Visualizing Electromagnetic Fields Using Gnuplot
Part 2
Field from Accelerating Charged Particle

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Abstract

This article is a continuation of our efforts to demonstrate how to plot the E field from a
time varying source, using Gnuplot. In the present case the source has been taken to be a
charged particle in accelerated motion, moving with relativistic speeds. Three specific
examples of acceleration have been taken, namely, (a) rectilinear acceleration, (b)
rectilinear deceleration, (b) centripetal acceleration. The formulas used for the plots have
been taken from a 1972 paper by Roger Y. Tsien. However, these formulas have been
re-derived in details to make the article reader friendly and pedagogical. We have
highlighted the lessons that the reader can derive from these plots: (1) Clarification of
concepts related to retarded time, and propagation of the E field from the source at the
retarded time t_r to the observer at the present time t. (2) Confirmation of the “Purcell
condition” \tan \phi_\beta = \gamma \tan \phi_0 in which \phi_\beta and \phi_0 are the angles that a given field line makes
with the direction of motion, after and before a period of acceleration. (3) Insight into the
phenomenon of synchrotron radiation, which is also an important component of the pulsed
radiations from pulsars and the Crab nebula. The commands used for writing the relevant
equations, and plotting them, have been copied from the Console into an Appendix, so
that an interested reader can replicate all the plots on his personal computer.
1 Introduction

This article is a follow-up of our earlier article published in this journal[1], in which we had shown application of Gnuplot for plotting Electromagnetic Fields originating from an oscillating Electric Dipole. In this article we shall concentrate on another important source of Electromagnetic fields, namely, a charged particle in accelerated motion. We shall take only three examples of this acceleration: (a) Rectilinear acceleration, (b) Rectilinear deceleration (c) Centripetal acceleration. In our examples, the particle under our consideration is moving with relativistic velocity \(c\beta\), where \(c\) is the speed of light.

Formulas of the Electric Field \(E\) and the magnetic field \(B\), originating from moving charges are familiar[2, 3]. However, plots of these fields are not so common. Purcell[4], in presenting a beginner’s course in Electricity and Magnetism, has drawn \(E\) field lines from a charged particle, originally at rest, and then picking up a relativistic velocity under an extremely large acceleration existing for a very short time. Roger Y. Tsien[5], in his 1972 paper has shown actual plots of field lines for a several interesting cases. These plots were drawn using “an IBM 360/65 computer programmed in Fortran IV and a ... drum plotter”. Such heavy tools are things of the past. Now any interested student can make all these plots on his desk using his PC and free software, like Gnuplot, and get wonderful, almost unbelievable results, in all colours.

Our objective in this article is two-fold. First, we want to demonstrate the power of Gnuplot in plotting such difficult fields with so much ease. The plot commands, copied from the Console to Appendix B, can encourage the reader to get more practice which he can use profitably in his study of Classical Electrodynamics. Secondly, we would like to illuminate some difficult concepts surrounding solution Maxwell’s equation, particularly retarded time \(t_r\), propagation of the field from the source at the retarded time \(t_r\) to the field point at the present time \(t\), covering the charge-to-field distance \(R\), abbreviated as CtF distance, with the speed of light \(c\), and illustrate how, with the plotting of the examples cited by us, he clarifies, and sharpens his understanding of these concepts.

Gnuplot can plot mathematical functions, even the most difficult ones, with ease, if we write the function clearly in the command line. There are two excellent guide books[6, 7] which the interested reader should keep as his constant companion for quick reference.

In our case the function is a parametric function of the form \(x = f(t), y = g(t)\) representing the \(E\) field line on the \(XY\) plane. The parameter \(t\) in our case is the CtF distance \(R\) having range \([0 : R_{max}]\). In the final example we have switched to retarded time \(t'\) as the parameter which is related to \(R\) as \(t' = -R/c\), and having the inverse range \([-R_{max}/c : 0]\).

We have written a sample of commands from the actual command line, in Appendix B, so that the reader can replicate all the plots presented in this article.

The mathematics of plotting is somewhat difficult. It requires not only a crisp understanding of the \((E, B)\) fields, their relation to the retarded time, but also a clear un-
derstanding of transformation of coordinates leading to the formulation of the differential equation for the field line, and its solution. To facilitate the reading of this article without too much abstraction, we have shifted most of this mathematics to Appendix A.

Sec. 2 begins with determination of the \((E, B)\) field from a point charge \(q\) in arbitrary acceleration. Our task in this article is to plot only the \(E\) field, disregarding the \(B\) field completely. The reason: (1) Whereas the motion of the charged particle \(q\) and the \(E\) field are confined to the same plane, taken as the \(XY\) plane, the \(B\) field is perpendicular to this plane. Hence we cannot show the geometry of the motion and the field arising from it on the same diagram. (2) The \(E\) field and the \(B\) field are intimately connected by the equation \(cB = n \times E\), where \(n\) is a unit vector in the direction of the CtF vector, lying on the \(XY\) plane. Hence \(E\) carries all the information about \(B\).

The next two sections, i.e., Secs. 3, 4 have been devoted to obtaining the differential equation of the field line and the boundary condition. These equations have been given by Tsien. However, we have spent extra efforts to re-derive these results in detailed steps which may be easier to follow.

Sec. 5 presents the case of the charge \(q\) moving under a constant force \(F\), acting in the positive \(x\)-direction. We have considered four examples:

1. \textit{Accelerated} motion on the positive \(x\)-axis, \(q\) moving \textit{away} from the origin towards \(+\infty\):
   - Ex.1(a) \(F\) continues for ever;
   - Ex.1(b) \(F\) starts and stops

2. \textit{Decelerated} motion on the positive \(x\)-axis, \(q\) moving \textit{towards} the origin from \(+\infty\):
   - Ex.2(a) \(F\) continues for ever;
   - Ex.2(b) \(F\) starts and stops

By measuring the angles made by the field lines in Ex.1(b) the reader can confirm the Purcell condition.

Each set of plottings teaches some lessons about how the field travels from the charge to the observer covering the CtF distance. These have been summarized by us.

Finally we arrive at the conclusions: (a) the field lines for the deceleration case are very similar to those for the acceleration case; (b) they curve down from straight lines, as if acted on by induced gravity, opposite to the direction of acceleration.

Sec. 6 presents the field from a charged particle moving in a circle with relativistic and ultra-relativistic speeds, and offers some useful insight into \textit{Synchrotron Radiation}, which is an important component of the pulsed radiations coming from pulsars and the Crab nebula.

### 2 Motion with Arbitrary Acceleration

#### 2.1 General Formula

We shall write the \((E, B)\) field of a charged particle in arbitrary motion as a first step towards making their plots. In Fig. 1(a) we have explained the configuration, with reference to the coordinate system \(XYZ\) of which \(O\) is the origin.

\(\Gamma\) is the trajectory of the particle, given by the parametric representation \(\vec{r} = \vec{r}(t')\), im-
plying that each point B on Γ has a radius vector \( \mathbf{\tilde{r}} = \xi(t') \mathbf{i} + \zeta(t') \mathbf{j} + \chi(t') \mathbf{k} \) in which \( \xi, \zeta, \chi \) are known functions of time \( t' \). We shall soon identify \( t' \) with \textit{retarded time}. A is the location of the particle at the \textit{present} time \( t \) (observer’s time), and P is the location of the observer (where the \((\mathbf{E}, \mathbf{B})\) field is measured).

The \((\mathbf{E}, \mathbf{B})\) field originates from the retarded location B of the particle at the retarded time \( t' \), reaching P at the present time \( t \), travelling with the speed \( c \), covering a distance \( R(t') \), which is the length of the radius vector \( \mathbf{R}(t') \) from B to P. We shall call the vector \( \mathbf{R} \) “\textit{charge to field vector}”, in brief “CtF” vector. Therefore,

\[
R(t') = c(t - t');
\]

\[
\Rightarrow \quad t' = t - \frac{R(t')}{c} = t - \frac{\mathbf{r} - \mathbf{\tilde{r}}(t')}{c}. \tag{1}
\]

When the particle is at B, it has velocity \( \mathbf{v}(t') = c\beta(t') \) and acceleration \( \mathbf{a}(t') = c\beta(t') = c\frac{\mathbf{\dot{\beta}}}{c} \). Even though \( \beta \) is a dimensionless vector, \( \mathbf{\dot{\beta}} \) is not. It has the dimension \( T^{-1} \).

Let \( \mathbf{n}(t') \) the unit vector in the direction of the CtF vector \( \mathbf{R}(t') \). Hence the relations.

\[
\mathbf{R}(t') = \mathbf{r} - \mathbf{\tilde{r}}(t') = R(t') \mathbf{n}(t'); \tag{a}
\]

\[
\mathbf{v}(t') = \frac{d\mathbf{R}(t')}{dt} = c\beta(t'); \tag{b}
\]

\[
\mathbf{a}(t') = \frac{d\mathbf{v}(t')}{dt} = c\mathbf{\dot{\beta}}(t'). \tag{c}
\]

The \((\mathbf{E}, \mathbf{B})\) field of the particle at the observer P at the time \( t \) is now given by the following formulas\cite{3, 2}.

\[
\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \left[ \left( \frac{(1 - \beta^2)(\mathbf{n} - \beta)}{\kappa^3 R^2} \right) + \frac{\mathbf{n} \times \{ (\mathbf{n} - \beta) \times \mathbf{\dot{\beta}} \}}{c\kappa^3 R} \right]_{r = t, c}. \tag{3}
\]

\[
c\mathbf{B}(\mathbf{r}, t) = \mathbf{n}(t_r) \times \mathbf{E}(\mathbf{r}, t).
\]

\[
\kappa(t') = 1 - \beta(t') \cdot \mathbf{n}(t'). \tag{3}
\]

2.2 Special Case: Motion confined to a plane

We shall specialize the \((\mathbf{E}, \mathbf{B})\) field for the special case in which the path \( \Gamma \), as well as the observer \( P \), exist on a plane, which we shall take as the \( XY \) plane. The equation of the path is

\[
\mathbf{\tilde{r}} = \mathbf{\tilde{r}}(t') \quad \Rightarrow \quad \mathbf{x} = \xi(t'), \quad \mathbf{y} = \zeta(t'). \tag{4}
\]

The field point P has Cartesian coordinates \((x, y)\) with respect to the \textit{fixed} origin O. However, for the purpose of drawing the field lines, we shall find it convenient to propose a set of alternative, curvilinear coordinates \((R, \Phi)\) with respect to the \textit{moving} origin \( B(t') \). Here \( R \) is the length of the CtF vector \( \mathbf{R} \) and \( \Phi \) is the angle this radius vector makes with the \( X \) axis.

Let \( (\mathbf{e}_R, \mathbf{e}_s) \) be the unit vectors associated with the new coordinates \((R, \Phi)\). As per convention, \( \mathbf{e}_R \) and \( \mathbf{e}_s \) are the directions in which the respective coordinates are increasing. In particular, \( \mathbf{e}_R \) is \textit{identical} with the unit vector \( \mathbf{n} \) introduced earlier and \( \mathbf{e}_s \) is perpendicular to \( \mathbf{e}_R \).
We have thus employed three sets of radius vectors, coordinates, unit vectors:

1. $\mathbf{r} = xi + yj$, Cartesian coordinates, for the observer P with respect to the fixed origin O;
2. $\mathbf{r}'(t') = \mathbf{i}(t') + \mathbf{j}(t')$, Cartesian coordinates, for the moving charge $q$ with respect to the fixed origin O;
3. $\mathbf{R}(t') = (R(t'), \Theta(t'))$, polar coordinates, and the corresponding unit vectors $(\mathbf{e}_R(t'), \mathbf{e}_\Phi(t'))$ of the observer P with respect to the moving point charge $q$.

