Inertial Versus Non-Inertial Frame of Reference

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Submitted on 17-01-2018

Abstract

In this paper, we have solved a rotational dynamics problem both in Inertial and Non-inertial frames of reference.
The work presents a comparison of the analysis in the two frames and comments on mathematical and physical concepts involved in solving the problem. The results are then represented in terms of simulations and solutions are thoroughly discussed, in an effort to give students, a clear picture of the importance of the two frames. The article is aimed at the undergraduate level students.

1 Introduction

Problem-Solving is the key to understanding Physics. There can be more than one way to solve a given problem and often the solutions arrived at by different approaches may not have a prima facie agreement with each other. This could lead to students remaining confused and their gradual loss of interest in problem-solving.
One of the initial steps involved in solving any problem in mechanics is to choose an appropriate frame of reference along with a suitable coordinate system to work with. Now, the frames of reference are broadly classified on the basis of their acceleration, viz, accelerated frame of reference (Non-inertial) and non-accelerated frame of reference (Inertial). Although these frames are introduced in higher secondary Physics, but their real analysis and comparison more or less remains difficult to comprehend even at undergraduate levels. Very frequently the analysis of a problem in the two frames is mixed up yielding erroneous results.
In this paper, a problem of rotational dynamics is designed and structured in such a way that after handling it in both the frames separately, the understanding of the physical parameters w.r.t. both the frames is enhanced.
Rigorous mathematics is involved while analysing the problem in inertial frame
using both the Cartesian and the Spherical polar coordinates. However, it gives an insight of the physical processes involved. The trajectory of the bead so obtained is illustrated to convey complete information about its motion. Contrary to this, the tedious mathematics involved in the inertial frame can be bypassed and the solution can be obtained in a much simpler manner in the non-inertial frame. Again, the trajectory in this frame is also illustrated and discussed. A detailed comparison of the analysis of the given problem in both the frames is also done.

Problem Statement:
A bead of mass ‘m’ slides without friction on a rigid rod rotating at constant angular speed \(\omega\), the rod being inclined at an angle \(\alpha\) with the axis of rotation, as shown in Figure 1. The problem is to find the trajectory of the Bead and compare the results in both the inertial and non-inertial frames. Assume the bead to be initially at rest, at a distance \(r_0\) from the pivoted end.

2 Non-Inertial Frame: A frame of reference attached to the rotating rod

Figure 2 shows the various forces acting on the bead as viewed by an observer in the Non-Inertial Frame. The rod is pivoted at O. \(\vec{r}(t)\) is the instantaneous position vector of the bead along the rod w.r.t the pivoted end and is directed away from the pivot. The forces acting on the bead are the normal contact forces \(N_1\) and \(N_2\) in two different planes perpendicular to each other; the Coriolis force \([-2m(\vec{\omega} \times \vec{\dot{r}})]\) acting perpendicular to the rod in a plane perpendicular to the plane containing \(\vec{\omega}\) and \(\vec{r}\) (the rod); the gravitational force, mg,
acting vertically downwards; the centrifugal force, \(m(r \sin \alpha)\omega^2\) acting radially outwards perpendicular to the axis of rotation.

At any instant, if \(\dot{\omega}\) and the rod are in the plane of the paper, i.e., the YZ-plane, then the Coriolis force acts perpendicular to the rod along the X axis. It is balanced by the normal force, \(N_2\), from the rod on the bead in the opposite (-X) direction. At this instant, the component of the centrifugal force, \(m(r \sin \alpha)\omega^2\), the component of gravitational force, \(mg \sin \alpha\), both acting on the bead perpendicular to the length of the rod and the Normal contact force \((N_1)\) experienced by the bead from the rod, all lie in the plane of the paper (the YZ-plane). Since the motion of the bead is along the rod, as seen by the non-inertial observer, there is no net force acting on the bead perpendicular to the length of the rod. So, we can write,

\[mg \sin \alpha = N_1 + m(r \sin \alpha)\omega^2 \cos \alpha \quad (2.1)\]

At all instants of time, the bead experiences no net force in any direction other than the one along the rod.

