Examining the Decay Constants of Radioactive Dice Experiment

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Abstract

Radioactive dice experiment is widely used as a pedagogical tool to demonstrate the phenomenology of radioactivity in classrooms. The decay constants obtained in such experiments are found to be consistently higher than the values predicted by the exponential nuclear decay law. It was suggested by some authors that the discrepancy could be minimized by using polyhedral dice having higher number of faces. In this article, some analytical attempts have been made to look for better numerical formulae which could minimize the discrepancy between the probabilistic prediction for dice experiment and the predictions based on exponential nuclear decay law. It was observed that the probabilistic prediction closely approaches the prediction based on exponential nuclear decay law under two different conditions: (i) when the data corresponding to a large number of throws are used, and (ii) when polyhedral dice having higher number of faces are used. Comparatively, the prediction based on the first condition yield better results than the second one.

1 Introduction

Radioactive decay of heavy nuclei is usually simulated in the laboratory or class room of higher secondary schools and undergraduate colleges by rolling dice [1–3]. The most common dice being a cube whose six faces are usually marked by numbers like 1 to 6. Rolling of one such dice would yield an outcome which has the probability of occurrence 1/6. Flipping a coin, which has two
faces, would yield an outcome which has the probability of occurrence 1/2. Polyhedral dice having 8,10,12 or more faces can also be used but these are relatively costlier and not readily available in the market. So, in general, cubic dice with six faces are the most widely used dice in the simulation of radioactive nuclei as these are relatively cheaper and readily available.

A large collection of six-faced dice is thrown simultaneously. The dice showing a particular number (say, for example a ‘3’) are deemed to have decayed like radioactive nuclei and all the other dice showing the numbers other than three (i.e., 1,2,4,5 and 6) are taken as ‘undecayed’ nuclei. All the dice with outcome ‘3’ are removed and the remaining ‘undecayed’ dice are counted. This number of ‘undecayed’ dice is recorded and represents the number of undecayed nuclei remaining after a certain interval of time. The ‘undecayed’ dice are then thrown and, again, those showing a ‘3’ are removed and the remainder counted. This is repeated for many times and the number of ‘undecayed’ dice is counted every time. Such dice rolling experiment is meant to represent decay of a particular species of radioactive nuclei with a certain decay constant $\lambda$. It was pointed out by Murray and Hart [1] that the ‘decay constant’ obtained from dice rolling experiment was consistently higher, on average, than the value predicted by the theory of nuclear disintegration. They have quantitatively shown that the cause of discrepancy in the dice decay experiment is the choice of six faced cube. If, instead, dice having more number of faces are used, then the discrepancy would be minimized. Greater the number of faces of dice, lesser would be the discrepancy.

In this article, we attempt to obtain numerical formulae which are mathematically consistent with the exponential decay law representing the real nuclear decay. Through simple analytical steps we arrive at two different results corresponding to two different mathematical conditions. One of the conditions leads us to the same conclusion as suggested in [1] while the other condition leads us to a new formula. The values of decay constants yielded by this new formula are found to be more closer to those predicted by the exponential decay law, provided, one compiles enough data corresponding to those dice throws whose ‘mass throw numbers’ are equal to or greater than the number of faces of the dice. For this, we need to take a very large number of dice at the beginning so that we can repeat the dice throw for a large number of times without being exhausted of the dice within a few intervals of time.

2 Theory of decay of real nuclei

In the decay of real nuclei, the rate of disintegration at any instant of time is directly proportional to the number of nuclei available at that instant. If $N$ be number of nuclei present at an instant of time $t$, $\Delta N$ be the decrease in the number of nuclei within a small
time interval $\Delta t$, then

$$\frac{\Delta N}{\Delta t} = \lambda N. \quad (1)$$

The decay constant $\lambda$ represents the probability per unit time that a nucleus can undergo disintegration. It is evident from (1) that the decay rate depends not only on $\lambda$ but also on $N$. If $N_0$ be the number of nuclei at the beginning and $N_t$ be the number of nuclei remaining after a time $t$, then we have