We shall rewrite the $\mathbf{E}$ vector, after breaking it up into two components, the “velocity field” $\mathbf{E}_v$ (depending entirely on the velocity) and the “acceleration field” $\mathbf{E}_a$ (involving both velocity and acceleration of the particle).

$$\mathbf{E} = \mathbf{E}_v + \mathbf{E}_a, \quad \text{where},$$

$$\mathbf{E}_v = \frac{q}{4\pi \varepsilon_0} \left[ \frac{(1 - \beta^2)(\mathbf{e}_R - \beta)}{\kappa^3 R^2} \right]_{t'=t}.$$

$$\mathbf{E}_a = \frac{q}{4\pi \varepsilon_0} \left[ \mathbf{e}_R \times \{ (\mathbf{e}_R - \beta) \times \beta \} \right]_{t'=t}.$$

It is seen that the acceleration field $\mathbf{E}_a$ is purely transverse, i.e., it has only $\mathbf{e}_\Phi$ component, perpendicular to the line BP.

### 2.3 Velocity and Acceleration of $q$ with Reference to the New Unit Vectors

We shall have to write the $\mathbf{E}$ field with reference to the new unit vectors $(\mathbf{e}_R, \mathbf{e}_\Phi)$. Let the...
velocity vector $c\beta$ make angle $\theta$ from the $X$-axis, and the CtF vector $R(t')$ make angle $\alpha$ from $\beta$, as illustrated in Fig 1(b), both angles taken positive if measured anticlockwise from the respective reference lines. Let us, temporarily, introduce two orthogonal unit vectors $(e_\beta, e_\phi)$, along and perpendicular to the direction of the velocity $\beta$, and write all the new unit vectors with reference to the Cartesian unit vectors $(i, j, k)$. Note that $\Phi = \alpha + \theta$.

\[
\begin{align*}
e_R &= \cos \Phi i + \sin \Phi j, \\
e_\phi &= -\sin \Phi i + \cos \Phi j, \\
e_\theta &= \cos \theta i + \sin \theta j. \\
e_\beta &= k \times e_2 = -\sin \theta i + \cos \theta j. \\
\end{align*}
\]

Either using (6), or straight from the drawings in Fig 1(b), one obtains $(e_\beta, e_\phi, e_\theta)$ in terms of $(e_R, e_\phi)$.

\[
\begin{align*}
e_\beta &= \cos \alpha e_R - \sin \alpha e_\phi, \\
e_\phi &= \sin \alpha e_R + \cos \alpha e_\phi, \\
e_\theta &= (-\sin \theta i + \cos \theta j) \dot{\theta} = \dot{\theta} e_\theta. \\
\end{align*}
\]

Hence,

\[
\begin{align*}
\beta &= \beta e_\beta = \beta (\cos \alpha e_R - \sin \alpha e_\phi) \\
\dot{\beta} &= \dot{\beta} e_\beta + \beta \dot{e}_\beta = \dot{\beta} e_\beta + \beta \dot{\theta} e_\theta \\
&= (\beta \cos \alpha + \beta \dot{\theta} \sin \alpha) e_R \\
&\quad + (\dot{\beta} \sin \alpha + \beta \dot{\theta} \cos \alpha) e_\phi.
\end{align*}
\]

### 2.4 The E field with reference to $(e_R, e_\phi)$

Let us now go back to Eq. (5), and write $\Omega = e_R \times \{(e_R - \beta) \times \beta\} = \Omega_1 + \Omega_2$, where

\[
\begin{align*}
\Omega_1 &= e_R \times (e_R \times \beta) = e_R (e_R \cdot \beta) - \beta \\
&= -e_\phi (e_\phi \cdot \beta) = (\beta \sin \alpha - \beta \theta \cos \alpha) e_\phi. \\
\Omega_2 &= -e_R \times (\beta \times \beta) = -e_R \times \beta^2 \hat{k} = \beta^2 \dot{\theta} e_\phi \\
\Omega &= [\dot{\beta} \sin \alpha + \beta \dot{\theta} (\beta - \cos \alpha)] e_\phi.
\end{align*}
\]

We thus get the following expression for the “acceleration field”:

\[
E_a = \frac{q}{4\pi \varepsilon_0} \left[ \frac{\dot{\beta} \sin \alpha + \beta \dot{\theta} (\beta - \cos \alpha)}{c^2 R^3} e_\phi \right]_{t=t_c}. 
\]

Back to Eq. (5), note that the expression in the numerator is $(1 - \beta^2)(1 - \beta \cos \alpha) e_R + \beta \sin \alpha e_\phi$, so that the “velocity field” is:

\[
E_v = \frac{q}{4\pi \varepsilon_0} \left[ \frac{(1 - \beta \cos \alpha) e_R + \beta \sin \alpha e_\phi}{k^2 \gamma^2 R^2} \right]_{t=t_c},
\]

where $1/\gamma^2 = 1 - \beta^2$.

Adding the two fields given in Eqs. (9) and (10), we get the complete field:
\[ E(r, t) = \frac{q}{4\pi \varepsilon_0} \left[ \frac{\beta \sin \alpha + \beta \dot{\theta}(\beta - \cos \alpha)}{ck^3 R} e_\theta + \frac{(1 - \beta \cos \alpha) e_r + \beta \sin \alpha e_\phi}{\kappa^3 \gamma^2 R^2} \right]_{t'=t_r} \]

\[ E_r = \frac{q}{4\pi \varepsilon_0} \left[ \frac{(1 - \beta^2)(1 - \beta \cos \alpha)}{\kappa^3 R^2} \right]_{t'=t_r} \tag{a} \]

\[ E_\phi = \frac{q}{4\pi \varepsilon_0} \left[ \frac{R \beta \sin \alpha + \beta \dot{\theta}(\beta - \cos \alpha) + (1 - \beta^2) \beta \sin \alpha}{\kappa^3 R^2} \right]_{t'=t_r} \tag{b} \]

\[ \kappa(t') = 1 - \beta(t') \cdot e_r(t')|_{t'=t_r} = 1 - \beta \cos \alpha(t'). \tag{11} \]

It can be observed from the above equations that “close” to the charge, the field is predominantly “velocity”, and “far” from it, predominantly “acceleration”. How close or how far? We shall set up a criterion in Sec. 5.2.

3 The differential equation for the Field Lines

Let us go back to Eq. (1), relating the time of observation \( t \) to the retarded time \( t' \). Given any pair of values of \( (r, t) \), determining the field \( E(r, t) \) at the location \( r \) at time \( t \), this equation can be solved to yield a unique value of \( t' \), and hence a unique retarded location \( \tilde{r}(t') \) of \( q \).

Let us consider a typical field line \( \Omega \) passing through the observation point \( P(r, t) \) at any arbitrary time \( t \), as shown in Fig. 2(a). If we take the observation time \( t \) as fixed, then there will exist a \( 1 \leftrightarrow 1 \) correspondence between \( r \) on \( \Omega \) and the retarded point \( \tilde{r}(t') \).

Different points on the field line \( \Omega: t, 1, 2, 3, 4 \), will correspond to different points on the path \( \Gamma \) at the corresponding retarded times: \( t' < t'_1 < t'_2 < t'_3 < t'_4 \), and different CtF vectors: \( R(t'), R(t'_1), R(t'_2), R(t'_3), R(t'_4) \), their magnitudes steadily increasing along the path: \( R(t') > R(t'_1) > R(t'_2) > R(t'_3) > R(t'_4) \), as shown in the figure. At A, \( t' = t'_A = t \) and \( R(t'_A) = 0 \).

Here it should be noted that the \( E \) field caused by the moving charge \( q \), and now existing over all space, can be pictured as field lines, all of which converge at A (present location of \( q \)). Infinite number of field lines, pointing in all directions, emanate from A. Only one of them, marked \( \Omega \), passes through the observer at P, and is tangential to the electric field \( E(r, t) \). Our immediate task is to obtain the differential equation, and the
boundary condition for the field lines. Applying the boundary condition we should be able to identify $\Omega$.

In Fig. 1b, and again in Fig. 2b, we have shown the following vector and angles at the retarded location $\tilde{r}(t')$: the velocity $c\beta(t')$ of the particle; the angle $\theta(t')$ between $\beta(t')$ and the X axis; and the angle $\alpha(t')$ between $\beta(t')$ and $R(t')$. It follows that

$$c\beta_x(t') = \tilde{x}; \quad c\beta_y(t') = \tilde{y};$$

$$\beta(t') = \sqrt{\beta_x^2 + \beta_y^2};$$

$$\tan \theta(t') = \frac{\beta_y(t')}{\beta_x(t')}.$$

Here $(\tilde{x}, \tilde{y})$ are derivatives of the functions $\xi(t'), \zeta(t')$ written in Eq. (4), with respect to $t'$. The $1 \leftrightarrow 1$ relation between $t$ and $t'$, and between $r(x, y)$ and $\tilde{r}(\tilde{x}, \tilde{y})$, will permit us to mark a point $r(x, y)$ on the field line $\Omega$ (at the present time $t$) in terms of its Curvilinear coordinates $(R, \alpha)$ (replacing $(R, \Phi)$). Noting that $r = \overrightarrow{OP}, R = \overrightarrow{BP}$, it is seen, either from Fig. 1(b), or from Fig. 2(b), that

$$r = \tilde{r}(t') + R(t').$$

$$R(t') = R(t')[\cos \Phi(t')i + \sin \Phi(t')j] = R(t')[\cos(\theta(t') + \alpha(t'))i + \sin(\theta(t') + \alpha(t'))j].$$

The above equations, written in terms of Cartesian components, takes the form:

$$x = \tilde{x}(t') + R \cos[\theta(t') + \alpha], \quad (a)$$

$$y = \tilde{y}(t') + R \sin[\theta(t') + \alpha], \quad (b)$$

where $t' = t - \frac{R}{c}$, $t$ is fixed. \hfill (14)
and the variable $\theta(t')$ is given by Eq. (12). The task now boils down to finding the differential equation and the boundary conditions from which the functional relationship between $R$ and $\alpha$ is to be obtained.

At this point some explanatory notes can be useful. Consider how we determine the trajectory of a particle under inverse square law force[8]. First we set up polar coordinates $(r, \theta)$ and obtain a differential equation involving the second derivative of $r$ with respect to $\theta$, solve that equation to obtain $r = f(\theta)$ and then use this solution to get the trajectory in the parametric form: $x = r \cos \theta = f(\theta) \cos \theta; y = r \sin \theta = f(\theta) \sin \theta$, taking $\theta$ as the parameter.

In the present case we take $R$ as the parameter, obtain a differential equation that will involve derivative of $\alpha$ with respect to $R$, solve the differential equation to obtain $\alpha = f(R)$, and then the parametric equation of the field line in the form (14), in which $R$ is the parameter.

Referring to Fig. 2(b), consider two neighbouring points $P$ and $Q$ on the field line, at radius vectors $r$ and $r + d\mathbf{r}$ respectively. Suppose we write the line element $d\mathbf{r}$ as

$$d\mathbf{r} = d\sigma \mathbf{e}_r + d\varepsilon \mathbf{e}_\phi,$$

where $d\sigma = d\mathbf{r} \cdot \mathbf{e}_r; d\varepsilon = d\mathbf{r} \cdot \mathbf{e}_\phi$. (15)

If the components of $\mathbf{E}(r, t)$ in the directions of $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are $E_r$ and $E_\phi$ respectively, then

$$\frac{d\sigma}{d\varepsilon} = \frac{E_r}{E_\phi}. \quad (16)$$

The above equation would become the desired differential equation for the field lines after expressing the differentials $d\sigma, d\varepsilon$ in terms of the new coordinates $(R, \alpha)$ and their differentials. The steps are long and difficult. One reason is that the displacement $d\mathbf{r} = \overrightarrow{PQ} = d\mathbf{r} + d\mathbf{R}$, and a look at Fig 2(b) will indicate that determination of the two vector differentials can be complicated. We have shifted this work to Sec. A.2 in Appendix A. What matters right now is the final differential equation for the field lines which we write as[5]:

$$\frac{d\alpha}{dR} = \frac{1}{c^2} \gamma^2 [\dot{\theta} - \dot{\beta} \cdot \mathbf{e}_\phi]. \quad (17)$$

### 4 The Boundary Condition

As with all differential equations, the equation (17) alone will not lead to the particular solution desired by us, unless we know how to evaluate the arbitrary constant(s) that come from integration of the differential equation. To get the particular solution of a differential equation one has to specify the initial condition(s), or the boundary condition(s) and evaluate the constant(s).

