have similar form to the Lorentz transformation. That is why it is expected that length contradiction and time dilation effects also apply to tachyonic rods and clocks. But it is shown that it is not exactly the same. There are some apparent differences in behavior between tachyonic and bradyonic rods and clocks which are discussed as follows:

Let us assume that there are two inertial reference frames S and S'. Frame S is bradyonic whereas frame S' is tachyonic. Let us consider a rod lying at rest along x' axis of the frame S'. The ends of the rod are at x'_1 and x'_2 so that its rest length is $x'_2 - x'_1$. Now suppose the rod is moving with velocity v > c along the x-axis relative to an observer in frame S, so that S considers the rod to be a tachyonic object. The apparent length of the tachyonic rod as viewed in bradyonic frame S is given by

$$|x_2 - x_1| = (x'_2 - x'_1)\sqrt{\frac{\mathrm{v}^2}{c^2} - 1} \qquad (11)$$

In SR, the equivalent expression for a bradyonic rod is given by

$$x_2 - x_1 = (x'_2 - x'_1)\sqrt{1 - \frac{\mathrm{v}^2}{c^2}} \qquad (12)$$

In SR, that is when v < c, we observe length contraction. But for tachyons (v > c) it is shown that (a) for $c^2 < \mathrm{v}^2 < 2c^2$ the length of the rod measured in frame S is shorter than its rest length in S', so the rod is contracted just as it is for v < c (SR). (b) For $\mathrm{v}^2 = 2c^2$ the rod appears to have the same length in both frames. (c) For $\mathrm{v}^2 > 2c^2$ the length of the rod appears to be dilated.

Now consider that there is a clock at rest in frame S' and the frame S' moves with a velocity v relative to frame S [1,6]. the time interval in S' is $t'_2 - t'_1$. The apparent time interval as measured by S is given by

$$|t_2 - t_1| = \frac{(t'_2 - t'_1)}{\left|\sqrt{\frac{\mathrm{v}^2}{c^2} - 1}\right|} \qquad (13)$$

In SR, the equivalent expression for time interval is given by

$$t_2 - t_1 = \frac{(t'_2 - t'_1)}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} \qquad (14)$$

In SR, that is when v < c, we observe time dilation i.e. the moving clock becomes slower in measuring time than an identical clock at rest.

Here, the Eq. (13) only gives the time interval between events in frame S. It does not indicate which event appears to occur first in that particular frame. (a) For $c^2 < \mathrm{v}^2 < 2c^2$ the clock appears to S to be slowed down just as it is for v < c (SR), (b) For $\mathrm{v}^2 = 2c^2$ the clock appears to run at the same rate in both frames, (c) For $\mathrm{v}^2 > 2c^2$ the clock as seen by S will appear to run fast.

Furthermore, it is also observed that a pair of tachyonic observers investigating tachyonic rods and clocks is equivalent to a pair of bradyonic observers investigating bradyonic rods and clocks [6].

4. Conclusions

A tachyon is a hypothetical particle that travels throughout its lifetime at a velocity greater than the velocity of light. That is why it is considered as a particle beyond the light barrier [9]. It has several peculiar properties such as velocity greater than that of light and can be arbitrarily large even velocity v = ∞ is allowed, energy of a tachyon decreases with increasing velocity, its energy can be negative, even though it has spin zero it is a fermion. Tachyons obey Fermi-Dirac statistics. Like other fermions, they must be created and annihilated in pairs. Ordinary particles can not cross the light-speed barrier. Similarly, tachyons can not slow down to below *c*. Because infinite amount of energy is required to cross the boundary. Tachyons lose energy as Cherenkov radiation. Special relativity (SR) is applied to the particles whose velocity is less than the velocity of light. Extended relativity (ER) can be used to describe the motion of tachyon particles [1,7]. Special relativity theory of tachyons is also known as *meta relativity* [2,9]. Such formulation does not change SR and gives similar results.

There is one fundamental question: Do tachyons exist in nature, and can they be detected? Albert Einstein [11] has said in his original paper on special relativity "…..velocities greater than that of light…have no possibility of existence." Many other physicists also think that faster than light particles can not exist. But Feinberg [4] has presented several arguments followed by reasons for the existence of faster than light particles. Experiments have also been conducted for searching them. But there is no compelling evidence for their existence [12]. Recently [13] Salem has discussed the possibility of existence of faster than light particles. According his new method simple particles (quarks and gluons) can travel faster than light. But there is no experimental proof for it. The recent experimental search for the superluminal neutrino at CERN, Switzerland, Geneva gives the negative result [14]. Hence, the existence of tachyons is one of the major challenges for both theoretical as well as experimental research. If tachyons exist we ought to find them. If they do not exist we have to find out the reason for their non-existence. So far we have found no valid reason why they could not exist. Still, much room for thought remains.

"There was a young lady named Bright

Whose speed was far faster than light

She went out one day

In a relative way

And returned home the previous night."

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Set Theoretic Approach to Resistor Networks

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Abstract

We study the properties of the sets of equivalent resistances arising in the resistor networks constructed from identical resistors. This enables us to obtain the bounds of the set of n equal resistors combined in series and parallel.

1. Introduction

The net resistance of n resistors having the values R_1, R_2, $\cdots$, R_n connected in series and parallel is given by the well-known relations

$$R_{series} = \sum_1^n R_i = R_1 + R_2 + \cdots + R_n .$$

$$R_{parallel} = \frac{1}{\sum_1^n \frac{1}{R_i}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}} .$$

The net resistance R_{series} is greater than the largest resistance among the resistances R_1, R_2, $\cdots$, R_n. The net resistance $R_{parallel}$ is less than the smallest resistance among the resistances R_1, R_2, $\cdots$, R_n. The net resistance of an arbitrary circuit (using any conceivable combination series, parallel, bridge or non-planar) must therefore lie between $R_{parallel}$ and R_{series} [1]. We shall use the symbols S and P to denote the series and parallel connections respectively. For two equal resistors R_0, there are two configurations R_0SR_0 and R_0PR_0 whose equivalent resistances are $2R_0$ and $(1/2)R_0$ respectively, giving rise to the set $\{(1/2)R_0, 2R_0\}$. We can omit R_0 without any loss of generality and write it as $\{1/2, 2\}$. For three resistors, there are 4 configurations, $(1S1)S1$, $(1S1)P1$, $(1P1)S1$ and $(1P1)P1$ giving the set $\{1/3, 2/3, 3/2, 3\}$. Continuing the exercise as in [2-3], using the series and parallel connections we obtain