If \(\vec{r}(t)\) is the instantaneous position vector of the bead along the rod with respect to the pivoted end, then, by Newton’s second law, the equation of motion of the bead is:

\[m\frac{\vec{r}''}{dt^2} = (m\omega^2 r \sin^2 \alpha + mg \cos \alpha) \vec{r} \quad (2.2)\]

The first term on the RHS of equation (2.2) is the component of centrifugal force along the rod and second term is the gravitational force component along the rod. The corresponding scalar equation is:

\[\ddot{r} = \omega^2 r \sin^2 \alpha + g \cos \alpha \quad (2.3)\]

This is a second order linear, non-homogeneous differential equation that can be solved by the method of undetermined coefficients.

**Solution of the differential equation:**

The characteristic equation is:

\[\Lambda^2 - \omega^2 \sin^2 \alpha = 0\]

The roots are \(\Lambda_1 = +\omega \sin \alpha\) and \(\Lambda_2 = -\omega \sin \alpha\)

Hence the solution of the homogeneous part is

\[r = Ae^{\omega \sin \alpha t} + Be^{-\omega \sin \alpha t}\]

where, A, B are arbitrary constants.

For the particular solution, using method of undetermined coefficient, let \(r(t)=c\) (constant)

Substituting this in equation (2.2) We get,

\[0 = g \cos \alpha + c\omega^2 \sin^2 \alpha\]

\[c = -\frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}\]

The complete solution is therefore

\[r(t) = Ae^{\omega \sin \alpha t} + Be^{-\omega \sin \alpha t} - \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}\]

Using the initial conditions,

At \(t=0\), \(r = r_0\)

\[r_0 = A + B - \gamma \quad \text{where} \quad \gamma = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}\]

At \(t=0\), \(\dot{r} = 0\)
\[ A - B = 0 \]

Therefore, \( A = B = \frac{r_0 + \gamma}{2} \)

Substituting the above values we get,

\[ r(t) = (r_0 + \gamma) \cosh((\omega \sin \alpha) t) - \gamma \] (2.4)

This is the equation of trajectory of the bead in the non-inertial frame. The non-inertial observer sees the bead moving in a straight line along the rod.

The Scilab plot[4] of equation (2.4) is shown in figure 3.

3 Inertial Frame: A frame of reference with respect to which the rod is rotating at an angular velocity \( \vec{\omega} \) making an angle \( \alpha \) with the rod

3.1 Cartesian Coordinates

The Cartesian system of coordinates is most often our default choice in problem solving and we are, somehow, hardwired to use it. The same is done here. Referring to Figure 4, \( \vec{r}(t) \) is the instantaneous position vector of the bead along the rod w.r.t the pivoted end and directed away from the pivot. So, the instantaneous cartesian coordinates of the bead are:

\[
x(t) = r(t) \cos \omega t \sin \alpha \\
y(t) = r(t) \sin \omega t \sin \alpha \\
z(t) = -r(t) \cos \alpha
\] (3.1)

\[ N_1 = mg \sin \alpha - m(r \sin \alpha) \omega^2 \cos \alpha \]
\[ N_2 = | -2m(\vec{\omega} \times \vec{r}) | = 2m\omega t \sin \alpha \] (2.5)

Figure 4: Orientation of the rod in cartesian coordinates. \( \vec{r}(t) \) is the instantaneous...
position vector of the bead from the pivot O. \( r(t) \sin \alpha \) is the instantaneous radius of curvature of the path of the bead. \( r(t) \sin \alpha \cos \omega t \) and \( r(t) \sin \alpha \sin \omega t \) are the instantaneous \( x(t) \) and \( y(t) \) coordinates of the bead respectively.

Figure 5: Forces acting on the bead as observed by an observer in the Inertial frame.

Figure 5 shows the various forces acting on the bead as observed in the inertial frame. The bead experiences the gravitational force, \( mg \), acting vertically downwards and the Normal contact forces \( N_1 \) and \( N_2 \) in two different planes perpendicular to each other. \( mg \cos \alpha \) and \( mg \sin \alpha \) are the components of the gravitational force along the rod and perpendicular to the rod respectively.

Figure 6: Acceleration experienced by the bead observed by an observer in the Inertial frame.