In this case there are infinite number of field lines that will satisfy Eq. (17). We shall have to identify one of them as the line. For this purpose we have to go to the the point $A$, the present location of the charge $q$, the fountain head from where all field lines spring out, and identify the one of our choice by specifying the boundary condition.

It can be seen from Eq. (5) that close to the location of the particle, $R \to 0$, and the
Figure 3: Fountain lines at the current location of $q$, for understanding the boundary condition

Field becomes a velocity field, with field lines issuing out as straight lines, in all directions [4]. We have reproduced the same field lines in Fig. 3, after rotating them in such a way that the 0-th line aligns with the direction of the particle velocity $c\beta$ at the time $t$. We shall call them fountain lines.

Let us imagine $N$ fountain lines, all of them confined to the $XY$ plane, issuing out from $A$, and tag them as $1, 2, ..., N$. (This number is 36 in the figure.) When $\beta \to 0$ the angular intervals between successive fountain lines becomes the same and equal of $2\pi/N$, so that the fountain line $\#n$ will make angle $\theta_n = 2\pi n/N$ with the $\beta$ vector. According to the Purcell condition[4], the angle $\phi_n$ that the same fountain line will make with the direction of $\beta$, when it is relativistic, is

$$\tan \phi_n = \gamma \tan \theta_n = \gamma \tan \left(\frac{2\pi n}{N}\right).$$

Equation (18)

We have illustrated this in Fig. 3.

We would like to plot $N$ field lines, each of them issuing out of the point $A$ (the present location of the particle) as fountain lines, such that the $n$-th field line ($n = 1, 2, ..., N$) is tangential to the fountain line $\#n$, making the angle $\phi_n$ with $\beta$ at the source point $A$, as required by Eq. (18). This is our boundary condition.

An example is shown in Fig. 3, in which the field line $\Omega$ issues out from the source point as the fountain line $\#15$. This figure sheds further light on the difference between the path $\Gamma$ of the particle, shown as a dotted
line, and the field line $\Omega$, shown as a solid line. The path $\Gamma$ does not exist as one piece at any given instant of time. Different points on this curve exist as dots at different times $t'_1, t'_2, t'_3, \ldots$. Also it is confined to the region in which the particle exists, either at rest or in motion. The field line $\Gamma$ exists as one piece at any given instant of time, issues out from the point A, and stretches out all the way to infinity, mainly due to Coulomb’s law.

It is obvious that $R = 0$ at A, and the tangent to the field line is

$$\frac{dy}{dx}\bigg|_{R=0} = \frac{E_y}{E_x}\bigg|_{R=0} = \tan(\phi_n + \theta)|_{R=0}. \tag{19}$$

It is seen from Eq. (5) that $E \to E_0$ as $R \to 0$. Therefore,

$$\frac{E_y}{E_x}\bigg|_{R=0} = \frac{E_{y0}}{E_{x0}}\bigg|_{R=0} = \frac{(\mathbf{e}_R - \mathbf{\beta}) \cdot \mathbf{e}_y}{(\mathbf{e}_R - \mathbf{\beta}) \cdot \mathbf{e}_x}\bigg|_{R=0} \tag{20}$$

$$= \sin(\theta + \alpha_n) - \beta \sin \theta \over \cos(\theta + \alpha_n) - \beta \cos \theta \bigg|_{R=0}. \tag{21}$$

Let us write $\alpha = \alpha_n$ for the $n$-th line at $R = 0$. Then from (19) and (20),

$$\tan(\phi_n + \theta)|_{R=0} = \frac{\sin(\theta + \alpha_n) - \beta \sin \theta}{\cos(\theta + \alpha_n) - \beta \cos \theta}, \tag{22}$$

from which it follows that (Sec.A.3, Appendix A)

$$\tan \phi_n = \frac{\sin \alpha_n}{\cos \alpha_n - \beta}. \tag{23}$$

The boundary condition (18) now reduces to the form

$$\gamma \tan \left(\frac{2\pi n}{N}\right) = \frac{\sin \alpha_n}{\cos \alpha_n - \beta}. \tag{24}$$

Some trigonometrical manipulations (Sec.A.3, Appendix A) will convert the above equation to the following form[5]

$$\tan \frac{\alpha_n}{2} = \sqrt{\frac{1 - \beta}{1 + \beta}} \tan \frac{\pi n}{N}; \; n = 1, 2, \ldots, N, \tag{25}$$

in which we shall apply the Boundary Condition.

We summarize as follows. A general field line $\Omega$, at a given instant $t$, will be expressed in the parametric form $\alpha = f(R)$. We shall obtain $N$ field lines $\{\Omega_n : \alpha_n = f_n(R); \; n = 1, 2, 3, \ldots, N\}$, springing out from the present location A of the point charge $q$, at the angles $\phi_n$ as given by Eq.(18). These $N$ lines $\{\Omega_n\}$ are to be obtained (and subsequently plotted) by the following procedure.

---

| Solve the differential equation | $\frac{d\alpha}{dR} = \frac{1}{c} \gamma^2[\dot{\theta} - \dot{\mathbf{e}} \cdot \mathbf{e}],$ | (a) |
| subject to the boundary condition | $\tan \frac{\alpha}{2} = \sqrt{\frac{1 - \beta}{1 + \beta}} \tan \frac{\pi n}{N},$ | (b) |
| to obtain | $\alpha_n = f_n(R); \; n = 1, 2, 3, \ldots, N$ | (c) |
| Insert this in Eq.(14): | $x = \tilde{x}(t') + R \cos[\theta(t') + \alpha_n],$ | (d) |
| | $y = \tilde{y}(t') + R \sin[\theta(t') + \alpha_n],$ | (e) |
| where | $t' = t - \frac{R}{c},$ (see Eq. 1) | (f) |
to get the parametric equations for the $N$ field lines, remembering that Eqs. (a),(b),(d),(e) involve $\beta$ which is a function of $t'$, and hence through Eq.(f), a function of $R$.

5 Specific Example 1: Particle Accelerating along a Straight Line

5.1 General Formula

Fig. 4 shows the particle moving along the $X$ axis, with acceleration $a$. We shall specialize Eqs. (25), for this special case, by setting $\theta = 0; \dot{\theta} = 0$. Note from Eq. (8b) that $\dot{\beta} \cdot e_m = -\dot{\beta} \sin \alpha$. The differential equation for the field lines, Eq. (25a), now reduces to:

$$\frac{d\alpha}{dR} = \frac{1}{c} \gamma^2 \dot{\beta} \sin \alpha$$

Or, $\csc \alpha \, d\alpha = \frac{1}{c} \gamma^2 \dot{\beta} \, dR = -\frac{d\beta}{1 - \beta^2}$. (26)

For the last equality in the last line we used Eq. (1), to get $dR/dt' = -c$.

We now integrate the last line of Eq. (26), and get

$$\ln \tan \frac{\alpha}{2} = \ln \sqrt{\frac{1 - \beta}{1 + \beta}} + \text{const}$$

Or, $\tan \frac{\alpha}{2} = K \sqrt{\frac{1 - \beta}{1 + \beta}}$. (27)

where $K$ is a constant. Going back to the boundary condition given in Eq. (25b) we evaluate this constant as $K = \tan \frac{\pi n}{N}$, leading to the solution given as

$$\tan \frac{\alpha_n}{2} = \sqrt{\frac{1 - \beta}{1 + \beta}} \tan \frac{\pi n}{N}; \quad n = 1, 2, \ldots, N.$$ (28)

The above equation represents the desired relation between $\alpha$ and $R$ for the field line that starts from the source point as the fountain line $\#n$, making angle $\theta_n = 2\pi n/N$ with the $X$-axis. We shall illustrate this for the special case of a particle moving under a constant force along the $X$ axis.

5.2 Special Case: Particle moving under a constant accelerating force

Before going far, we shall write the expression for the $E$ field from rectilinear motion of the charge $q$ specializing Eq. (11) to this case, by setting $\theta = 0$.

$$E_{Rv} = \frac{q}{4\pi \varepsilon_0} \left[ \frac{(1 - \beta \cos \alpha)}{\kappa^3 R^2} \right]_{r = r_v}$$ (a)

$$E_{\phi v} = \frac{q}{4\pi \varepsilon_0} \left[ \frac{\beta \sin \alpha}{\gamma^2 \kappa^3 R^2} \right]_{r = r_v}$$ (b)

$$E_{\phi a} = \frac{q}{4\pi \varepsilon_0} \left[ \frac{\dot{\beta} \sin \alpha}{\gamma^2 \kappa^3 c R} \right]_{r = r_v}$$ (b)

We have broken up the field into its "velocity" components $E_{Rv}, E_{\phi v}$, and "acceler-
as emphasized following Eq. (11), the field is predominantly “acceleration” far from the charge (where our interest lies). Taking $\beta \sim 1$ the transverse components of the two field components are nearly equal at $R = c^2/a$. Hence the criterion is that

$$E \approx E_a, \quad \text{for } R \gg c^2/a. \quad (30)$$

Page and Adams[9] have shown plots of the acceleration component $E_a$, which are circles passing through the present location of $q$, with centres on the the plane $x = -c^2/a$. In our case we shall plot the entire $E$ field.

For some of the examples we are going to consider soon, (i) $a = 0.17 \times 10^{18}$ m/s$^2$, and $c^2/a = 0.529$ m, same as the unit of distance $\tau c$ we shall adopt, (ii) $a = 0.17 \times 10^{20}$ m/s$^2$, and $c^2/a = 0.005$ m, far less than the unit of distance we shall adopt.

Now, let us consider the rectilinear motion of particle of charge $q$ and rest mass $m_0$, under the influence of a uniform electric field $\mathbf{E} = E \mathbf{i}$. This electric field induces a constant acceleration $a = qE/m_0 = F/m_0$ in the $X$ direction, as measured in its instantaneous rest frame. The particle will ultimately achieve relativistic velocity.

The displacement $\tilde{x}(t')$ and the velocity $c\beta(t')$, both as functions of the retarded time $t'$, are given by the following formulas[10, 11, 12]:

$$\tilde{x}(t') = c\tau \left[ \sqrt{1 + \left( \frac{t'}{\tau} \right)^2} - 1 \right]$$

$$\beta(t') = \left[ 1 + \left( \frac{t'}{\tau} \right)^2 \right]^{-\frac{1}{2}}; \quad t' > 0. \quad (31)$$

In the above equations we have defined a characteristic time $\tau$, such that the particle would reach the velocity $c$ in this time, if non-relativistic mechanics had been applicable. That is,

$$a\tau = c. \quad \text{Or, } \tau = c/a \quad (32)$$

Figure 4: Particle in linear acceleration
Replacing \( t' \) by \( t - R/c \) in Eqs. (31) (see Eq. 1), we get expressions for \( \tilde{x} \) and \( \beta \) in terms of the parameter \( R \), at the present time \( t \).

The equation of the field line is now obtained by going back to Eqs. (25d,e), then setting \( \tilde{y} = 0, \theta(t') = 0 \):

\[
\begin{align*}
\dot{x} &= \tilde{x}(R) + R \cos \alpha_n. \quad (a) \\
y &= R \sin \alpha_n. \quad (b)
\end{align*}
\]

(33)

Let us consider a charged particle, e.g., an electron, which is accelerated in a 30m long linear accelerator (e.g., pelletron) to 30 MeV. The electric field through which the particle is accelerated is assumed to be uniform, and equal to \( E = 10^6 \) \( \text{V/m} \). Because of our preference for positive quantities, we shall assume the particle to be positively charged, but having the same magnitude of charge as that of an electron or proton, viz. \( e = 1.6 \times 10^{-19} \) C, and mass that of an electron or positron, viz., \( m_0 = 9.11 \times 10^{-31} \) kg. The acceleration is then

\[
\begin{align*}
a &= \frac{eE}{m_0} = \frac{1.6 \times 10^{-19} \times 10^6}{9.11 \times 10^{-31}} \\
 &= 1.76 \times 10^{10} \text{m/s}^2 \\
\tau &= (3 \times 10^8)/(0.17 \times 10^{18}) \\
 &= 17.6 \times 10^{-10} \text{s.} \\
\tau c &= 17.6 \times 10^{-10} \times 3 \times 10^8 \\
 &= 0.528 \text{m.}
\end{align*}
\]

(34)

The particle is at the point \( A \) at the current time \( t \). At this point \( t' = t \). Also note from Eq. (25f) that (i) \( R = ct \) at \( t' = 0 \), when the particle is at \( O \), and (ii) \( R = 0 \) at \( t' = t \), when the particle is at \( A \).