$$A(1) = \{1\},$$

$$A(2) = \left\{\frac{1}{2},\ 2\right\},$$

$$A(3)=\left\{\frac{1}{3},\frac{2}{3},\frac{3}{2},\ 3\right\},$$

$$A(4)=\left\{\frac{1}{4},\frac{2}{5},\frac{3}{5},\frac{3}{4},\ 1,\frac{4}{3},\frac{5}{3},\frac{5}{2},\ 4\right\},$$

$$A(5)=\left\{\frac{1}{5},\frac{2}{7},\frac{3}{8},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{5}{8},\frac{5}{7},\frac{4}{5},\frac{5}{6},\frac{6}{7},\frac{7}{6},\frac{6}{5},\frac{5}{4},\frac{7}{5},\frac{8}{5},\frac{7}{4},\ 2,\frac{7}{3},\frac{8}{3},\frac{7}{2},\ 5\right\},$$

$$A(6)=\left\{\frac{1}{6},\frac{2}{9},\frac{3}{11},\frac{3}{10},\frac{1}{3},\frac{4}{11},\frac{5}{13},\frac{5}{12},\frac{4}{9},\frac{5}{11},\frac{6}{13},\frac{7}{13},\frac{6}{11},\frac{5}{9},\frac{7}{12},\frac{8}{13},\frac{7}{11},\frac{2}{3},\frac{7}{10},\frac{8}{11},\frac{3}{4},\frac{7}{9},\frac{4}{5},\frac{5}{6},\frac{9}{10},\frac{10}{11},1,\frac{11}{10},\frac{10}{9},\frac{6}{5},\frac{5}{4},\frac{9}{7},\frac{4}{3},\frac{11}{8},\frac{10}{7},\frac{3}{2},\frac{11}{7},\frac{13}{8},\frac{12}{7},\frac{9}{5},\frac{11}{6},\frac{13}{7},\frac{13}{6},\frac{11}{5},\frac{9}{4},\frac{12}{5},\frac{13}{5},\frac{11}{4},\ 3,\frac{10}{3},\frac{11}{3},\frac{9}{2},\ 6\right\}.$$

The sets of higher-order need not contain the elements from a set of lower-order. For example, 1/3 is in $A(3)$ but not in $A(4)$ and $A(5)$. The element 1 is not present in $A(2)$, $A(3)$ and $A(5)$. These three sets have even cardinality. We shall shortly prove that these are the only three exceptional sets, which do not have 1 as its element and have an even cardinality. The resistor networks do have other sets. For instance, if we use at most three resistors, then the set $A(3)$ is replaced by the larger set $C(3)=\{1/3,1/2,2/3,1,3/2,2,3\}$, which is a union of the three sets $A(1)$, $A(2)$ and $A(3)$. The $C(n)$ of higher order contains all the $C(n)$ of lower orders. For five or more resistors, it is possible to include the bridge connections, giving rise to the sets $B(n)$. The order of the sets $A(n)$ (denoted by $|A(n)|$) grows for $n=1,2,3,\cdots$, as 1, 2, 4, 9, 22, 53, $131,\cdots$, and this sequence is known by the unique identity A048211 in *The On-Line Encyclopedia of Integer Sequences* (OEIS), created and maintained by Neil Sloane [4]. The $|C(n)|$ grow as 1, 3, 7, 15, 35, $77,\cdots$ [A153588]. The $|B(n)|$ grows as 1, 2, 4, 9, 23, $57,\cdots$ [A174283]. The problem for $n\le 16$ has been addressed computationally in [2] and suggests that $|A(n)|\sim 2.53^n$. How does $|A(n)|$ behave for large n? Does the base 2.53 increase or decrease? In this article, we shall address these questions through the properties of the set $A(n)$ and arrive at the bounds $(0.25)(2.41)^n<|A(n)|<2.73^n$. The number of configurations [A000084] is much larger than the number of equivalent resistances [A048211] and in this article we are not concerned about it. The computer memory has restricted the numerical studies of resistor networks to $n=23$. Hence the analytical studies are of extra significance.

2. Set Theoretic Properties

We shall derive some properties of $A(n)$ and the other sets arising in the resistor networks.

The Scaling Property:

If a/b is a member of $A(m)$, then we can construct the resistances $k(a/b)$ and $(1/k)(a/b)$ using k such blocks in series and parallel respectively, using km number of unit resistors. Hence, $kA(m)\in A(km)$ and $(1/k)A(m)\in A(km)$.

A block of i equal resistors in series has an equivalent resistance i. If i such blocks are combined in parallel we get back the unit resistance. From this we conclude that $1 \in A(i^2)$. The same result can equivalently be obtained by taking i blocks in series, each containing i unit resistors in parallel. Once the unit resistor has been obtained, using i^2 resistors (or much less as we shall soon see), we can use it to construct other equivalent resistances. Every set $A(m)$ is made from m unit resistors. The same set can be replicated by using m number of unit resistors constructed with i^2 resistors. So, $A(m) \subset A(i^2 m)$. Whenever 1 belongs to some set $A(i)$, we label it as 1_i to indicate that it has been constructed from i number of basic unit resistors, R_0.

The Translation Property:
It is the statement that $1 \in A(i)$ implies $1 \in A(i+3)$. This can be seen by taking either of the following two combinations of 1_i with 3 basic unit resistors: $(1S1)P(1S1_i) = 2P2 = 1$, or $(1P1)S(1P1_i) = (1/2)S(1/2) = 1$. Note, that we have consumed $(i+3)$ resistors. So whenever, $1 \in A(i)$, it follows that $1 \in A(i+3)$. We shall use this translation property to prove the theorem, that 1 belongs to all $A(n)$ barring three exceptions.

Theorem-1:
We have $1 \in A(n)$, $n \neq 2$, $n \neq 3$, *and* $n \neq 5$.