Figure 6 shows the various components of accelerations experienced by bead during its motion. The bead experiences a centripetal acceleration, \( r \sin \alpha \omega^2 \), acting radially inwards towards the axis of rotation. \((r \sin \alpha \omega^2) \sin \alpha\) and \((r \sin \alpha \omega^2) \cos \alpha\) are the components of centripetal acceleration experienced by the bead along the rod and perpendicular to the rod respectively. \( \ddot{r} \) is the instantaneous acceleration of the bead along the rod, directed away from the pivot.

The component of gravitational force, \( mg \cos \alpha \), experienced by the bead along the rod is partly used to provide the necessary centripetal acceleration component, \((r \sin \alpha) \omega^2 \sin \alpha\), to the bead along the rod and the remaining part of the gravitational force component accelerates it down the rod away from the pivot. Hence, the expression for the acceleration of the bead along the
rod is:

$$[\ddot{r} - (r \sin \alpha) \omega^2 \sin \alpha] = g \cos \alpha \hat{r} \quad (3.2)$$

where, $\hat{r}$ is the unit vector along $\vec{r}(t)$.

The corresponding scalar equation can be written as:

$$\ddot{r} = g \cos \alpha + \omega^2 r \sin^2 \alpha \quad (3.3)$$

The equation (3.3) is identical to equation (2.3).

So, substituting the expression for $r(t)$ obtained in equation (2.4) in equations (3.1), we get the parametric equations of the trajectory of the bead as observed in the inertial frame. On plotting the equations (3.1) along with equation (2.4) using Scilab [4], the trajectory of the bead comes out to be helical with a changing radius, as shown in Figure 7.

Thus, we see that both the inertial and the non-inertial observers agree on the values of the acceleration experienced by the bead along the length of the rod [Equations 3.3 and 2.3]. However, unlike the non-inertial observer, the motion of the bead is not linear. Since, the trajectory of the bead is helical in the inertial frame of reference [Figure 7], the inertial observer sees additional components of acceleration other than the component of acceleration $\ddot{r}$, of the bead along the rod.

To determine these components of acceleration we need to write down the equations of motion of the bead in directions normal to the rod.

One of these equations is quite obvious, i.e.,

$$(mg \sin \alpha - N_1) = m(r \sin \alpha \omega^2 \cos \alpha) \quad (3.4)$$

Thus, we get the component of centripetal acceleration $[(r \sin \alpha \omega^2 \cos \alpha)]$ experienced by the bead perpendicular to the length of the rod.

The Normal force $N_2$ acting on the bead also accelerates it. This component of acceleration can be obtained by dividing the magnitude of the normal force $N_2$ in equation (2.5) with the mass of the bead.

This component of acceleration comes out to be equal to $2\omega \dot{r} \sin \alpha$. It is obtained by dividing the magnitude of the normal force in equation (2.5) with mass of the bead.

$$N_2/m = 2\omega \dot{r} \sin \alpha \quad (3.5)$$

Figure 7: Helical trajectory of the bead as observed in the Inertial frame. It has an
increasing radius as the bead slides along the rod away from the pivoted end.

3.2 Spherical Polar Coordinates

Alternatively, the motion of the bead, in the inertial frame can be analysed using Spherical polar coordinates which quite simplifies the mathematics involved in the analysis.

Figure 8: Orientation of the rod in Spherical Polar Coordinates

Figure 8 shows the orientation of the rod in Spherical polar coordinates. The relation between the Spherical polar and Cartesian coordinates is given by:

\[
\begin{align*}
x(t) &= r(t) \sin \theta \cos \phi \\
y(t) &= r(t) \sin \theta \sin \phi \\
z(t) &= r(t) \cos \theta
\end{align*}
\]  

(3.6)

where, \(\theta\) is the polar angle \([0 \leq \theta \leq \pi]\) and \(\phi\) is the azimuthal angle \([0 \leq \phi \leq 2\pi]\).