We shall consider two variations of this exercise. For each case we shall assume that the particle is originally sitting at rest at the origin \( O \), until the \( E \) field is turned on at the time \( t = 0 \).

**Case 1.** The \( E \) field continues for ever. It is at the origin at \( t' = 0 \) and reaches a point \( A \) on the \( X \) axis at \( t' = \tau \), which we set equal to the present time \( t \) (in the Ex.I to follow). That is, \( t = \tau \).

**Case 2.** The \( E \) field is switched off at \( t = \tau \), when it is at \( A_1 \), then continues to move with constant velocity and reaches the point \( A_2 \) when \( t' = 2\tau \), which we set equal to the present time \( t \). That is, \( t = 2\tau \).

### 5.2.1 Plotting the \( E \) field for Case 1

To avoid any confusion we shall remind the reader that we have been, and shall be, using two symbols for time, viz., \( t' \), to mean time measured along the track of the particle, and \( t \), to mean present time, i.e., the time when the entire \( E \) field, and the field lines are viewed. Both times are measured in the "Lab frame" (No relativity is involved.)

We shall take three successively increasing values of the present time \( t \), resulting in successively increasing values of \( \beta \) at \( t \), and get the required values at \( A \) and \( O \), using Eqs. (31).

**Ex. I:** \( t = \tau = 17.6 \times 10^{-10} \text{ s.} \)

\[
\begin{align*}
\beta(A) &= 1/\sqrt{2} = 0.707. \\
\tilde{x}(A) &= c\tau [\sqrt{2} - 1] = 0.218 \text{ m.} \quad (35) \\
R(O) &= \tau c = 0.528 \text{ m.}
\end{align*}
\]

**Ex. II:** \( t = 2\tau = 35.2 \times 10^{-10} \text{ s.} \)

\[
\begin{align*}
\beta(A) &= 2/\sqrt{5} = 0.894. \\
\tilde{x}(A) &= c\tau [\sqrt{5} - 1] = 0.653 \text{ m.} \quad (36) \\
R(O) &= 2\tau c = 1.056 \text{ m.}
\end{align*}
\]
Figure 5: E field line at the $t = \tau$ due to charge $q$ moving with constant acceleration

Figure 6: E field line from a particle moving with constant acceleration at two different instants of time: (a) $t = 2\tau$, (b) $t = 3\tau$. 

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Ex. III: \( t = 3 \tau = 52.8 \times 10^{-10} \) s.

\[
\begin{align*}
\beta(A) &= 3/\sqrt{10} = 0.949. \\
\tilde{x}(A) &= c \tau [\sqrt{10} - 1] = 1.14 \, \text{m}. \quad (37) \\
R(O) &= 3 \tau c = 1.584 \, \text{m}.
\end{align*}
\]

We shall plot the field at the present time, by the following steps.

- Insert the values of \( a \) and \( \tau \) as obtained in Eqs.(34) in Eqs. (31) to get \( \tilde{x}(t') \), \( \beta(t') \) as functions of \( t' \) only;

- Obtain \( \alpha(t') \) as a function of \( t' \), from Eq.(28);

- Use the expressions of \( \tilde{x}(t'), \alpha(t') \) in Eqs. (33), in which \( x, y \) now look like functions of \( t' \);

- Change the variable from \( t' \) to \( R \) using the relationship (25f), i.e., by setting \( t' = t - R/c \), treating \( t \) as a constant. We now get the equations of the field lines in the form of the parametric equations: \( x = x(R); y = y(R) \), where \( R \) is the parameter;

- \( R = (0, ct) \), at \((A,O)\). We shall therefore take the range of \( R \) as \([0 : ct]\).

We have plotted three sets of field lines, taking \( N = 16 \) in all cases, and corresponding to \( t = \tau \) in Fig.5, and corresponding to \( t = 2\tau, 3\tau \) in Fig.6.

### 5.2.2 Plotting the E field for Case 2

We consider the following situation. The same particle, as described in Case 1, originally sitting at the origin \( O \), undergoes the same acceleration \( a \) from \( t' = 0 \) to \( t' = \tau \), reaching the point \( A1 \), then moves with constant velocity. We would like to plot the \( E \) field emanating from the particle at the present time \( t = t' = 2\tau \) when it is instantaneously located at the point \( A2 \), as shown in Fig. 7.

Here it should be noted that during the time \( t' < 0 \), when the particle was sitting at \( O \), it was giving out the Coulomb field continuously from \( t' = -\infty \) to \( t' = 0 \), spreading over all space, all the way to infinity. Our plots will also take a glimpse of a part of this Coulomb field. For this purpose we shall take the \textit{retarded time} in the range \( -\tau < t' \leq 2\tau \), corresponding to the CtF vector \( 3\tau > R \geq 0 \).

We shall therefore divide the time zone into 3 parts and write down expressions for velocity \( c\beta \) and displacement \( \tilde{x} \) and for these 3 parts.

For this purpose we shall represent the the functions \( \beta(t'), \tilde{x}(t') \) written in Eqs.(31) by new symbols \( \bar{\beta}(t'), \bar{x}(t') \).

\[
\begin{array}{|c|c|c|}
\hline
\text{Time range} & -\tau \leq t' \leq 0 & 0 < t' \leq \tau & \tau < t' \leq 2\tau \\
\hline
\beta(t') = & 0 & \bar{\beta}(t') & \bar{\beta}(\tau) \\
\bar{x}(t') = & 0 & \bar{x}(t') & \bar{x}(\tau) + c\bar{\beta}(\tau)(t' - \tau) \\
\hline
\end{array}
\]
Taking $N = 16$, we have plotted 16 field lines emanating from $q$ at the time $t' = t = 2\tau$, when it is located at A2, following the same steps as given for Case 1. The plots of the field lines are displayed in Fig.8(a).

5.3 Lessons from the field lines of accelerating charge

The plots for the Case 2 can give some valuable insight into the propagation of the electromagnetic field, if we examine the field lines with some interest and care. For this purpose we have reproduced the plots of Fig.8(a) in Fig.8(b), and divided the space into 3 zones: (i) Coulomb zone, (ii) acceleration zone, (iii) velocity zone.

The zone (i) is all space, all the way to infinity, but lying beyond a sphere of radius $R_0 = c\tau = 1.056$ m, centred at O, and labelled $S_0$. The field lines in this zone are straight lines, spreading out isotropically in all directions. If these lines are extended backward, they will all meet at the point O, the original rest house of the charge $Q$. These lines are the typical Coulomb field lines.

The sphere $S_0$ arises at $t = 0$ when the particle starts moving, and keeps expanding with the speed of light. In the short time $t = 2\tau = 35.2 \times 10^{-10}$ s, this sphere has expanded from zero radius to the radius $R_0 = 2c\tau = 1.056$ m, and keeps expanding forever. Observers outside this sphere, thousands of kilometers away, even light years away from the origin O, see the particle unmoved, see only its “static” Coulomb field, with the radial field lines passing by him.

The zone (iii) lies within another sphere labelled $S_1$ with centre at the fixed point A1. Its radius of has expanded from zero radius at $t = \tau$ to radius $R_1 = c\tau = 0.528$ m at $t = 2\tau$.

Observers within this sphere see the $E$ field of the charge $q$ moving with constant velocity $0.707c$, and the field lines issuing out radially from the moving point A2, the instantaneous location of the charge $q$. These lines are similar to the lines shown in Fig. 3, bunched more
densely transverse to the direction of the velocity vector \( c\beta \), than in the longitudinal direction.

The zone (ii) lies between the two spheres \( S_0 \) and \( S_1 \). Observers in this zone see the particle moving in the \( X \) direction with constant acceleration \( a \).

Fig. 9 in which we have focussed on the time evolution of only one field line, corresponding to \( n = 3 \), may shed further light on the concept of field propagation. Fig (a) shows how this field line has evolved from \( t = \tau \) to \( 2\tau \) to \( 3\tau \). The Coulomb field that had been emanating from \( O \) from \( t' = -\infty \) to \( t' = 0 \), and propagating along the the radial line \( Oc \), is plotted as the segment \( b_1 - b_2 - b_3 - c \). The acceleration field, the source of which moved from \( O \) to \( A_1 \), is plotted at the present times \( t = \tau, 2\tau, 3\tau \), as the shifting arcs \( \Omega_1, \Omega_2, \Omega_3 \), respectively, with the points \( O \) mapped onto \( b_1, b_2, b_3 \), and \( A_1 \) onto \( a_1, a_2, a_3 \), at the same respective present times. The velocity field that ensued from the moving point \( A_1 \rightarrow A_2 \rightarrow A_3 \) as \( t' = \tau \rightarrow 2\tau \rightarrow 3\tau \), is shown as straight lines changing from \( A_2-a_2 \) to \( A_3-a_3 \) as the present time changed from \( t = 2\tau \) to \( t = 3\tau \).

Since the Coulomb lines for three lines are merged in Fig (a), we have separated them into three individual lines in Figs. (b), (c), (d) for a clearer look at them.

The boundary condition (18) is valid for all
Figure 9: Field lines corresponding to $n = 3$: at $t = \tau, 2\tau, 3\tau$ in (a), at $t = \tau$ in (b), at $t = 2\tau$ in (c), at $t = 3\tau$ in (d).
the three field lines. The reader should verify this by setting $n = 3$, $\gamma = \sqrt{2}$, so that $\theta_3 = \frac{6\pi}{16} = 67.5^0$, $\phi_3 = 73.65^0$. Now if he makes a printout of this page and measures the angles $(\theta_3, \phi_3)$, with a protractor, he will verify these angles. This lends further confirmation of our faith in the amazing power of Gnuplot.

### 5.4 Special Case: Particle moving under a constant decelerating force

#### 5.4.1 Plot of the Field Lines

We shall now consider the deceleration case. The same constant electric field $\mathbf{E}$, resulting in the same acceleration vector $\mathbf{a}$, pointing in the $x$-direction will apply in this case as well. However, the particle will be moving in the negative $x$ direction, approaching the origin from $+\infty$, during particle time $t' = -\infty$ to $t' = 0$, (instead of moving away from the origin, from $t' = 0$ to $t' = +\infty$ examined in the case of accelerating particle, considered in the previous section). The $\mathbf{E}$ field is acting like a break (as in bremsstrahlung radiation) bringing $q$ to complete stop at O. We shall modify the velocity and displacement functions of Eq. (31) to its new forms:

\[
\begin{align*}
\ddot{\mathbf{x}}(t') &= c\tau \left[ \sqrt{1 + \left( \frac{t'}{\tau} \right)^2} - 1 \right]; \quad t' \leq 0 \quad (a) \\
\beta(t') &= -\left[ 1 + \left( \frac{t'}{\tau} \right)^2 \right]^{-\frac{1}{2}}; \quad t' < 0. \quad (b) \\
&= 0; \quad t' = 0 \quad (c)
\end{align*}
\]

Note that the velocity function has undergone a change in sign, whereas the displacement function remains the same. We have presented a picture of the configuration in Fig.10.

Fig.10(a) shows the path of the particle, coming from $+\infty$ with velocity $c\beta_\infty = \beta(-\tau) = -\frac{\sqrt{2}}{\sqrt{2}}$, undergoes deceleration from the point B1 to O, and stops at O. Fig.10(b) shows the configuration on the $t'$ scale.