From an exhaustive search (or otherwise) we know that 1 belongs to $A(6)$, $A(7)$ and $A(8)$. Using the translational property, 1 also belongs to $A(9)$, $A(10)$ and $A(11)$; $A(12)$, $A(13)$ and $A(14)$; and so on. Thus we conclude that 1 belongs to all $A(n)$ for $n \geq 6$. As for the lower $A(i)$, 1 belongs to $A(1)$ and $A(4)$; and 1 does not belong to $A(2)$, $A(3)$ and $A(5)$. Hence, the theorem is proved. In passing, we note, that an exhaustive search is not always required. Two resistors in parallel lead to1/2; and two such blocks in series lead to 1 and hence, $1 \in A(4)$. The set $A(3)$ contains 1/3 and 2/3; combining these two blocks in series gives 1, implying $1 \in A(6)$. Combining the 1/2 present in $A(2)$ with the 1/2 in $A(5)$ in series, we conclude that $1 \in A(7)$. Similarly 1/4 and 3/4 are present in $A(4)$ and lead to $1 \in A(8)$. From references [5-7], we know that all elements in $A(n)$ have a reciprocal pair (a/b and b/a) and 1 is its own partner; presence of element 1 implies that the order of $A(n)$ is always odd with the exception of $|A(2)| = 2$, $|A(3)| = 4$, and $|A(5)| = 22$.

Corollary-1:
We have $1/2 \in A(n)$, $n \neq 1$, $n \neq 3$, $n \neq 4$ *and* $n \neq 6$.

The parallel combination of 1 basic unit resistor, R_0 with 1_i ($i = 4$ and $i \geq 6$) results in an equivalent resistance of 1/2, (since, $1P1_i \equiv (1 \times 1_i)/(1 + 1_i) = 1/2$), which implies that $1/2 \in A(i+1)$, for $i = 4$ and $i \geq 6$. The corollary is proved for $n = 5$ and for all $n \geq 7$. Resorting to the exhaustive search, we note, that 1/2 belongs to $A(2)$; the four exceptional sets are $A(1)$, $A(3)$, $A(4)$ and $A(6)$, which do not contain the element 1/2.

We constructed 1_i from i basic unit resistors ($i = 4$ and $i \geq 6$). Any set $A(m)$ can be constructed from m number of 1_i, using mi number of resistors, consequently $A(m) \subset A(mi)$ for $i = 1$, $i = 4$ and $i \geq 6$. The above statement is silent about $i = 2$, 3 and 5. The argument, mi is multiplicative, giving no information about the near or immediate neighbours of $A(m)$. Additive statements have the arguments of the type $(m+i)$ and when they exist, they provide information about the neighbours of

$A(m)$. In the present context the additive statements are more informative and override the multiplicative statements. We examined the occurrence of 1 in $A(i)$, since 1 is the basic unit and all other resistances can be constructed from it. The element 1/2 was examined as a special case. Remaining elements (infinite in number) shall be discussed collectively using the modular property.

Theorem-2 (Modular Theorem):
We have $A(m) \subset A(m+3)$ *and* $A(m) \subset A(m+i)$ *for* $i \geq 5$.

Every set $A(m)$ is constructed from m basic unit resistors R_0. If we replace any one of these basic unit resistors with 1_i ($i = 4$ and $i \geq 6$), we will reproduce the complete set $A(m)$ using $(m+i-1)$ resistors. Consequently, $A(m) \subset A(m+i-1)$ for $i = 4$ and $i \geq 6$. Thus the modular theorem is proved stating that every set $A(m)$ is completely contained in all the subsequent and larger sets, $A(m+3)$ along with the infinite and complete sequence of sets $A(m+5)$, $A(m+6)$, $A(m+7)\cdots$, and so on. However, it is very curious to note, that the infinite range theorem is silent about the three important sets: the *nearest neighbour*, $A(m+1)$; *next-nearest neighbour*, $A(m+2)$ and the *near-neighbour* $A(m+4)$. From the modular relation, $A(m) \subset A(m+i)$ for $i \geq 5$, we conclude that $A(n-5) \subset A(n) \cap A(n+1)$ for $n \geq 6$. This is the closest we can get to know the overlap between $A(n)$ and its *nearest neighbour* $A(n+1)$. An immediate consequence of the modular theorem is on the sets $C(n)$, obtained by taking the union of $A(i)$

$$\begin{aligned} C(n) &= \bigcup_{i=1}^{n} A(i) = \bigcup_{i=n-2}^{n} A(i) \\ &= A(n-2) \cup A(n-1) \cup A(n). \end{aligned}$$

It suffices to consider only the last three sets $A(n-2)$, $A(n-1)$ and $A(n)$ in the union. Hence, it is not surprising that the ratios $|C(n)|/|A(n)|$ are close to 1.

Decomposition of $A(n)$:
The set $A(n)$ can be constructed by adding the $n-th$ resistor to the set $A(n-1)$. This addition can be done in three distinct ways and results in three basic subsets of A(n). Treating the elements of $A(n-1)$ as single blocks, the $n-th$ resistor can be added either in series or in parallel. We call these two sets as *series set* and *parallel set* and denote them by $1SA(n-1)$ and $1PA(n-1)$ respectively. The $n-th$ resistor can also be added somewhere within the $A(n-1)$ blocks, and we call this set as the *cross set* and denote it by $1 \otimes A(n)$. The set $A(n)$ is the union of the three sets formed by different ways of adding the $n-th$ resistor. The decomposition $A(n) \equiv 1PA(n-1) \cup 1SA(n-1) \cup 1 \otimes A(n-1)$, is very illustrative, and enables us to understand some of the properties of $A(n)$. All the elements of the parallel set are strictly less than 1 (since $1P(a/b) = a/(a+b) < 1$) and that of the series set are strictly greater than 1 (since $1S(a/b) = (a+b)/b > 1$). So, $1PA(n-1) \cap 1SA(n-1) = \varnothing$ and the element 1 necessarily belongs to the cross set alone.

The series and the parallel sets each have exactly $|A(n-1)|$ number of configurations and the same number of equivalent resistances. Let c/d and d/c be any reciprocal pair (ensured by the reciprocal theorem in [5-7]) in $A(n-1)$, then it is seen that $1P(c/d) = c/(c+d)$ and $1P(d/c) = d/(c+d)$ belong to the set $1PA(n-1)$; and $1S(c/d) = (c+d)/d$ and

$1S(d/c) = (c+d)/c$ belong to the set $1SA(n-1)$. Thus all the reciprocal partners of $1PA(n-1)$ always belong to $1SA(n-1)$ and vice versa. These two disjoint sets contribute $2|A(n-1)|$ number of elements to $A(n)$. The order of the cross set, $1 \otimes A(n)$ is $(|A(n+1)| - 2|A(n)|)$ and results in the sequence, 0, 0, 0, 1, 4, 9, 25, 75,$\cdots$ [A176497]. For $n \geq 7$, all the three basic sets have odd number of elements, since $A(n)$ is odd for $n \geq 6$.