On mapping the spherical coordinates with our problem, we have:

\[
\begin{align*}
\theta &= \pi - \alpha \\
\phi(t) &= \omega t
\end{align*}
\]  

(3.7)

Since \(\alpha\) is a constant as the rod is inclined at a fixed angle, \(\theta\) also does not change with time. The rate of change of the azimuthal angle \(\phi\) is equal to the angular speed of rotation of the rod, \(\omega\), and is also a constant.

The expression for acceleration in spherical polar coordinates is:

\[
\ddot{\mathbf{r}} = (\ddot{r} - r \omega^2 \sin^2 \theta - r \dot{\theta}^2) \hat{e}_r + (r \dot{\omega} \sin \theta + 2r \omega \cos \theta \dot{\phi}) \hat{e}_\phi + (r \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \omega^2 \cos \theta \sin \theta) \hat{e}_\theta
\]  

(3.8)

where, \(\hat{e}_r, \hat{e}_\phi\) and \(\hat{e}_\theta\) are the radial, azimuthal and polar unit vectors respectively. On mapping with our ongoing analysis, \(\hat{e}_r\) is identical to \(\dot{r}\).

So, the radial equation of motion is:

\[
m[\ddot{r} - r \omega^2 \sin^2 \theta - r \dot{\theta}^2] = mg \cos \alpha
\]  

(3.9)

Using equations (3.7) the above equation reduces to:

\[
\ddot{r} = g \cos \alpha + r \omega^2 \sin^2 \alpha
\]  

(3.10)
As expected, the equation (3.10) is identical to equation (3.3) and (2.3). On going back to equation (3.8), we see that it explicitly gives the other components of acceleration experienced by the bead in the inertial frame. We may refer to the component of acceleration along $\hat{e}_\theta$ as the "polar acceleration", $a_\theta$, of the bead and the component along $\hat{e}_\phi$ as the "azimuthal acceleration", $a_\phi$, of the bead. These are given by:

$$a_\phi\hat{e}_\phi = 2\dot{r}\omega \sin \theta \hat{e}_\phi$$
$$a_\theta\hat{e}_\theta = -r\omega^2 \sin \theta \cos \theta \hat{e}_\theta$$  \hspace{1cm} (3.11)$$

(Since, $\dot{\theta} = 0$ and $\dot{\omega} = \phi(t) = 0$, using $\theta = (\pi - \alpha)$)

The scalar form of above equations can be re-written as:

$$a_\phi = 2\dot{r}\omega \sin \alpha$$
$$a_\theta = r\omega^2 \sin \alpha \cos \alpha$$  \hspace{1cm} (3.12)$$

The so termed "polar acceleration" can be identified with the component of centripetal acceleration experienced by the bead normal to the rod. The "azimuthal acceleration" can be identified with the acceleration experienced by the bead due to the Normal contact force $N_2$ acting on it [as shown in the equation (3.5)].

### Torque and Angular Momentum:

If $\vec{L}$ is the angular momentum of the bead about the pivot, then we can write

$$\vec{L} = \vec{r} \times \vec{m}\ddot{r}$$  \hspace{1cm} (3.13)$$

Differentiating the above equation w.r.t. time, we get the rate of change of angular momentum of the bead as,

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{m}\ddot{r} + \vec{r} \times \vec{m}\dddot{r}$$

which reduces to

$$\dot{\vec{L}} = \vec{r} \times \vec{m}\dddot{r}$$  \hspace{1cm} (3.14)$$

Using $\vec{r} = r\hat{e}_r$ and substituting for $\dddot{r}$ from equation 3.8, the expression for $\dot{\vec{L}}$ becomes,

$$\dot{\vec{L}} = r\hat{e}_r \times [m((\dot{r} - r\omega^2 \sin^2 \theta)\dot{e}_r + (2r\omega \sin \theta)\dot{e}_\phi - (r\omega^2 \sin \theta \cos \theta)\dot{e}_\theta)]$$

$$\dot{\vec{L}} = mr[2r\omega \sin \theta(\dot{e}_r \times \dot{e}_\phi) - r\omega^2 \sin \theta \cos \theta(\dot{e}_r \times \dot{e}_\theta)]$$  \hspace{1cm} (3.15)$$