Before plotting the $\mathbf{E}$ field lines for the decelerating charge, let us take stock of what we should expect, in comparison with the plots of the accelerating charge viewed at length in Sec. 5.2. For this purpose we go back to Eqs. (29).

The acceleration vector (in the guise of deceleration) continues to be in the positive $x$ direction. Just as $\beta$ becomes negative, the angle $\alpha$ changes to angle $\alpha + \pi$, so that $\beta \sin \alpha$ does not change. Of the three components, $E_{\Phi v}$ and $E_{\Phi a}$ do not change at all. Only the component $E_{R v}$ changes sign, but has negligible effect in regions perpendicular to the path. The net result is that the field lines for the deceleration case should not be drastically different from the acceleration case.

However, there is one important difference. The decelerating charge has stopped (momentarily) at the present time $t = 0$, whereas the accelerating charge is moving with speed $\beta = 0.707$ at the present time $t = \tau$, and the field lines are issuing out from $q$ at the present times, in all the examples. Therefore the field lines from the accelerating charge will make steeper angles with the $x$-axis at its source point, compared to the decelerating one, in accordance with the Purcell’s condition.

Let us now turn to plotting. We shall con-
Figure 10: path of charge q moving from infinity towards the origin O under deceleration. Fig (a) shows the direction of motion and the location of q on the x-axis; Fig (b) shows the retarded time $t'$ axis, divided into three segments.
sider the following three cases.

**Case I.** Deceleration \( a = a_o = 0.17 \times 10^{18} \text{ m/s}^2 \) is forever, never stops. Present time is \( t = t' = 0 \) at the event “the particle reaches O”. Plots of the field lines are relatively simple, and are shown in Fig. 11.

**Case II.** The particle passes B2 at \( t' = -2\tau \), reaches B1 at \( t' = -\tau \), when deceleration \( a = a_o \) starts, reaches O at \( t' = 0 \), when deceleration stops. The particle stays at O from \( t' = 0 \) to \( t' = +\infty \). Present time is \( t = t' = \tau \).

The velocity and displacement as functions of \( t' \) are given in the table below, which is analogous to the equation table (40).

<table>
<thead>
<tr>
<th>Time range ( \rightarrow )</th>
<th>(-2\tau \leq t' \leq -\tau )</th>
<th>(-\tau &lt; t' &lt; 0 )</th>
<th>( 0 \leq t' \leq \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta(t') = )</td>
<td>( \beta(-\tau) )</td>
<td>( \beta(t') )</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{x}(t') = )</td>
<td>( \bar{x}(-\tau) + c\beta(-\tau)(t' + \tau) )</td>
<td>( \bar{x}(t') )</td>
<td>0</td>
</tr>
</tbody>
</table>

We have plotted the \( \beta(t') \) and \( \bar{x}(t') \) functions in Fig. 12. Plots for the field lines are shown in Fig. 13.

**Case III.** The situation is the same as in Case 2, with the following changes:

1. The velocity is now increased from \( \beta_\infty = -0.707 \) to \( \beta_\infty = -0.949 \), as in the last example in Sec. 5.2.1. This value has been decided by setting \( (t'/\tau) = 3 \) in Eq.(39b). Hence, the previous time unit \( \tau \), is now replaced by the new unit \( 3\tau \).

2. The acceleration is increased to 100 times of the previous value. That is \( a_{\text{old}} = a_o = 0.17 \times 10^{18} \text{ m/s}^2 \rightarrow a_{\text{new}} = 100a_o = 0.17 \times 10^{20} \text{ m/s}^2 \).

3. The particle passes B2 at \( t' = -6\tau \), reaches B1 at \( t' = -3\tau \), when deceleration starts, reaches O at \( t' = 0 \), when deceleration stops. The particle stays at O from \( t' = 0 \) to \( t' = +\infty \). Present time is \( t = t' = 3\tau \).

Plots for the field lines are shown in Fig. 14.

### 5.5 Lessons from the field lines of decelerating charge

We shall point out some important features of the field lines for the decelerating charge, with reference to Fig. 15. Fig. (a) shows the bare field lines for the Case 2. Fig. (b) has two circles S1, S2 superimposed on the bare field lines, similar to what we had done in Sec. 5.3.

The circles represent two spherical surfaces S1 and S2, dividing the space into 3 zones: (i) **Coulomb zone**, i.e., the region inside the sphere S1; (ii) the **Deceleration zone**, i.e., the region between the spheres S1 and S2; (iii)
the Velocity zone, i.e., the region outside the sphere S2.

1. The particle was moving with the constant velocity $c\beta_\infty = -1/\sqrt{2}c$ during its displacement $B_1 \leftarrow B_2$. The field lines are radial straight lines, making “steeper angles” $\phi_n$ with the X axis, as given by the Purcell condition. This uniform motion came to end at $B_1$, at the time $t' = -\tau$, but the current time is $t' = \tau$. Hence there is a time gap of $2\tau$ between the end of uniform motion and now. Observers beyond the radius $2\tau c$ about $B_1$ see the particle still in uniform motion with the constant velocity $c\beta_\infty = -1/\sqrt{2}c$. The sphere $S_2$ of radius $R_2 = 2\tau c$ with centre at $B_1$, marks the boundary beyond which the field is still the velocity field. However, the velocity field lines would emanate from the “present” location of the particle, had the particle been in constant uniform motion. Where is that “present” position? It is the point $V$, the virtual location of the particle, if the particle had been moving with the velocity $-c\beta_\infty$ unchanged.

What will be the x-coordinate of $V$?

The coordinate of the point $B_1$ is $x(B_1) = (\sqrt{2} - 1)\tau c$. Let $\Delta x = $ distance the particle will move in the negative x direction in time $2\tau$, which equals $c\beta(-\tau) \times 2\tau = -1/\sqrt{2} \times 2\tau c = -\sqrt{2}\tau$. Hence, $x(V) = x(B1) + \Delta x = -\tau c$.

Therefore, the velocity field lines seem to be emanating from $V$, at $x = -\tau c$, as shown in Fig 15.

2. The particle is sitting at $O$ from $t' = 0$
to the present time $t = \tau$. The Coulomb field has, in this time, propagated up to a radius of $R_1 = \tau c$. The sphere $S_1$ marks that sphere. The field inside this is purely Coulomb.

5.6 Comparison between Acceleration field lines and Deceleration field lines

In Fig. 16 we have compared the plots of the Case I of deceleration with those of Case I of acceleration considered in Sec. 5.2.1, by superimposing the two sets of plots one upon another with a common origin $O$. The lines with arrowheads represent the decelerating case, and the ones without the accelerating case. They appear to match the predictions made in Sec.5.4.1.

However, there is one difference. The field lines for the deceleration case appear to be longer in the +x direction, and those for acceleration in the -x direction. This may lead to an erroneous conclusion that the field is stronger in the +x direction in the deceleration case and its opposite in the acceleration case.

To dispel this notion let us stress that the length of a field line has no relation to the field strength. Secondly, the furthest point on the field line receives its contribution from...
the corresponding retarded point, same as the farthest point B on the track of the particle for the given range of $R$ considered (see Fig. 2.)

The track of the particle in the present case the $X$-axis. In the acceleration case the present location is at $x = c\tau$ and the farthest point is $x = 0$. In the deceleration case the present location is $x = 0$ and the farthest point is $x = c\tau$. Let us mark the farthest point on the field line as C. Going back to Eq. (13), we can now write the radius vector for the point C as

$$
\mathbf{r}_c = -\tau \mathbf{i} + \mathbf{R}_c. \quad \text{acceleration}
$$

$$
\mathbf{r}_c = +\tau \mathbf{i} + \mathbf{R}_c. \quad \text{deceleration} \quad (41)
$$

The additive vectors $\mp \tau \mathbf{i}$ can be considered as “bias vectors”. We have replotted the deceleration field lines by removing this bias, i.e., by plotting the bare CTF vector $\mathbf{R}(t')$, as shown in Fig. 17. In this modified plot the field lines on the $x$-axis have the same length on either side of the origin.

5.7 Gravity Effect?

According to Einstein’s Principle of Equivalence an accelerating frame of reference replicates gravity[11]. A particle in a frame of reference which is accelerating in the positive $x$ direction feels the effect of gravity in the negative $x$ direction. Gravity pulls down ev-
everything, including light, forcing it to deviate from its straight line path. A cursory look at the plots of the \( E \) field may give the impression that the field lines are bending down, i.e., in the direction of the induced gravity, whenever the particle is in an accelerated state of motion\[13\] (for both acceleration and deceleration). In the absence of acceleration, the field lines betray their normal character, i.e., straight lines originating from the instantaneous location of the charge, whether at rest or in relativistic motion.

Whether the formulas for the EM fields from an accelerating charge, as obtained from Maxwell’s equations, can be traced to the Principle of Equivalence is a question on which experts in General Relativity can enlighten us.

6 Specific Example 2: \( E \) field from Circular Motion

6.1 Expression for the \( E \) field

Referring to Fig. 18, we shall describe the configuration as follows. A charged particle carrying charge \( q \) is moving in a circle of radius \( a \) on the \( XY \) plane with angular velocity \( \omega \) in the clockwise direction. The observer P...

Figure 15: Features of the Field lines from decelerating charge
Figure 16: Acceleration field lines (lines without arrows) superimposed on deceleration field field lines (lines with arrows) for comparison

Figure 17: Deceleration field lines without bias
is sitting on this XY plane, at a distance \( r \) from the origin.

It should be noted that we have taken the rotational angular velocity negative, so that the angle \( \theta \) that the velocity \( \beta \) makes with the \( X \) axis, at the retarded point \( B \) is a positive acute angle. The same with the angle \( \alpha \). We have chosen the coordinate axes in such a way that the observer \( A \) and the location of the particle at the present time \( t \) (set equal to 0) lie on the \( Y \) axis. With these choices of positive acute angles in the drawings derivations of trigonometrical relations becomes easier.

Our objective is to plot the \( E \) field as at the present time \( t = 0 \).

Following Eq. (4) the radius vectors \( \mathbf{r} = \hat{x}i + \hat{y}j \) from the origin to the location \( Q \) of the particle at any arbitrary time \( t \) is given by the equations

\[
\hat{x} = a \sin \omega t; \quad \hat{y} = a \cos \omega t. \quad (42)
\]

We need to evaluate the required quantities. Let \( t' \) stand for the retarded time \( t' \), \( B \) the location of the particle at \( t' \), and \( \mathbf{R}(t') \) the radius vector from \( B \) to \( P \) at the time \( t \). Then

\[
\mathbf{R}(t, t') = r \mathbf{j} - \mathbf{\hat{r}} = -a \sin \omega t' \mathbf{i} + (r - a \cos \omega t') \mathbf{j}. \quad (a)
\]

\[
R^2(t, t') = (a \sin \omega t')^2 + (r - a \cos \omega t')^2 = a^2 + r^2 - 2ra \cos \omega t'. \quad (b)
\]

where \( t = t' + \frac{R(t, t')}{c} = t' + \frac{1}{c} \sqrt{a^2 + r^2 - 2ra \cos \omega t'}, \quad (c)
\]

\[
c \beta = \hat{x} \mathbf{i} + \hat{y} \mathbf{j} = \omega a (\cos \omega t' \mathbf{i} - \sin \omega t' \mathbf{j}) \quad (d)
\]

\[
c \beta = \omega a; \quad \dot{\beta} = 0; \quad \theta = -\omega t'; \quad \dot{\theta} = -\omega, \quad (e)
\]

\[
\mathbf{R} \cdot c \beta = -r \omega a \sin \omega t', \quad (f)
\]

\[
\cos \alpha = \frac{\mathbf{R} \cdot c \beta}{R c \beta} = -\frac{r \sin \omega t'}{\sqrt{a^2 + r^2 - 2ra \cos \omega t'}} = \frac{r \sin \theta}{\sqrt{a^2 + r^2 - 2ra \cos \theta}} \quad (g)
\]

\[
(43)
\]
We now go back to Eqs. (11) and insert the values just determined: \( \dot{\beta} = 0; (R/c)\beta \dot{\theta} = -(R/a)\beta^2 \). Set \( \frac{q}{4\pi\varepsilon_0} = 1, \ a = 1 \).