The cross set is not straightforward, as it is generated by placing the $n-th$ resistor anywhere within the blocks of $A(n-1)$. It is the source of all the extra configurations, which do not always result in new equivalent resistances. For, $n > 6$, the cross set has at least $|A(n-2)|$ elements, since $A(n-1)$ has $|A(n-2)|$ connections corresponding to $1 \otimes A(n-2)$; this leads to the recurrence relation $|A(n+1)| > 2|A(n)| + |A(n-1)|$, for $n \geq 6$. Similar, arguments lead to the relation $|A(n+1)| < 2|A(n)| + 2|A(n-1)|$. From the decomposition, we note, that the element 1 can belong only to the cross set and not the other two (since all elements of $1PA(n-1)$ are less than 1 and all the elements of $1SA(n-1)$ are greater than 1). We noted that the two disjoint sets, $1PA(n-1)$ and $1SA(n-1)$ are reciprocal to each other. Consequently all elements in $1 \otimes A(n-1)$, have their reciprocal partners in $1 \otimes A(n-1)$ itself; 1 is its own partner. The cross set is expected to be dense near 1 with few of its elements below half (recall that 1/2 is contained in $1PA(n)$ for $n \geq 6$, and not a member of the cross sets). This is reflected in the fact that the cross sets up to $1 \otimes A(7)$ do not have a single element below half. The successive cross sets have, 1, 6, 9, 24, 58, 124,$\cdots$ elements respectively [A176498], a small percentage compared to the size of the cross sets, 195, 475, 1265, 3125,$\cdots$ [A176497].

It is straightforward to carry over the set theoretic relations to the bridge circuits sets; since, $A(n) \subset B(n)$. Unlike the sets $A(i)$, the sets $B(i)$ have the additional feature $1 \in B(5)$. So, the various statements must be modified accordingly. In particular, we have

$$1 \in B(n) \text{ for } n \neq 2 \text{ and } n \neq 3,$$

$$\frac{1}{2} \in B(n) \text{ for } n \neq 1,\ n \neq 3 \text{ and } n \neq 4,$$

$$B(m) \subset B(mi) \text{ for } i = 1 \text{ and } i \geq 4,$$

$$B(m) \subset B(m+i) \text{ for } i \geq 3,$$

$$B(n-3) \subset B(n) \cap B(n+1) \text{ for } n \geq 4.$$

Complementary Property:
It is the statement that every set $A(n)$ with $n \geq 3$ has some complementary pair such that their sum is equal to 1. As an example, in $A(3)$ we have the pair $(1/3, 2/3)$; in $A(4)$ we have two pairs $(1/4, 3/4)$ and $(2/5, 3/5)$; and so on. By virtue of the Corollary-1, we see that 1/2 can be treated as its own complementary partner. We shall soon conclude that each element of the set $1PA(n-1)$ has a complementary partner in $1PA(n-1)$ itself. By the reciprocal theorem, the elements c/d and d/c occur as reciprocal pairs in every $A(n-1)$ (see [2-3, 5-7] for details and proofs). So in $1PA(n-1)$ we have

$$\left(1P\frac{c}{d}\right)+\left(1P\frac{d}{c}\right)=\frac{c}{c+d}+\frac{d}{c+d}=1.$$

Thus all the elements (except the element 1/2) of $1PA(n-1)$ have a complementary partner in $1PA(n-1)$ itself. For $n \geq 7$, the number of such pairs in the set $1PA(n-1)$ is $(|A(n-1)|-1)/2$, since $|A(n-1)|$ is odd for $n \geq 7$ and $1/2 \in A(n)$ for $n \geq 7$. It is obvious that the set $1SA(n-1)$ does not have complementary pairs.

3. Bounds of $|A(n)|$:

The decomposition of $A(n)$ enabled us to obtain the relations, $|A(n+1)| > 2|A(n)| + |A(n-1)|$, for $n \geq 6$ and $|A(n+1)| < 2|A(n)| + 2|A(n-1)|$. The solution of these two relations provides the strict bounds

$$(0.25)(1+\sqrt{2})^n < |A(n)| < (1+\sqrt{3})^n.$$

The numerically obtained result $|A(n)| \sim (2.53)^n$ in [2], for $n \leq \mathbf{16}$. is consistent with the strict bounds presented here.

4. Concluding Remarks

Several set theoretic relations among the sets $A(n)$ and $B(n)$ were derived using simple arguments. The decomposition of the set $A(n)$ into three basic subsets derived from $A(n-1)$ leads to the strict lower and upper bounds analytically. The set theoretic relations point to the complexity of estimating the upper bound of the order of $A(n)$ and other sets using combinatorial arguments. The Haros-Farey sequence approach presented in [3, 5-8], is another method to estimate the upper bounds of the various sets occurring in the resistor networks. The strict upper bound $|A(n)| < 0.318(2.618)^n$, obtained using the Haros-Farey sequence is also valid for the sets $B(n)$, $C(n)$ and other sets (using any conceivable combination series, parallel, bridge or non-planar). A comprehensive account of the resistor networks along with the computer programs using the symbolic package MATHEMATICA [9] is available in [5-8]. The program to generate the order of the set $A(n)$, using the symbolic package MATHEMATICA is presented in the Appendix.

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generate additional terms. The sequences cited in this article are: Sequence A00084, Sequence A048211, Sequence A153588, Sequence A174283, Sequence A176497 and Sequence A176498. Additional sequences at http://oeis.org/wiki/User:Sameen_Ahmed_Khan