Since $\theta = \pi - \alpha$,

$$\dot{\vec{L}} = mr[-2r\omega \sin \alpha \dot{e}_\theta + r\omega^2 \sin \alpha \cos \alpha \dot{e}_\phi]$$  \hspace{1cm} (3.16)$$

If $\vec{\tau}$ is the torque experienced by the bead about the pivot, then,

$$\vec{\tau} = \dot{\vec{L}}$$

Therefore,

$$\vec{\tau} = r[-2m\dot{r}\omega \sin \alpha \dot{e}_\theta + m(r \sin \alpha)\omega^2 \cos \alpha \dot{e}_\phi]$$
The negative sign in the first term shows that this component of torque acts to decrease the angular momentum of the bead. We re-write the above equation as

\[ \vec{\tau} = -\tau_\theta \hat{e}_\theta + \tau_\phi \hat{e}_\phi \]  \hspace{1cm} (3.18)

where,

\[ \tau_\theta = r^2 m \dot{r} \omega \sin \alpha \]
\[ \tau_\phi = rm (r \sin \alpha) \omega^2 \cos \alpha \]  \hspace{1cm} (3.19)

The \( \tau_\phi \) component can be shown to be equal to the resultant torque experienced by the bead about the pivot due to the gravitational force component, \( mg \sin \alpha \), and the normal contact force \( N_1 \).

\[ \tau_\phi = r (mg \sin \alpha - N_1) \]  \hspace{1cm} (3.20)

Substituting for \( N_1 \) from equation (2.5) in equation (3.20), we get

\[ \tau_\phi = rm (r \sin \alpha) \omega^2 \cos \alpha \]  \hspace{1cm} (3.21)

which is identical to its expression in equation (3.19).

Similarly, the \( \tau_\theta \) component can be shown to be equal to the torque experienced by the bead about the pivot due to the Normal contact force \( N_2 \) acting on it.

\[ \tau_\theta = rN_2 \]  \hspace{1cm} (3.22)

Substituting for \( N_2 \) from equation (2.5), we get,

\[ \tau_\theta = r^2 m \dot{r} \omega \sin \alpha \]

which is identical to its expression in equation (3.19).

4 RESULTS AND DISCUSSIONS

Arriving at a detailed solution of the motion of the bead, equations of trajectories are obtained in both the frames. The non-inertial observer sees the bead moving along the rod, away from the pivot, i.e., the trajectory of the bead is linear [Figure 3]. On the other hand, the inertial observer sees the rod rotating, and the bead sliding along it away from the pivot. The trajectory of the bead in the inertial frame is helical with increasing radius [Figure 7].

Both the observers agree on the acceleration experienced by the bead along the rod and rightly so. However, the inertial observer also sees the bead accelerating in other directions in addition to the one along the rod. Having a closer look at equation (3.12), we see that \( a_\phi \) multiplied by the mass of the bead corresponds to the Coriolis force acting on the bead in the non-inertial frame. Similarly, \( a_\theta \) multiplied by mass of the bead corresponds to the component of centrifugal force experienced by the bead normal to the rod in the non-inertial frame.

This illustrates an important concept: “The only difference in writing equation of motion in the two frames is that the acceleration term in the inertial frame turns into a fictitious force term in the accelerating frame and appears on the other side of the equation”.

It can be seen that the analysis of trajectory
of the bead in the inertial frame requires quite an elaborate mathematical skill. The mathematics is somewhat simplified when we use spherical polar coordinates to analyze acceleration of the bead and other dynamical aspects of its motion like torque and angular momentum in the inertial frame. The advantage of tackling the problem in the non-inertial frame lies in the ease of evaluating acceleration of the bead along the length of the rod, in terms of the mathematical complexity. The determination of the Normal Contact forces is also quite straightforward in the non-inertial frame.

5 Conclusion

The comparison of the analysis of the given problem in both inertial and non-inertial frames highlights the significance of both frames with the pivotal elements of each in focus. The choice of a reference frame could be made on the basis of what aspect of the problem is to be analyzed. The present work attempts to reveal the advantage of using a suitable coordinate system that makes use of the symmetry (if any) of a given problem. This goes a long way in easing out the mathematics needed to solve the problem.

6 References