\[
E_R = \frac{(1-\beta^2)(1-\beta \cos \alpha)}{\kappa R^3} \quad E_\phi = \frac{R\beta \sin \alpha + \beta (\beta \cos \alpha) + (1-\beta^2)\beta \sin \alpha}{\kappa R^3} = -R\beta^2(\beta \cos \alpha) + (1-\beta^2)\beta \sin \alpha.
\]

\[
\dot{\beta} \cdot e_\phi = \beta \dot{\theta} \cos \alpha \\
\frac{d\alpha}{dR} = \frac{1}{c}\gamma^2 \dot{\theta} (1 - \beta \cos \alpha). \quad [\text{from (25 a)}] \\
(1 - \beta \cos \alpha)^{-1} \ d\alpha = \frac{1}{c}\gamma^2 \dot{\theta} dR \\
= \frac{1}{c}\gamma^2 d\theta (dR/dt') \\
= -\gamma^2 d\theta. \quad [\text{using Eq. (1)}],
\]

6.2 Equations for the Field Lines

Let us be reminded that the independent variables are \( (R, \alpha) \). However, we have \( \theta \) on the right side. It is a function of \( R \) through for this special case. In this case \( \dot{\beta} = 0. \) the relation \( \theta = \theta(t') \) and \( t' = t - R/c = \)

Figure 18: Charge \( q \) in circular motion
\[ -\frac{\alpha}{c} \text{. From integral table} \]
\[
\int \frac{d\alpha}{1 - \beta \cos \alpha} = \frac{2}{\sqrt{1 - \beta^2}} \times \tan^{-1} \left( \sqrt{\frac{1 + \beta}{1 - \beta}} \tan \frac{\alpha}{2} \right). \quad (46)
\]

Therefore, integrating both sides of the last line of Eq. (45)
\[
\sqrt{\frac{1 + \beta}{1 - \beta}} \tan \frac{\alpha}{2} = \tan \left[ -\frac{\gamma \theta}{2} + \frac{k}{2\gamma} \right]. \quad (47)
\]

where \( k \) is a constant. Therefore,
\[
\tan \frac{\alpha}{2} = \sqrt{\frac{1 - \beta}{1 + \beta}} \tan \left[ -\frac{\gamma \theta(t - R/c)}{2} + \frac{k}{2\gamma} \right]. \quad (48)
\]

At the present location of the particle \( R = 0 \).

Setting this in (48), and going back to the boundary condition given in Eq. (25b), we get for the \( n \)-th field line,
\[
-\frac{\gamma \theta(t)}{2} + \frac{k}{2\gamma} = \frac{\pi n}{N}. \quad (49)
\]

\[
\dot{x}(t') = a \sin \omega t'; \quad \dot{y}(t') = a \cos \omega t'. \quad (a)
\]
\[
c\beta_x(t') = \omega a \cos \omega t'; \quad c\beta_y(t') = -\omega a \sin \omega t'. \quad (b)
\]
\[
\omega = \beta c/a; \quad \omega' = (\beta c/a)(-R/c) = -\beta R/a. \quad (c)
\]
\[
\tan \theta(t') = \beta_y(t')/\beta_x(t') = -\tan \omega t' = \tan(\beta R/a). \quad (d)
\]
\[
\theta(t') = \beta R/a = -\omega t'. \quad (e)
\]

The required parametric equations are given in Eqs. (25d,e), which we adopt for the present case.

\[
x = -a \sin(\beta R/a) + R \cos[(\beta R/a) + \alpha], \quad (52)
\]
\[
y = a \cos(\beta R/a) + R \sin[(\beta R/a) + \alpha], \quad (53)
\]
with $\alpha$ determined from Eq. (50), and $R \geq 0$.

### 6.3 Plot of the Field Lines

We shall plot several examples of field lines from a particle moving in a circle of radius $a = 1$ meter, with relativistic velocity $c\beta$. For each example we shall plot $N$ field lines originating as spring lines from the instantaneous present location of the particle, marked A, at the present time which has been taken as $t = 0$. The values of $\beta, N$, and the range of the parameter $R$, for the 7 exercises we have undertaken, are specified in columns 2,3,4 of the table below. The plot label includes the values of $\beta$ (b5 for $\beta=0.5$, b95 for $\beta=0.95$, etc), $N$, the number of “samples” (400, 1000, 3000), and the date when the plot was made.

If we denote the angular coordinate of the particle at the retarded time $t'$ as $\phi = \omega t'$, measured from the Y axis in the direction of its motion, i.e., clockwise, then it is seen from (52e) that $R = -\phi/\beta$. Hence $\pi$ can be a convenient unit of the linear distance

![Field line from a synchrotron accelerator](image-url)
In all plots we have use \( \pi \) (in meters), as an alternative the unit of displacement, and set the tic marks along the \( X \) and \( Y \) axes in integral multiples of \( \pi \). These plots are shown in Figs. 20 - 26.

In the final plot, shown in Fig. 27, \( \beta \to 1 \), and \( R = -\phi \), so that positive range of \( R \) will correspond with the same numerical range of \( \phi \), but with negative values.

<table>
<thead>
<tr>
<th>Ex.</th>
<th>( \beta )</th>
<th>( N )</th>
<th>Range</th>
<th>Plot label</th>
<th>Figure label</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>0.1</td>
<td>16</td>
<td>[0:60( \pi )]</td>
<td>Qcir-b1-16-800-160720A.fig</td>
<td>Fig. 20</td>
</tr>
<tr>
<td>2</td>
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<td>16</td>
<td>[0:13( \pi )]</td>
<td>Qcir-b5-16-400-160706.fig</td>
<td>Fig. 21</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>8</td>
<td>[0:6( \pi )]</td>
<td>Qcir-b5-16-400-160717.fig</td>
<td>Fig. 22</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
<td>8</td>
<td>[0:6( \pi )]</td>
<td>Qcir-b8-8-1000-160706.fig</td>
<td>Fig. 23</td>
</tr>
<tr>
<td>5</td>
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<td>8</td>
<td>[0:5( \pi )]</td>
<td>Qcir-b9-8-3000-160707.fig</td>
<td>Fig. 24</td>
</tr>
<tr>
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<td>Qcir-b9-4-3000-160707.fig</td>
<td>Fig. 25</td>
</tr>
<tr>
<td>7</td>
<td>0.95</td>
<td>8</td>
<td>[0:5( \pi )]</td>
<td>Qcir-b95-8-3000-160707.fig</td>
<td>Fig. 26</td>
</tr>
</tbody>
</table>

### 6.4 Notable features and Lessons from the Field Lines; Synchrotron Radiation

As we progress from Fig. 20 to 26, we witness changes in the field pattern as the velocity moves up from “slow” non-relativistic to “very fast” ultra-relativistic.

When a charged particle moves in a circle with non-relativistic velocity, it emits electromagnetic field, and hence radiation, which is nearly isotropic, but oscillates with the frequency of rotation. At ultra-relativistic velocity, it emits what is often referred to as *Synchrotron radiation*. Such radiation can be “seen” in a proton synchrotron in famous Nuclear Research Laboratories[14]. However, synchrotron radiation is also an Astronomical phenomenon, and has been observed in the light coming from pulsars and the Crab nebula[15].

Coming back to the plots of the \( \mathbf{E} \) field, which is our current interest, we go to Fig. 20 to see the “slow example”, the charged particle moving in a circle of radius \( a \), with velocity \( c/10 \). We have shown 16 field lines issuing out from the present location \( A \) of the particle, at equal angular intervals of \( 2\pi/16 \). The charge at the source being positive, \( \mathbf{E} \) vector is pointing away from the source, as indicated with arrowheads. (We shall avoid arrows in subsequent plots.) As the field propagates, it oscillates with the same angular frequency...
ω as that of rotation and hence carries wavelength \( \lambda = 2\pi c/\omega \). Now, \( \omega = \beta c/r = \beta c \), since the radius of the circle is \( a = 1 \) m. Hence, \( \lambda = 2\pi/\beta \). For \( \beta = 1/10 \), \( \lambda = 20\pi \), also manifested in the figure. In one cycle the “crests” at \( P_1, P_2 \) are displaced to \( P_2, P_3 \), respectively, each displacement being equal to one wavelength \( 20\pi \).

Figs. 21 and 22 correspond to the same values of \( \beta, N \), equal to 0.5, 16 respectively. However, the range of \( R \) used in the latter is \( 1/3 \) of the former and presents a magnified picture of the field lines close to the synchrotron.

In the same way, even though Figs. 24 and 25 correspond to the same value of \( \beta \), equal to 0.9, the former uses 8 field lines, whereas the latter only 4. As a consequence the latter presents a clearer picture of the formation of the sharp zig zag kinks. As we have shown on the field line \( n = 2 \), it encounters sharp reversals in its direction (kinks) at the points \( G \) and \( H \), in quick succession, and then proceeds onward. If we watch the kink regions of \( n = 1, \ldots, N \), in Figs. 24 and 25, they appear to merge or smear into a spiral band.

Fig. 26 presents an ultra-relativistic picture: \( \beta = 0.95 \). The particle moves in the same circle with velocity \( 0.95c \). Here the formation of the spiral band is conspicuous.

We have plotted the spiral band in Fig. 27(a), by first obtaining its parametric equation. First note that at the kink points \( E_\beta = 0 \). Assuming that the observer is far away, the field varies as \( 1/R \). Hence from Eq. (44), \( \beta - \cos \alpha = 0 \). For the ultra-relativistic case \( \beta \to 1 \Rightarrow \alpha = 0 \).

Now we go back to Eq. (53), set \( a = 1 \), \( \alpha = 0 \).
Figure 21: Field line: $\beta=0.5$, $N=16$, $R \epsilon [0 : 13\pi]$

Figure 22: Field line: $\beta=0.5$, $N=16$, $R \epsilon [0 : 6\pi]$
Figure 23: Field line: $\beta=0.8$, $N=8$, $R \epsilon [0 : 6\pi]$

Figure 24: Field line: $\beta=0.9$, $N=8$, $R \epsilon [0 : 5\pi]$
0, $\beta = 1$. Adopt a new variable $\phi = \omega t'$. From Eq. (52c) $R = -\phi$. Change the variable from $R$ to $\phi$ to get the locus of the kink points as

$$x = \sin \phi - \phi \cos \phi; \quad y = \cos \phi + \phi \sin \phi. \quad \phi \leq 0.$$  \hfill (55)

This is the equation of the involute of a unit circle, which can be visualized as the path of the free end $P$ of a rope which is unwound from a circular cylinder of unit radius to which it had been tightly wound[16], as illustrated in Fig. 27b. The locus of $P$ can be easily obtained from the diagram as

$$x = \sin \theta - \theta \cos \theta; \quad y = \cos \theta + \theta \sin \theta. \quad \theta \geq 0.$$  \hfill (56)

which is the same as (55), except that the two angles $\phi$ and $\theta$, both measured from the $Y$ axis, are clockwise and anti-clockwise respectively. The spiral shown in Fig. 27(a) is a plot of (55), but drawn with a thick line (line width = 4).

Fig. 27(a) can provide a valuable insight into the nature of synchrotron radiation. Let the particle be located at some point $Q$ on the circle at some retarded time $-t'$, corresponding to the present time $t = 0$. Let there be observers $P_1, P_2, K$, all of them located on the plane of the synchrotron. Draw the directed straight line $QP_1P_2$ tangential to the circle, in the direction of the instantaneous velocity $\beta c$ at $-t'$, such that the points $P_1, P_2$ lie on the two nearest arms of the spiral. These two points receive sharp pulses at the present time, but coming from the location $Q$ at re-
Figure 26: Field line: $\beta=0.95$, $N=8$, $R \in [0 : 5\pi]$
Figure 27: Concentration of $\mathbf{E}$ field along a spiral, indicating synchrotron radiation

Consider an arbitrary observer $K$ sitting somewhere on this plane. Like the observers at $P_1, P_2$, he receives periodic radiation pulses at those moments when the spira-
ral, turning with the same angular velocity \( \omega \), intercepts the point K.