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Appendix

Computer Program in MATHEMATICA

The problem of resistor networks is intrinsically a computational problem. The following programs have been written using the symbolic package MATHEMATICA [9]. They just need to be run on a faster computer. The results obtained can be shared at *The On-Line Encyclopedia of Integer Sequences* (OEIS) [4].

```
(* n Equal Resistors connected in Series and/or Parallel*)
NumberResistors = 4;
ClearAll[CirclePlus, CircleTimes];
SetAttributes[{CirclePlus, CircleTimes}, {Flat,Orderless}];
SeriesCircuit[a_, b_]:= a⊕b;
ParallelCircuit[a_, b_]:= a⊗b;
F[a_,b_]:= Flatten[Outer[SeriesCircuit, a, b]⋃Outer[ParallelCircuit, a, b], 2];
S={{R}, {R⊕R, R⊗R}};
Do[SX = F[S[[1]], S[[i-1]]];
    Do[SX = Flatten[SX⋃F[S[[k]], S[[i-k]]], 2]; , {k, 2, i/2}];
    S = S⋃{SX}; , {i, 3, NumberResistors}];
S[[NumberResistors]] (*This line displays the Full Set of Configurations*)
Print[StringForm["NumberResistors = ``, NumberConfigurations = ``", NumberResistors,
Dimensions[S[[NumberResistors]]]]]
SetAttributes[{CirclePlus, CircleTimes},{NumericFunction, OneIdentity}];
a_⊕b_:= a + b;
a_⊗b_:= a*b/(a+b);
CirclePlus[x_]:= x;
CircleTimes[x_]:= x;
(*Print[S[[NumberResistors]]/.R→1]*) (*This line displays the Set of Equivalent Resistances
corresponding to the Set of Configurations*)
(*Print[Union[S[[NumberResistors]]/.R→1]]*) (*This line displays the Full Set of Equivalent
Resistances*)
Print[StringForm["NumberResistors = ``, NumberConfigurations = ``,  NumberEquivalentResistances =
``, CPU time in seconds = ``", NumberResistors, Dimensions[S[[NumberResistors]]],
Dimensions[Union[S[[NumberResistors]]]], TimeUsed[%]]]
{R⊕R⊗(R⊕R), R⊕R⊗R⊗R, R⊗R⊕R⊗R, R⊕R⊕R⊗R, R⊕R⊕R⊕R,
 R⊗(R⊕R⊗R), R⊗(R⊕R⊕R), (R⊕R)⊗(R⊕R), R⊗R⊗(R⊕R), R⊗R⊗R⊗R}
NumberResistors = 4, NumberConfigurations = {10},
NumberEquivalentResistances = {9}, CPU time in seconds = 0.031`
```

This Program in MATHEMATICA is designed to compute the "Set of Configurations" [A00084] and the "Set of Equivalent Resistances" [A048211] of n Equal Resistors connected in Series and/or Parallel. The output is shown for $n = 4$.

Verification of laws of reflection and refraction from quantum model of light

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Abstract

Light exhibits dual characteristics – wave and particle. Although the laws of reflection and refraction were verified in corpuscular theory, wave theory and electromagnetic theory of light no explanation of it was given in old quantum model of light. Here a simplified version of verification has been dealt with using conservation of linear momentum and continuity of tangential component of electric and magnetic fields across the interface between two media.

1. History

Light is something that allows us to see the objects around us [1-4]. About 300 BC Euclid of Alexandria in his book *Optica* noted that light travels in straight lines and described the law of reflection. Aryabhatta, an Indian philosopher, reiterated that it was light arriving from an external source that illuminated the world around us. Ibn al-Haytham (965 – 1039) in his *Book of Optics* gave a lucid description of the optical system of the eye which led to the belief that light consists of rays which originate in the object seen and not in the eye, a view contrary to that of the ancient Greek and Indian philosophers. Willebrord Snell Van Royen (1580 – 1626) discovered the law of refraction in 1621 that is now popularly known as the Snell's Law.

2. The laws

In brief the laws of reflection and refraction are as follows:

(i) The first law of reflection and refraction state that the incident ray, the reflected ray, the refracted ray and the normal to the point of incidence at the interface between two different optical media with the refractive indices (n_1) *and* (n_2) *respectively, lie in one plane for any wavelength of incident light.*

(ii) The second law of reflection states that the angle of incidence (θ) *is equal to the angle of reflection* (θ') *for any wavelength of incident light.*

(iii) Finally the second law of refraction or Snell's law states that for a particular wavelength of incident light the sine of angle of incidence (θ) *bears a constant ratio to the sine of angle of refraction* (θ_m). Thus there are different constants for different colours and mathematically,

$$\sin\theta / \sin\theta_m = n_2 / n_1 \qquad - (1)$$

Clearly this is the relative refractive index of second medium with respect to the first and for free space as first medium it is absolute refractive index n_m. Of course the refractive index of free space is unity.

3. The corpuscular theory

However, true nature of light was not clear. It is invisible, but it influences growth of plants by transfer of energy. But, how this energy transfer takes place between source and object was not clear. The interesting point is that René Descartes (1596 – 1650) first gave corpuscular model of light in 1637 and Sir Isaac Newton (1643 – 1727) presented this model of light in 1675 in a Royal Society paper. Later on Newton popularized this concept in the book 'opticks' and thus this theory is very often, attributed to him. According to this theory, light is a stream of minute, invisible material particles called corpuscles which shot out from a luminous source with a very high speed in a straight line much like the shots fired from a gun. Corpuscles have no mass and can spread in all directions. Different colours are due to different sizes of corpuscles. Reflection occurs due to repulsion between the reflecting surface and the corpuscles. Refraction is due to attraction between the corpuscles and the refracting surface. For refraction trajectory of the corpuscle is determined by the conservation of the horizontal component of momentum along the interface. If the angles of incidence and refraction are θ and θ_m with the momentum of the corpuscle in the two media be p and p_m respectively, then $p\sin\theta = p_m \sin\theta_m \Rightarrow \sin\theta/\sin\theta_m = p_m/p = c_m/c$. This gives the incorrect explanation of Snell's law and predicts higher speed if the ray moves towards normal and inconsistence with the experimental results that speed of light in free space or vacuum is maximum [1-4].

4. The wave theory

In 1678 by geometrical construction, Christian Huygens (1629 – 1695) proposed the wave theory of light in a communication to the Academie des Science in Paris and in particular demonstrated how waves might interfere to form a wave front in a straight line. This theory showed that speed of light in denser medium is less than that measured in free space which agrees well with the experimental value of speed of light or $\sin\theta/\sin\theta_m = c/c_m$. It also explained satisfactorily the laws of reflection and refraction. However the main drawback of this theory was that light was assumed as a mechanical wave and it was not clear how light came from the sun through the space. Huygens assumed that light is transmitted through all pervading ether medium that is made up of small elastic particles, which was quite unsatisfactory. So the wave nature was not really accepted until the interference experiments by Thomas Young (1773 – 1829) and Augustin-Jean Fresnel (1788 – 1827). The final death blow to the corpuscular theory and the universal acceptance of the wave theory was achieved by Foucault in 1850, who determined the speed of light in different media and showed conclusively that its speed is definitely greater in a rarer medium - a fact supported by the wave theory and contrary to the corpuscular theory [1-4].