$$N_t = N_0 e^{-\lambda t} \quad (2)$$

The half life period of the radioactive nuclei $t_\frac{1}{2}$ is defined as

$$t_\frac{1}{2} = \frac{ln2}{\lambda} \quad (3)$$

3 Theory of the decay of the radioactive dice

A large collection of dice, each having ‘$s$’ number of surfaces, is thrown simultaneously. The dice showing a particular face are deemed to have decayed like radioactive nuclei and all the other dice are taken as ‘undecayed’ nuclei. We choose a constant time interval $\Delta t$ between any two successive throws. Let us throw $N_0$ dice simultaneously by the end of first time interval. Since we are taking polyhedral dice having $s$ faces, the probability that a dice decays in time interval $\Delta t$ is $1/s$. From the theory of probability, then, the number of remaining undecayed dice after the first throw is given by $N_1 = N_0(1 - 1/s)$ .After n simultaneous mass throws the number is

$$N_n = N_0(1 - \frac{1}{s})^n. \quad (4)$$

Now, the probability per unit time and hence, by definition, its decay constant $\lambda$ is given by

$$\lambda = \frac{1}{s\Delta t}. \quad (5)$$

Since $\Delta t$ is the time interval between two successive throws, the total time elapsed $t$ after $n$ mass throws is given by $t = n\Delta t$. Therefore, $n = s\lambda t$. Substituting this in (4) we obtain

$$N_n = N_0(1 - \frac{1}{s})^{s\lambda t}. \quad (6)$$

Now, we will make use of a well known formula of limits which goes as

$$\lim_{s \to \infty} (1 - \frac{x}{s})^s = e^{-x} \quad (7)$$

If we choose dice having large $s$, then we can employ (7) so as to express (6) in the approximate form

$$N_n = N_0 \left[ \lim_{s \to \infty} (1 - \frac{1}{s})^s \right]^{\lambda t} = N_0 [e^{-1}]^{\lambda t} = N_0 e^{-\lambda t} \quad (8)$$

Thus, we find, for large value of $s$, that the prediction of (4) would be close to that of (2). Recall that (2) represents real decay of nuclei. In other words, the discrepancy can be reduced by using polyhedral dice having higher number of faces. This result is in agreement with the findings of [1]. Alternatively, we can use the relation $n = s\lambda t$ and express (4) as

$$N_n = N_0 \left( 1 - \frac{\lambda t}{n} \right)^n. \quad (9)$$
For large values of \( n \), one can use Eq.(7) to write Eq.(9) approximately as

\[
N_n = N_0 \lim_{s \to \infty} \left( 1 - \frac{\lambda t}{n} \right)^n = N_0 e^{-\lambda t}.
\]  

(10)

This means that the decay constant measured from dice experiment approaches the value of decay constant predicted by the real nuclear decay law when \( n \) is large. In other words, we can say that the discrepancy can also be minimized by throwing the dice a large number of times instead of using dice having large \( s \).

4 Numerical estimation of decay

In the large \( s \) approximation, the number of undecayed dice can be predicted by using (4). In the case of large \( n \) approximation, one cannot employ (9) directly to make numerical predictions as this equation involves \( \lambda \) and \( t \). We make the following rearrangements in (10), again, by using (7) with \( x = 1 \):

\[
N_n = N_0 e^{-\lambda t} = N_0 e^{-1} \lambda t = N_0 \left[ \lim_{n \to \infty} (1 - \frac{1}{n})^n \right]^{\lambda t}.
\]

On dropping the limit and using \( \lambda t = n/s \) in the above result, we obtain the numerical formula for large \( n \) approximation as

\[
N_n = N_0 \left( 1 - \frac{1}{n} \right)^{n^2}.
\]  

(11)

Obviously, the formula given in (11) would be valid for large values of \( n \). The values of \( \lambda \) obtained by using (11) are henceforth termed as those obtained under large \( n \) approximation (LnA). On the other hand, the results obtained by using (4) are termed large \( s \) approximation (LsA) results. On the same footing, decay constants given by the exponential nuclear decay law, that is (2), are called real nuclear decay (RnD) results. Note that (4) and (11) are expressed in terms of the variables \( n \) and \( s \) while (2) involves the variable \( t \). So, for the sake of uniformity of variables used in (2), (4) and (11), the exponential law of (2) can be written in terms of \( n \) and \( s \) instead of \( t \). Using in (2), we simply get

\[
N_n = N_0 e^{-n/s}.
\]  

(12)

Now, we compare the numbers of undecayed dice or nuclei as predicted by Eq.(4) (LsA), Eq.(11) (LnA) and Eq.(12) (RnD) and obtain their respective decay constants.