Jackson has shown the same spiral band with the following observation[17], “The form of the angular distribution of radiation ... corresponds to a narrow cone or search light beam of radiation directed along the instantaneous velocity vector of the charge.” He has presented a picture of this search light beam[18]. This picture of the search-light beam can also be found in many other references[15].

We shall summarise some of the lessons we derive from our exercise for the special case \( \beta \to 1 \).

1. An observer located on the plane of the synchrotron at a point K will receive a burst of radiation pulse every time the spiral band intercepts the point K.

2. These radiation pulses will reach him at the regular time interval \( T \), coming from the retarded locations of the particle corresponding to its velocity vector \( c\beta \) pointing towards him. A very important example of this is the radiation coming from astronomical objects, like pulsars, and the Crab Nebula.

3. The radius vector \( \mathbf{R} \) from the retarded location \( Q \) to its corresponding field point \( P \) at the present time, has the following properties: (a) it is tangent to the synchrotron circle, (b) it is perpendicular to the spiral arc at the point \( P \).

4. Hence, the Pointing’s vector, which represents the electromagnetic energy flux density, is perpendicular to every segment of the spiral arm at each of its points.

5. Hence, the arm of the spiral, at any point \( P \), is expanding outward along the radius vector \( \mathbf{R} \) with the velocity \( c \), even as it rotates with the angular velocity \( \omega \).

The reader should prove the statements made in items 3,4,5. He can prove the property 3, using the equation of the involute as given in (56). For proving properties 3 and 4, he can use theorems and formulas of classical electrodynamics.

A Appendix A

A.1 New Differentials in terms of Old Ones

We shall go back to the definition of the new coordinates \((R, \alpha)\), as given in Eqs.(14), and construct the required differentials:

\[
\begin{align*}
dx &= \frac{\partial x}{\partial R} dR + \frac{\partial x}{\partial \alpha} d\alpha \\
dy &= \frac{\partial y}{\partial R} dR + \frac{\partial y}{\partial \alpha} d\alpha
\end{align*}
\] (A.1)

Using Crammer’s formula:

\[
dR = \frac{1}{J} \left| \begin{array}{cc}
\frac{dx}{dy} & \frac{\partial x}{\partial \alpha} \\
\frac{dy}{dx} & \frac{\partial y}{\partial \alpha}
\end{array} \right| ; \quad d\alpha = \frac{1}{J} \left| \begin{array}{c}
\frac{\partial x}{\partial R} \\
\frac{\partial y}{\partial R}
\end{array} \right| dx
\] (A.2)

where \( J \) is the Jacobian of the transformation. To find the derivatives and \( J \), we proceed as follows, going back to Eqs. (14).
\[ \frac{\partial x}{\partial R} = \frac{\partial x}{\partial R} \left( -1/c \right) + \cos(\theta + \alpha) - R \sin(\theta + \alpha) \frac{d\theta}{dR}. \]
\[ = c \beta_x \frac{d\theta}{dR} \left( -1/c \right) + \cos(\theta + \alpha) - R \sin(\theta + \alpha) \frac{d\theta}{dR} \]
\[ = -\beta_x + \cos(\theta + \alpha) + (R \theta/c) \sin(\theta + \alpha). \] (A. 3)

Similarly,

\[ \frac{\partial x}{\partial \alpha} = -R \sin(\theta + \alpha). \]

\[ \frac{\partial y}{\partial R} = -\beta_y + \sin(\theta + \alpha) - (R \theta/c) \cos(\theta + \alpha). \]
\[ \frac{\partial y}{\partial \alpha} = R \cos(\theta + \alpha). \]

\[ J = \left| \begin{array}{cc} \frac{\partial y}{\partial R} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial R} & \frac{\partial x}{\partial \alpha} \end{array} \right| \]
\[ J_1 = \left| \begin{array}{cc} -\beta_x & -R \sin(\theta + \alpha) \\ -\beta_y & R \cos(\theta + \alpha) \end{array} \right| = -\beta_x R \cos(\theta + \alpha) - \beta_y \sin(\theta + \alpha) \]
\[ = -\beta_x R \cos(\Phi + \beta_y \sin(\Phi)) = -R \beta \cdot e_R = \beta \cdot e_{\beta} \cdot e_R = -R \beta \cos \alpha. \] See Eq. 7(a). (A. 4)

\[ J_2 = \left| \begin{array}{cc} \cos(\theta + \alpha) & -R \sin(\theta + \alpha) \\ \sin(\theta + \alpha) & R \cos(\theta + \alpha) \end{array} \right| = R \]
\[ J_3 = \left| \begin{array}{cc} (R \theta/c) \sin(\theta + \alpha) & -R \sin(\theta + \alpha) \\ -(R \theta/c) \cos(\theta + \alpha) & R \cos(\theta + \alpha) \end{array} \right| = 0 \]
\[ J = (1 - \beta \cos \alpha) R. \]

\[ dR = \frac{1}{J} \left( \frac{\partial y}{\partial \alpha} dx - \frac{\partial x}{\partial \alpha} dy \right) = \frac{R \cos(\theta + \alpha) dx + R \sin(\theta + \alpha) dy}{(1 - \beta \cos \alpha) R} = \frac{dr \cdot e_R}{1 - \beta \cos \alpha}. \] (A. 5)

\[ d\alpha = \frac{1}{J} \left( \frac{\partial x}{\partial R} dy - \frac{\partial y}{\partial R} dx \right) \]
\[ = \frac{\left\{ -\beta_x + \cos(\theta + \alpha) + (R \theta/c) \sin(\theta + \alpha) \right\} dy - \left\{ -\beta_y + \sin(\theta + \alpha) - (R \theta/c) \cos(\theta + \alpha) \right\} dx}{(1 - \beta \cos \alpha) R}. \] (A. 6)

We shall simplify Eq. (A. 6), using (6). A part of the numerator simplifies as follows.

\[ \left\{ -\beta_x dy + \beta_y dx \right\} + \left\{ \cos(\theta + \alpha) dy - \sin(\theta + \alpha) dx \right\} = (1 - \beta \cos \alpha) (dr \cdot e_{\beta}) - (\beta \sin \alpha) (dr \cdot e_R). \] (A. 7)
Proof: The RHS of (A. 7)

\[ d\mathbf{r} \cdot \mathbf{e}_\phi - \beta [\cos \alpha (d\mathbf{r} \cdot \mathbf{e}_\phi) + \sin \alpha (d\mathbf{r} \cdot \mathbf{e}_\beta)] \]
\[ = d\mathbf{r} \cdot \mathbf{e}_\phi - \beta [\cos \alpha \{- \sin (\theta + \alpha)dx + \cos (\theta + \alpha)dy\} \quad \text{[from (6b) ]} \]
\[ + \sin \alpha \{\cos (\theta + \alpha)dx + \sin (\theta + \alpha)dy\} \quad \text{[from (6a) ]} \]
\[ = d\mathbf{r} \cdot \mathbf{e}_\phi - \beta \{- \cos \alpha \sin (\theta + \alpha) + \sin \alpha \cos (\theta + \alpha)\}dx \]
\[ + \{\cos \alpha \cos (\theta + \alpha) + \sin \alpha \sin (\theta + \alpha)\}dy \]
\[ = d\mathbf{r} \cdot \mathbf{e}_\phi + \beta [\sin (\theta + \alpha)dx - \cos (\theta + \alpha)dy] \]
\[ = d\mathbf{r} \cdot \mathbf{e}_\phi + \beta \gamma dx - \beta \gamma dy \]
\[ = - \sin (\theta + \alpha)dx + \cos (\theta + \alpha)dy + \beta \gamma dx - \beta \gamma dy \]

\[ Q.E.D. \]

The remaining part of the numerator in Eq. (A. 6) is

\[ (R\dot{\theta} / c) \{\sin (\theta + \alpha)dy + \cos (\theta + \alpha)dx\} \]
\[ = (R\dot{\theta} / c) (d\mathbf{r} \cdot \mathbf{e}_n). \quad \text{(A. 8)} \]

Hence, from (A. 6),

\[ d\alpha = \frac{1}{R} [(d\mathbf{r} \cdot \mathbf{e}_\phi) + (1 - \beta \cos \alpha)^{-1} \times \{(R\dot{\theta} / c) - \beta \sin \alpha\}(d\mathbf{r} \cdot \mathbf{e}_n)]. \quad \text{(A. 9)} \]

A.2 Differential Equation for the Field Line

As shown in Fig. 2(b), \( d\mathbf{r} \) is the displacement vector from P to Q, and \( ds \) is the corresponding infinitesimal arc length. The differential equations we propose to write are based on the following equations.

\[ d\mathbf{r} \cdot \mathbf{e}_R = \frac{E E}{q} ds; \quad d\mathbf{r} \cdot \mathbf{e}_\phi = \frac{E E}{q} ds = \frac{E E}{q} ds. \quad \text{(A. 10)} \]

Inserting these in (A. 5) and (A. 9) we get

\[ dR = \frac{d\mathbf{r} \cdot \mathbf{e}_R}{1 - \beta \cos \alpha} = \frac{E R}{E (1 - \beta \cos \alpha)} ds. \quad \text{(a)} \]
\[ d\alpha = \frac{1}{R} \left[(d\mathbf{r} \cdot \mathbf{e}_\phi) + (1 - \beta \cos \alpha)^{-1} \times \{(R\dot{\theta} / c) - \beta \sin \alpha\}(d\mathbf{r} \cdot \mathbf{e}_n)\right] \]
\[ = \frac{1}{RE} \left[\frac{E_\phi}{E} + (1 - \beta \cos \alpha)^{-1} \times \{(R\dot{\theta} / c) - \beta \sin \alpha\} \right] ds. \quad \text{(b)} \]

We go back to the expression for \( E_R \) and \( E_\phi \) in Eq. (11), rewrite them in a reduced from,

\[ \frac{e^2 R^2 q}{q} E_R = (1 - \beta^2)(1 - \beta \cos \alpha) \quad \text{(a)} \]
\[ \frac{e^2 R^2 q}{q} E_\phi = R \left\{ \beta \sin \alpha + \beta \dot{\theta} (\beta - \cos \alpha) \right\} \]
\[ + (1 - \beta^2) \beta \sin \alpha \quad \text{(b)} \]

We go back to (A. 11a), and using (A. 12a) get

\[ \frac{e^2 R^2 q}{q} \frac{dR}{ds} = \frac{(1 - \beta^2)(1 - \beta \cos \alpha)}{(1 - \beta \cos \alpha)} = 1 - \beta^2. \quad \text{(A. 13)} \]

Similarly, going back to (A. 11b), and using (A. 12a,b) we get
\[
\frac{k^3 R^2 E}{\dot{q}} \frac{d\alpha}{ds} = \frac{1}{R} \left[ \frac{R}{c} \{ \dot{\beta} \sin \alpha + \beta \dot{\theta} (\beta - \cos \alpha) \} + (1 - \beta^2) \beta \sin \alpha \right] \\
+ (1 - \beta \cos \alpha)^{-1} \{ (R \dot{\theta} / c) - \beta \sin \alpha \} (1 - \beta^2) (1 - \beta \cos \alpha) \right] \tag{A. 14}
\]

Finally, dividing the Eq. (A. 14) with (A. 13) we get the required differential equation:

\[
\frac{d\alpha}{dR} = \frac{1}{c} \gamma^2 [\beta \sin \alpha - \beta \dot{\theta} \cos \alpha + \dot{\theta}], \tag{A. 15}
\]

where \( \gamma^2 = 1/(1 - \beta^2) \). See the line following Eq. (10).