5. The electromagnetic theory

In the year 1873 James Clark Maxwell (1831 – 1879) developed a completely different nature of light while working with the variation of electric and magnetic field intensities in space due to oscillatory electric currents. The nature of transfer of energy was thought of as mechanical one at that time. But Maxwell developed wave theory further from his theoretical research and put forward the hypothesis that light was not a mechanical wave but electromagnetic wave and thus transfers only electromagnetic energy. He theoretically deduced that electric and magnetic field are in phase and are at right angles to each other and also to the direction of propagation. The electric vector plays the role of light vector. He called it electromagnetic field wave and deduced the speed of this wave, which is numerically equal to the speed of light. In 1845 Faraday first obtained the definite indication of electromagnetic nature of light when plane of polarization of plane polarized light was rotated in presence of strong magnetic field parallel to beam of light. Hertz remarkably confirmed experimentally Maxwell's predictions in the year 1888. He produced and detected electromagnetic wave and at the same time he demonstrated that these waves could be reflected, refracted, focused, polarized, etc. similar to light wave. Also, the necessity of all pervading medium called ether is not required here for the propagation of this wave and Maxwell's electromagnetic wave can move through free space or vacuum with speed governed by the relation $c = 1/\sqrt{\varepsilon_0 \mu_0}$. Here ε_0 is the permittivity and μ_0 the permeability of free space and it correctly describes the speed of light. So Huygens' and Fresnel's ideas did not meet any opposition in electromagnetic theory of light but only the mechanical wave was replaced by electromagnetic wave and interpretation was done in terms of electrodynamics in place of mechanics. Thus absolute refractive index of a medium will be [1-6] $n_m = c / c_m = \sqrt{\varepsilon_m \mu_m / \varepsilon_0 \mu_0} = \sqrt{\varepsilon_r \mu_r}$.

6. The quantum theory

In 1887, Heinrich Rudolf Hertz (1857 – 1894) discovered the photoelectric effect which did not agree with the established wave theory of light. In 1905 Albert Einstein (1879 – 1955) interpreted the photoelectric effect by putting forward the famous photon theory according to which light consisted of quanta of energy $E = h\nu$. Here ν is the frequency of incident light and h is the Planck's constant. Einstein also showed that the photons in addition to having an energy $h\nu$ should have momentum in free space given by $p = h\nu / c$. This was verified experimentally in 1923 by Arthur Holly Compton (1892 – 1962). Thus dual nature of light was established. One is the particle nature supported by the photoelectric effect and the other is the wave nature supported by interference, diffraction and polarization experiments [7-8].

7. The necessity

Although the refractive index of a medium was not explained properly in corpuscular theory using particle nature of light still the laws of reflection and refraction are simply verified in corpuscular theory. They are also verified in wave theory and also in electromagnetic theory of light. But in quantum theory of light this is not explained properly [9]. Here a simplified and elementary explanation of refractive index of a medium has been given from analogy with other classical concepts and ideas from quantum theory of light.

8. Analogy with other physical phenomena

(i) de Broglie hypothesis and Compton effect
Nature manifests itself in the form of matter and energy and so by symmetry matter possess the same dual nature of particle and wave just like energy. This was de Broglie's hypothesis. Compton effect used the momentum and energy conservation for photon and electron collision [7-8].

(ii) Classical mechanics and optics
The fundamental laws governing mechanics and optics are also very similar. The principle of least action in mechanics (a moving particle always chooses a path of minimum action) is similar to Fermat's principle in optics (light ray always chooses the least path) [1-4,10].

(iii) Conservation of linear momentum in classical mechanics
It can be stated as the time integral of force or impulse is equal to the integral of momentum. Mathematically we can write it as $\vec{p} = \int d\vec{p} = \int \vec{F} dt$. *Linear momentum conservation theorem* can be verified from this theorem. If there is no applied force then $\vec{F} = 0$ and from it we have $\vec{p} = \int d\vec{p} =$ conserved.

(iv) Continuity of electric and magnetic field across the interface separating two different media

A charge free and current free region of space is considered with $\oint \vec{E} \cdot d\vec{l} = 0$. Thus, the tangential component of the electric field is continuous across the boundary $\vec{E}_1 \cdot \hat{n} = \vec{E}_2 \cdot \hat{n} \Rightarrow E_{1t} = E_{2t}$, where $\hat{n}$ is a unit vector along the tangent to the surface. Thus, the tangential component of the electric field is continuous across the boundary. Next, magnetic field $(\vec{H})$ of electromagnetic wave is taken and Ampere's Circuital Law is applied, $H_{1t} = H_{2t}$. Thus, in the absence of any surface current, the tangential components of magnetic intensity are continuous across the interface. Hence, the tangential components of these fields are continuous across the interface or the boundary of the two medium [7-8].

(v) Pressure exerted by radiation.

An electromagnetic radiation exerts a small but finite pressure on the incident surface. It was first observed by Kepler, when the tail of comets continuously veers round so as to be always opposite to the sun. Maxwell's electromagnetic theory gives a theoretical proof while quantum theory of light relates it to the momentum.

(vi) Momentum of electromagnetic wave

The momentum of electromagnetic wave in free space can be easily identified using Poynting's theorem because of known values of electric field $\vec{E}$, electric displacement vector $\vec{D} = \varepsilon_0 \vec{E}$, magnetic field $\vec{H}$ and magnetic induction vector $\vec{B} = \mu_0 \vec{H}$. But within a medium, we have to make a choice between $(\vec{E}, \vec{D})$ and $(\vec{H}, \vec{B})$ because there are two entirely reasonable and rival forms for the momentum of light in a medium [11-13]. These are the Minkowski (1908) momentum $\vec{p}_{Minkowski} = \vec{D} \times \vec{B}$ and the Abraham (1909, 1910) momentum $\vec{p}_{Abraham} = \frac{\vec{E} \times \vec{H}}{c^2}$. It is 100 years since Minkowski $(p_{Minkowski} = \hbar \omega n_m / c)$ and Abraham $(p_{Abraham} = \hbar \omega / c n_m)$ first gave rival expressions for the momentum of light in a material medium. At the single-photon level, these correspond, respectively, either to multiplying or dividing the free-space value $\hbar \vec{k}$ by the refractive index n_m. The conclusion is that both the Abraham and Minkowski forms of the momentum are correct, with the former being the kinetic momentum and the latter the canonical momentum.