Table 1: Numbers of undecayed dice/nuclei with \( N_0 = 10000 \) and \( s = 6 \)

<table>
<thead>
<tr>
<th>Throw number</th>
<th>Numbers of undecayed dice/nuclei</th>
<th>Throw number</th>
<th>Numbers of undecayed dice/nuclei</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>(LsA)</td>
<td>(LnA)</td>
<td>(RnD)</td>
</tr>
<tr>
<td>1</td>
<td>8033</td>
<td>8465</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>6944</td>
<td>6565</td>
<td>21</td>
</tr>
<tr>
<td>3</td>
<td>5795</td>
<td>5443</td>
<td>22</td>
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<td>4</td>
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<td>4346</td>
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</tr>
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<td>1938</td>
<td>2231</td>
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<td>1613</td>
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<td>29</td>
</tr>
<tr>
<td>10</td>
<td>1412</td>
<td>1566</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 1 shows the numbers of undecayed dice/nuclei after various throws for \( N_0 = 10000 \) and \( s = 6 \). A glance at table 1 reveals that (i) LnA predictions are closer to the RnD predictions than those predicted by LsA when \( \lambda \), (ii) LnA cannot predict \( N_1 \), that is the number of undecayed dice after the first throw. But, this drawback is insignificant in the calculation of decay constant,
and (iii) the number of undecayed dice after every throw in both, LnA and LsA, are smaller than that in RnD. For \( n > s \), the number of undecayed dice in LnA is more than that in LsA. We have plotted \( \ln \frac{N_0}{N} \) for three different ranges of \( n \) in figure 1. For uppermost graph \( n \) runs from 2 to 10, for middle graph \( n \) goes from 7 to 17 and for lowermost graph \( n \) is from 30 to 40. Note that for the last two graphs, we have \( n > s \). These are straight lines whose slopes give the values of decay constants \( \lambda \). The LsA values of \( \lambda \) for all the three ranges of \( n \) are consistently higher (\( \lambda = 0.18 \)) than the RnD values (\( \lambda = 0.167 \)) . The LnA yields relatively poor result (\( \lambda = 0.163 \)) in the uppermost graph while it gives better results in the middle (\( \lambda = 0.166 \)) and lowermost (\( \lambda = 0.166 \)) graphs. This is expected as the condition of validity of LnA is \( n > s \). In order to check the validity of LsA (4) and LnA (11) for other values of \( n \) and \( s \), we generated data similar to table 1 for \( s = 10, 20 \) and 40 and obtained the value of \( \frac{\ln(N_0/N)}{n} \) for each value of \( n \) and plotted them. The resulting graphs are presented in figure 2. Thus, we observe that the LnA-formula (11) appears to predict reasonably good results for \( n > s \) while it is not reliable for \( n < s \). The accuracy of LsA results increases with increasing values of \( s \). For \( s = 6 \), the difference between LsA and RnD result is about 9% and this difference gradually decreases to 5%, 2.6% and 1.2% for \( s \) equal to 10, 20 and 40 respectively. Interestingly, LnA yield better results than the LsA for \( n > s \).

Figure 1: Plots of \( \ln(N_0/N) \) for three different ranges of mass throw number \( n \). Here, \( N_0 = 10000 \) and \( s = 6 \). The throw number \( n \) runs from 2 to 10 in the uppermost panel, it goes from 7 to 17 in the middle panel and it runs from 30 to 40 in the lowermost panel. The red solid line represents the real nuclear decay (RnD), the blue dotted line denotes dice decay in large \( n \) approx.(LnA) and the green dashed line shows dice decay in large \( s \) approximation (LsA).
5 Conclusion

The accuracy of predictions of the LsA-formula given in (4) is found to increase with increasing values of s. Thus, the discrepancy between the rolling dice experiment and the theoretical formula (4) can be minimized by using polyhedral dice having large number of faces.

But, polyhedral dice having 8,10,12 or more faces are relatively costlier and not readily available in the market. Therefore, cubic dice with six faces are the most widely used dice in the simulation of radioactive nuclei.

In the preceding sections, we have seen that the LnA-formula (11) predict reasonably good results for \( n > s \). Therefore, it appears to be possible to perform rolling dice experiment even with six-faced cube and still obtain a better value of decay constant by using LnA-formula given in (11). This means that in the actual dice experiment one has to take a large number of data so that she is left with enough data even after discarding the data of the first \( s \) throws. In this scenario, one can use even cubic dice and obtain still better result. However, in order to consider only those data for which the throw numbers are greater than the number of faces of dice, i.e., data satisfying the condition \( n > s \), one has to begin with a large number of dice and record a large number of data. This result (11) may be also extended to other simulation activities involving exponential decay.

References