It is seen from (8) that

\[
\dot{\beta} \cdot e_\phi = - \dot{\beta} \sin \alpha + \beta \dot{\theta} \cos \alpha. \tag{A. 16}
\]

Hence, the differential equation for the field lines takes the following form, written by Tsien:

\[
\frac{d\alpha}{dR} = \frac{1}{c} \gamma^2 [\dot{\theta} - \dot{\beta} \cdot e_\phi]. \tag{A. 17}
\]

### A.3 Boundary Condition

From Eq. (21) we shall prove the identity:

\[
\text{If, } \tan(\phi_n + \theta) = \frac{\sin(\theta + \alpha_n) - \beta \sin \theta}{\cos(\theta + \alpha_n) + \beta \cos \theta}, \tag{a}
\]

\[
\text{then, } \tan \phi_n = \frac{\sin \alpha_n}{\cos \alpha_n - \beta}. \tag{b}
\]

**Proof:** Let us write

\[
\mu = \phi_n + \theta, \quad \tan \mu = \frac{\sin(\theta + \alpha_n) - \beta \sin \theta}{\cos(\theta + \alpha_n) - \beta \cos \theta},
\]

\[
\tan \phi_n = \tan(\mu - \theta) = \frac{\tan \mu - \tan \theta}{1 + \tan \mu \tan \theta} = \frac{\sin(\theta + \alpha_n) - \beta \sin \theta - \tan \theta[\cos(\theta + \alpha_n) + \beta \cos \theta]}{1 + \sin(\theta + \alpha_n) - \beta \sin \theta \cos(\theta + \alpha_n) - \beta \cos \theta} = \frac{\text{num}}{\text{den}}
\]

\[
\text{num} \times [\cos(\theta + \alpha_n) - \beta \cos \theta] = \sin(\theta + \alpha_n) - \beta \sin \theta - \tan \theta[\cos(\theta + \alpha_n) + \beta \cos \theta]
\]

\[
= \sin \theta \cos \alpha_n + \cos \theta \sin \alpha_n - \tan \theta[\cos \theta \cos \alpha_n - \sin \theta \sin \alpha_n]
\]

\[
= \cos \theta \sin \alpha_n + \sin^2 \theta \sin \alpha_n = \frac{\sin \alpha_n}{\cos \theta}
\]

\[
\text{den} \times [\cos(\theta + \alpha_n) - \beta \cos \theta] = \cos(\theta + \alpha_n) - \beta \cos \theta + [\sin(\theta + \alpha_n) - \beta \sin \theta] \tan \theta
\]

\[
= \cos \theta \cos \alpha_n - \sin \theta \sin \alpha_n - \beta \cos \theta + [\sin \theta \cos \alpha_n + \cos \theta \sin \alpha_n - \beta \sin \theta] \tan \theta
\]

\[
= \frac{\cos^2 \theta \cos \alpha_n - \cos \theta \sin \theta \sin \alpha_n - \beta \cos^2 \theta + \sin^2 \theta \cos \alpha_n + \sin \theta \sin \alpha_n - \beta \sin^2 \theta}{\cos \theta}
\]

\[
= \frac{\cos \alpha_n - \beta}{\cos \theta}.
\]

Hence, \( \tan \phi_n = \frac{\text{num}}{\text{den}} = \frac{\sin \alpha_n}{\cos \alpha_n - \beta} \).
Combining Eqs. (18), (A. 18) we get the boundary condition as
\[ \gamma \tan \left( \frac{2\pi n}{N} \right) = \frac{\sin \alpha_n}{\cos \alpha_n - \beta} \quad (A. 19) \]

Let us now prove a useful inversion formula:

If, \[ \tan \theta = \sqrt{\frac{1 - \beta}{1 + \beta}} \tan \phi, \quad (a) \]

Then, \[ \tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{2 \sqrt{\frac{1 - \beta}{1 + \beta}} \tan \theta}{1 - \left( \frac{1 - \beta}{1 + \beta} \right) \tan^2 \theta} \]

where, \[ \gamma = \sqrt{\frac{1 - \beta}{1 + \beta}} \quad (c) \]

**Proof:** From (A. 20a),
\[ \tan \phi = \frac{\sqrt{1 - \beta}}{1 - \beta} \tan \theta. \]

\[ \tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{2 \sqrt{\frac{1 - \beta}{1 + \beta}} \tan \theta}{1 - \left( \frac{1 - \beta}{1 + \beta} \right) \tan^2 \theta} = \frac{2 \sqrt{1 - \beta} \tan \theta}{(1 - \tan^2 \phi) - (1 + \beta) (1 + \tan^2 \phi)} \]

\[ = \frac{1}{\gamma} \frac{2 \tan \theta}{\cos^2 \phi - \sin^2 \phi} \]

\[ = \frac{1}{\gamma} \frac{2 \tan \theta}{\cos^2 \phi} = \frac{1}{\gamma} \frac{2 \sin \theta \cos \theta}{\cos^2 \phi} = \frac{1}{\gamma} \frac{\sin 2\theta}{\cos 2\theta - \beta}. \]

Q.E.D.

Due to Eq. (A. 19) and identity between the equations (A. 21a) and (A. 21b), the former will now serve as the reduced boundary condition.

### B Appendix B

**B.1 Plot Commands for Gnuplot**

In our previous article we had written the commands used in Gnuplot in full, to encourage the reader to replicate all the plots in that article. In this article we shall be economical, avoid the Preamble parts and go straight into the main commands, copying only six examples from the Console. Using them as a guide, the reader should be able to replot all the field lines.

Note that all plot functions are parametric functions of the form: \( x = f(t), y = g(t) \), in which \( t \) is the parameter. In the first five examples this parameter \( t \) represents the CtoF distance \( R \). In the final example, \( t \) represents the retarded time.

**B.1.1 Example 1.** E field line from a particle moving under constant acceleration. See Fig.5

\[ \text{ac} = 0.17 \times 10^{18} \]
\[ c = 3 \times 10^{18} \]
\[ \text{tau} = c/\text{ac} \]
print tau

1.76470588235294e-09

set term fig color portrait size 16 20 metric points max 1000 solid font "Times-Roman, 12" depth 50
set size square
set parametric
t0=tau
tr(t) = t0-t/c
RO = tau*c
b(t)=1/sqrt( 1+( tau/tr(t) )**2 )
print b(0.001*RO), b(0)

0.706752962498301 0.707106781186547
print b(0.999*RO), b(0)
0.0009999999500000358 0.707106781186547
xt(t)=c*tau*( sqrt(1+( tr(t)/tau)**2 ) -1 )
print xt(RO), xt(0)

0.0 0.21928953302105
a(n,t) =2*atan( sqrt( (1-b(t))/(1+b(t) ) ) )
)tan( pi*n/16 ) )
x(n,t) = xt(t) + t* cos(a(n,t) )
y(n,t) = t* sin(a(n,t) )
set xrange [-RO:RO]; set yrange [-RO:RO]
set trange [0:RO]
set samples 200
set xtics RO/2; set mxtics 5
set ytics RO/2; set mytics 5
set title "Erect-161028A.fig"
set out "Erect-161028A.fig"
do for [n=1:16] plot x(n,t), y(n,t) lt n-1

B.1.2 Example 2. E field lines from a particle accelerating from $t = 0$ to $t = \tau$, then moving with constant velocity. See Fig.8(a)

This exercise invokes piecewise functions $\beta(t')$, $\tilde{x}(t')$, each defined for 3 different ranges in Eq. (38). Page 29 of Gnuplot Manual[6] tells us how to do it. Notice how we have defined the functions $b(c,t)$ and $xt(c,t)$ to represent $\beta(t')$, $\tilde{x}(t')$.

ac = 0.17*10**18
ac=3*10**8
tau = c/ac
t0=2*tau
set parametric
tr(c,t)=t0-t/c
RA2=0
RA1= c*tau
RO= 2*RA1
RB= 3*RA1
bet(c,t)=1/sqrt( 1+ ( tau/tr(c,t) )**2 )
print bet(c,RA1), bet(c,RO), bet(c,RB)

0.707106781186547 0.707106781186547 0 0
print bet(c,RA2), bet(c,RA1), bet(c,RO), bet(c,RB)
0.707106781186547 0.707106781186547 0 0
xta(c,t)=c*tau*( sqrt(1+( tr(c,t)/tau)**2 ) -1 )
print xta(c,RO), xta(c,0)

0.593640181884517 0.21928953302105 0 0
a(n,c,t) = 2*atan( sqrt(1-b(c,t))/(1+b(c,t)) )*tan( pi*n/16 )

x(n,c,t) = xt(c,t) + t*cos(a(n,c,t))
y(n,c,t) = t*sin(a(n,c,t))

set term fig color portrait size 16 20 metric pointsmax 1000 solid font "Times-Roman, 12" depth 50

set size square
set xrange [-1.5:1.5]; set yrange [-1.5:1.5]
set samples 300
set xtics 0.5; set mxtics 5
set ytics 0.5; set mytics 5
set grid
set title "Erect-160522A.fig"
set out "Erect-160522A.fig"
do for [n=1:16] plot x(n,c,t), y(n,c,t) lt n-1

B.1.3 Example 3. E field line from q moving under constant deceleration. See Fig.11

ac = 0.17*10**18
c=3*10**8
tau(ac,c) = c/ac

set parametric dummy variable is t for curves, u/v for surfaces
t0(ac,c) = 0
tr(ac,c,t) = t0(ac,c) - t/c
b(ac,c,t) = 1/sqrt(1+(tau(ac,c)/tr(ac,c,t))**2)
x(t) = c*tau/(sqrt(1+(tau/ac,c,tr(ac,c))**2)-1)
print x(t), x(t,0.528) 0.0 0.218291931024583

set term fig color portrait size 16 20 metric pointsmax 1000 solid font "Times-Roman, 12" depth 50

set size square
set xrange [-0.8:0.8]; set yrange [-0.8:0.8]
set trange [0:0.528]
set samples 200
set xtics 0.4; set mxtics 4
set ytics 0.4; set mytics 4
set grid
set title "Edcel-161003A.fig"
set out "Edcel-161003A.fig"
do for [n=1:16] plot x(n,ac,c,t), y(n,ac,c,t) lt n-1

B.1.4 Example 4. E field lines from q decelerating from $\beta_\infty = -0.707$ during $t = -\tau$ to $t = 0$, then sitting still. See Fig.13

ac = 0.17*10**18
c=3*10**8
tau = c/ac

print tau
1.76470588235294e-09

set parametric
tr(t) = tau - t/c
R2=3*c*tau; R1=2*c*tau; RO=c*tau

print R2, R1, RO
1.58823529411765 1.05882352941176 0.529411767405882

bet(t) = -1/sqrt(1+(tau/tr(t))**2)

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Example 5. Field line from particle in circular motion with $\beta = 0.5$, $N = 16$, $R \in [0 : 13\pi]$. See Fig. 21

$$g(b) = \frac{1}{\sqrt{1-b^2}}$$

defines $\gamma$

$$a(N,n,b,r,t) = 2\tan \left( \frac{\pi n}{N} \right)$$

defines $\alpha$

$$x(N,n,b,r,t) = r \sin (b t / r) + t \cos \left( \frac{b t}{r} + a(N,n,b,r,t) \right)$$

$\# x$ coordinate of field line

$$y(N,n,b,r,t) = r \cos (b t / r) + t \sin \left( \frac{b t}{r} + a(N,n,b,r,t) \right)$$

$\# y$ coordinate of field line

set term fig color portrait size 16 20 metric pointsmax 1000 solid font "Times-Roman, 12" depth 50

set parametric

set trange [0:6*pi]

set xrange [-6*pi:6*pi]; set yrange [-6*pi:6*pi]

set samples 400

set xtics 4*pi; set mxtics 4

set ytics 4*pi; set mytics 4

N=16; b=0.5; r=1

set title "Qcir-b5-16-400-160717.fig"

set out "Qcir-b5-16-400-160717.fig"

for [n=1:N] plot x(N,n,b,r,t) , y(N,n,b,r,t) lt n-1

Example 6. Concentration of E field along a spiral. See Fig. 27

set parametric

set term fig color portrait size 16 20 metric pointsmax 1000 solid font "Times-Roman, 12" depth 50

for [n=1:N] plot x(n,t), y(n,t) lt n-1
\[ x(t) = \sin(t) - t^* \cos(t); \quad y(t) = \cos(t) + t^* \sin(t) \]

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References


[10] Ref. 8, pp. 591-593


[13] Hint from the referee