9. Verification of laws of reflection

A photon of frequency ν and wavelength λ falls on an interface MN separating two media as shown in figure 1. We know that light exhibits dual nature – wave like (i.e., as an electromagnetic wave) and particle like (i.e., as photon). Initially it has momentum $h\nu / c$ along the direction AO according to De Broglie hypothesis. Its tangential component will be $(h\nu / c)\sin\theta$ before reflection and $(h\nu' / c)\sin\theta'$ after reflection, where ν' is the frequency of photon after reflection and c is the speed of photon in free space. Since frequency remains unaltered on reflection, so $\nu = \nu'$. Applying linear momentum conservation principle along the tangent to the surface or along MN, we get

$$\frac{h\nu}{c}\sin\theta = \frac{h\nu'}{c}\sin\theta'$$

Thus $$\theta = \theta' \qquad -(2)$$

We arrive at the conclusion that angle of incidence is equal to the angle of reflection. Now, the path followed by incident photon and reflected photon and the normal at the point of incidence lie on the plane of the paper. Thus, the laws of reflections are verified using quantum model of light.

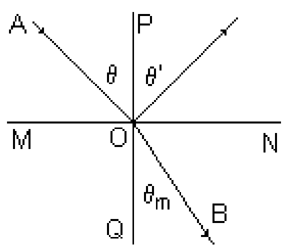

Figure 1 Verification of laws of reflection and refraction

10. Verification of laws of refraction

A photon of frequency ν and wavelength λ falls on an interface MN separating two media as shown in figure 1 of refractive indices, n and . We know that light exhibits dual nature – wave like (i.e., as an electromagnetic wave) and particle like (i.e., as photon). Initially the photon has momentum $h\nu/c$ according to De Broglie hypothesis. The photon will bend towards the normal as $n < n_m$ and away as $n > n_m$. We apply linear momentum conservation principle along the tangent to the interface MN as

$$\frac{h\nu}{c}\sin\theta = \frac{h\nu'}{c_m}\sin\theta_m$$

Thus $$\frac{\sin\theta}{\sin\theta_m} = \frac{c\nu'}{c_m\nu} = \frac{c}{c_m} = \frac{n}{n_m} \qquad - (3)$$

It is because the colour of light does not change on refraction so $\nu = \nu'$. Here $n = 1$ for free space and $n_m = c/c_m$ is the refractive index of the second medium. Thus Snell's law is verified using quantum model of light. Now, the path followed by incident photon and refracted photon and the normal at the point of incidence lie on the plane of the paper. Thus, the laws of refraction are also proved.

11. Conclusions

Thus as like the corpuscular theory, the wave theory and the electromagnetic theory of light the idea of photon momentum in quantum theory of light can be used to verify the laws of reflection and refraction although there is a debate for photon momentum for over hundred years. The photon momentum increases in a medium in comparison to that in free space as like refractive index. While going through the undergraduate syllabus I found it missing and a link is demonstrated here although elementary yet uses the gray area between the classical and quantum ideas. Emphasis is given so that undergraduate students can think of new ideas and nurture with problems.

Acknowledgements

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Conference Report: Breaking the Glass Ceiling

Deepti Sidhaye

In the past 112 years, 195 individuals have received a Nobel Prize in Physics out of which only 2 are women. They too got it in 1903 and 1963 respectively [1]. This means not a single woman physicist became a Nobel Laureate in the past 50 years. These numbers speak a lot about the current situation of women in physics. Though Nobel prizes are not the only and the best criteria to judge the situation, unfortunately the general scenario shows similar trends. In this age of equality, when women are excelling in almost every field, the number of women taking up Physics professionally is alarmingly low. Many of those who do take up Physics are faced with many difficulties and hence have to give up their career eventually. Of course there are some women who are not deterred by any adversities and successfully overcome them to have an illustrious and fulfilling career as a physicist. In order to boost the number of such women physicists, certain steps need to be taken. It is advisable to have a forum where women 'sailing in the same boat' can meet and interact with each other. Accomplished women physicists can serve as role models in the forum and can guide the new entrants by sharing their experiences. I was fortunate to attend one such forum in form of 'Career Development Workshop for Women in Physics'. It was conducted in Abdus Salam International Center for Theoretical Physics (ICTP), Trieste, Italy in September 2013. S. Narasimhan (India), E. Simmons (USA) and E. Coppola (Italy) were the organizers.

The workshop began with an eye-opener session by Dr. Rachel Ivie from American Institute of Physics, USA. She presented results from Global Survey of Physicists [2]. This multinational study (around 15000 respondents across 130 countries) broadly covered four aspects (i) Education (ii) Work and Family (iii) Opportunities and (iv) Resources. As per the survey, women physicists in developing countries lagged behind their male counterparts in their career growth. The chief reason being household responsibilities and having children. One may argue that this may be true only in less developed countries but contrary to the expectations, the situation was similar even in developed countries. The findings of the survey underlined the need of honing professional skills in order to make optimum use of available resources and time. For this, the workshop offered sessions providing professional guidance to the participants.

Writing grant proposals, setting up a

research lab, publishing research in peer-reviewed journals of high repute, making presentations in lectures, conferences and meetings are some of the key elements of academic career. Sessions devoted to improving each of these skills were included in the workshop. Dr. Assunta Vigliante, Head of Business Development for Semiconductor Solutions at Bruker AXS, talked about funding opportunities and writing grant proposals. Sara Laschever, a cultural critic and author of books like ‘Women don’t ask’ demonstrated the importance of ‘negotiation’ for resources, a skill that may be undermined by many. Mr. Samindranath Mitra, Editor of Physical review Letters, explained the nitty-gritties of the publishing world. Dr. Alison Hatt from Lawrence Berkeley National laboratory and Sinead Griffin from ETH, Switzerland conducted a session about presentation skills. Prof. Elizabeth Simmons from Michigan State University provided Hands-on-Training about preparing an effective CV.

Another important aspect of the workshop was of sharing of experiences. There were proficient women physicists like Prof. Young-Kee Kim and Prof. Setsuko Tajima who spoke about their work and the journey of their lives. From being a small village girl to being a Deputy Director of Fermi lab, particle Physicist Prof. Kim has come a long way. In her talk, she laid emphasis on significance of developing a strong sense of community, power of collaboration and role of a mentor. Prof. Tajima’s career was interrupted several times due to family problems and other external factors. She accepted these forced breaks as destiny and resumed her career with fresh enthusiasm every time, doing her best in the given boundary conditions.

The workshop also comprised Panel discussions on a variety of topics relevant to women physicists. These included (i) Aptly named 'The Two Body Problem': where scientist couples shared their experiences about balancing their professional and personal life. (ii) How to Combine Career and Family: The most crucial aspect dealing with time management while juggling to attain career-family balance, handling commitments and fulfilling as many responsibilities as possible while shuttling between workplace and home. (iii) ‘Women in Physics in Africa’: This gave us a glimpse of the challenges and hardships faced by African women physicists and the solutions they sought.

A highlight of the workshop was an interactive session conceptualized and monitored by Anuradha Narasipur Srinivasan. It was called ‘tapestry of cultures’ wherein the participants were divided into dynamic groups (they kept changing over time) and were given several topics to discuss. The topics were in tune with theme of the workshop. It was indeed wonderful to interact with participants from all across the globe. The bottom-line of the discussion was that

'Irrespective of geographical, cultural and development-wise differences, women physicists have similar experiences in all the countries. They face hurdles of similar kind and show the same grit to overcome them.' Apart from this session, each of the other sessions also had room for some discussions and deliberations. In addition all the participants were given opportunity to showcase their research work through posters and hence consider possible collaborations.

I have attended different workshops and conferences but all of them dealt with a core subject like Physics or Nanotechnology. This workshop was unique as it dealt with the social and cultural aspect of being a woman Physicist. The content of the workshop was highly relevant for day-to-day life. The workshop was a distinct opportunity to meet and interact with women physicists from different continents. It was one of the most enriching and memorable experiences in my life and I would definitely want to thank the organizing team and host institute ICTP for it. I feel it is necessary to conduct such workshops frequently. It is not just about women emancipation. It is about empathizing, encouraging, sharing, caring and progressing. If some permanent support groups emerge out from these workshops, it would be a valuable benefit. The most celebrated women physicist, two times Nobel laureate Marie Curie once said, "Life is not easy for any of us. But what of that? We must have perseverance and above all confidence in ourselves. We must believe that we are gifted for something and that this thing must be attained [3]." It is time we have many more Marie Curies…

1. http://www.nobelprize.org/nobel_prizes/ lists/ women.html.
2. Physics Today 65(2), 47 (2012); doi: 10.1063/PT.3.1439.
3. http://www.mariecurie.co.uk/marie-curie-quotes.htm

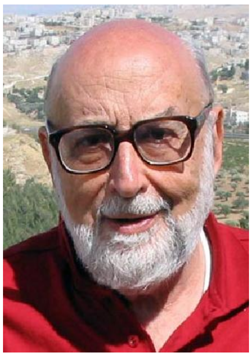
News: Nobel Prize in Physics 2013

honours

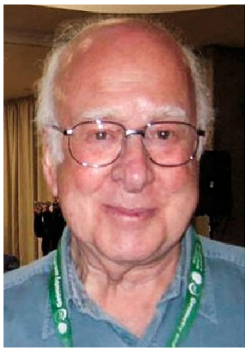
FrançoisEnglert and Peter Higgs

Abstract

Nobel Prize for 2013 celebrates the culmination of the “Standard Model”of Particle Physics and goes to **François Englert**, UniversitéLibre de Bruxelles, Brussels, Belgium and **Peter Higgs**, University of Edinburgh, Edinburgh, United Kingdom



François Englert



Peter W. Higgs

This year's Nobel Prize in Physics celebrates the completion of the theory of basic building blocks of all matter and the forces of interaction among them (apart from gravity) that goes by the prosaic name **the Standard Model.**

This theory has earned Nobel award on at least five earlier occasions, starting from 1979 forSheldon Glashow, Abdus Salam and Steven Weinberg, when the Electro-weak part of the theory could be confirmed by the experiments at CERN whichsaw clear signals of weak neutral currents. In 1984, Carlo Rubbia and Simon van der Meer were honoured, when W and Z bosons could be seen in $P\bar{P}$ collisions in a remarkable Sp$\bar{p}$S Collider experimentagain at CERN. The proof of renormalization of such theories gave prizes toGerard ‘tHooft and M Veltman in 1999 and the nature of asymptotic freedom inherent in the QCD component of the theory earned recognitions for David G Gross, H David Politzer and Frank Wilczek in 2004. The crucial ingredient of spontaneous symmetry breaking in these theories was recognized by YoichiroNambu, who received the prizein 2008, sharing with M Kobayashi and T Makagawa, who saw the need for three generations of quark - lepton families.

This year the story continues andthe Prize goes to the two (among 6 or perhaps 9 distinguished Physicists) whoplayed a role in unravelling a key ingredient of the theory;a precise mechanism of the spontaneous symmetry breaking, popularly referred to as Higgs Mechanism. The London – Anderson – Brout – Englert – Higgs – Hagen – Guralnik - Kibble - Weinberg mechanism invokes Spontaneous

Symmetry Breaking by supplementing to known fields of particle ingredients an all-pervading Scalar Field, named as Higgs field. The interaction of this field with the vector bosons that are force carriers ensures that they differentiate between weak and electromagnetic interactions. While photon that mediates long range electromagnetic interactions remains massless, the weak bosons $W^{\pm}$ and Z gain mass through such mechanism, thereby making them both short ranged and weaker forces compared to the *em*interaction. Higgs field is also responsible for all elementary constituents of matter (which to begin with are necessarily massless in *gauge theories*) derive their mass content. The ultimate tell-tale signature of the mechanism is a prediction of a Scalar Higgs meson. Two dedicated experiments at CERN saw signals for the presence of it at mass value 126 GeV thus completing the last piece of what may be termed as the modern set of a-toms (meaning indivisible entities) with which all matter is made up. Six flavours of quarks and leptons as basic fermion set together with $W^{\pm}$, Z and γ as force mediating vector bosons and the scalar neutral boson H, the Higgs meson make up the **Standard Model.**

R Ramachandran
Physics Education

Announcement

A series on Computational Physics

Computational physics has emerged as a new discipline. Large scale Numerical simulations are one of the most powerful tools do deal with complex systems, perhaps in many cases the only way to handle real life situations. We wish to bring to readers a series of articles dealing with simple yet useful algorithms and numerical techniques. The articles will be written by experts in the field with easy to understand flowcharts/ algorithms. It is hoped that the students/ teachers will be able to code the methods and use the developed codes for enhancing their understanding. Some of the topics planned are : Numerical solution of Schrödinger's equation - Bound states and scattering states, classical molecular dynamics, Monte Carlo methods, two state problems, Model Hamiltonians via exact diagonalization, Lanczoz method etc. The list is just representative and completely open. We invite experts to submit articles.

The series will be a continuation of the sequence named as "Physics through Computation" in Volume 22 –24 of Physics Education and will now be edited by Dr D G Kanhere of Pune, who joins our Team of Editors..

Pramod S Joag
Chief Editor