Vibrations of a Circular Membrane - Some Undergraduate Exercises

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Abstract

In this paper we revisit the vibrational modes of a circular membrane with different boundary conditions. This we hope will serve as an exercise to get more insight into the study of percussion instruments. The displacement for various modes are found for two initial velocities and two initial displacements. The first three modes are plotted for both cases. We found that the different initial velocities and initial displacements does not change the frequency or shape of different modes. Such an exercise, we believe, will help the students to understand the importance of the concept of modes associated with vibrations.

1 Introduction

The essential component in the sound production of percussion drums is the vibration of a circular membrane. When the drum head is struck, the circular membrane vibrates in different modes. The basic equation governing the vibration of a circular membrane is the wave equation, which is given by [1]

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)
\] (1)

where \(u = u(r, \theta, t)\) is the displacement of the membrane and the velocity of sound wave \(c = \sqrt{T/\rho}\) where \(T\) is the tension on the membrane and \(\rho\) is the uniform mass density. Using the separation of variable technique, the solution obtained is [2]

\[
u(r, \theta, t) = \sum_{m=1}^\infty \sum_{n=0}^\infty J_n(k_{mn}r) \cos(n\theta) [a_{mn} \cos(ck_{mn}t) + b_{mn} \sin(ck_{mn}t)]
\] (2)

Here \(m\) and \(n\) are integers, where \(m\) represents the number of nodal circles and \(n\) represents the number of nodal lines. There are certain regions on the membrane where
there is no motion or vibration. When the non vibrating region is a circle it is called a nodal circle and when it is a line, the same is called a nodal line. \( J_n(k_{mn}r) \) is the \( n^{th} \) order Bessel function where \( k_{mn} \) is the wave number and \( r \) is the radius. \( a_{mn} \) and \( b_{mn} \) are the constants to be determined. For an axis symmetric circular membrane the displacement \( u(r, \theta, t) \) is independent of \( \theta \) and then we modify our wave equation (1) as

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \tag{3}
\]

For axis symmetric membrane no nodal lines are present and hence we fix our \( n = 0 \) and then equation (2) changes to

\[
u(r, t) = \sum_{m=1}^{\infty} J_0(k_m r) \left[ a_m \cos(ck_m t) + b_m \sin(ck_m t) \right] \tag{4}
\]

where \( a_{m0}, b_{m0} \) and \( k_{m0} \) are redefined as \( a_m, b_m \) and \( k_m \).

2 **Values of the coefficients** \( a_m \) and \( b_m \)

At time \( t = 0 \), we get the displacement of the membrane from Eq (4) as

\[
u(r, 0) = \sum_{m=1}^{\infty} J_0(k_m r)a_m
\]

Let

\[
u(r, 0) = f(r)
\]

We have a theorem given in the book “Fourier Series and Boundary Value Problems” by James Brown and Ruel Churchill (pp 275, theorem 1) which states that when \( F(q) \) is a continuous function in the interval \( 0 < q < p \) and if \( \alpha_j \) are the positive roots of equation

\[
J_0(\alpha_j p) = 0
\]

then the function \( F(q) \) can be written as a Fourier-Bessel series

\[
F(q) = \sum_{j=1}^{\infty} A_j J_0(\alpha_j q)
\]

where

\[
A_j = \frac{2}{p^2 J_1(\alpha_j p)^2} \int_0^p q F(q) J_0(\alpha_j q) dq
\]

Our drum has a radius \( R \) and we can define an interval \( 0 < r < R \). At \( R \), the boundary is fixed and there is no vibration. The radial part of the solution in (4) at the boundary becomes

\[
\sum_{m=1}^{\infty} J_0(k_m R) = 0 \tag{5}
\]

Then for that region by the above theorem

\[
a_m = \frac{2}{R^2 J_1(k_m R)^2} \int_0^R r f(r) J_0(k_m r) dr
\]

On differentiating \( u(r, t) \) in (4) with respect to \( t \) we get

\[
\frac{du(r, t)}{dt} = \sum_{m=1}^{\infty} J_0(k_m r) [-ck_m a_m \sin(ck_m t) + c k_m b_m \cos(ck_m t)]
\]

For \( t = 0 \), let \( \frac{du}{dt} = g(r) \) and we get

\[
g(r) = \sum_{m=1}^{\infty} J_0(k_m r)ck_m b_m
\]
Using the theorem stated above, we get
\[ b_m = \frac{2}{c k_m R^2 [J_1(k_m R)]^2} \int_0^R r g(r) J_0(k_m r) dr \]
For the equation (5), let the positive roots be \( \alpha_m \), then
\[ k_m R = \alpha_m \]
\[ k_m = \frac{\alpha_m}{R} \] (6)
Thus we get
\[ a_m = \frac{2}{R^2 [J_1(\alpha_m)]^2} \int_0^R r f(r) J_0(\frac{\alpha_m}{R} r) dr \] (7)
\[ b_m = \frac{2}{c \alpha_m R [J_1(\alpha_m)]^2} \int_0^R r g(r) J_0(\frac{\alpha_m}{R} r) dr \] (8)
From the above equations (4), (7) and (8) we get
\[ u(r, t) = \sum_{m=1}^\infty J_0(\frac{\alpha_m}{R} r) \left[ \frac{2}{R^2 [J_1(\alpha_m)]^2} \int_0^R r f(r) J_0(\frac{\alpha_m}{R} r) dr \cos(c k_m t) + \frac{2}{c \alpha_m R [J_1(\alpha_m)]^2} \int_0^R r g(r) J_0(\frac{\alpha_m}{R} r) dr \sin(c k_m t) \right] \]

3 Defining Cauchy conditions

For a circular membrane, the initial displacement indicates the shape of the drum at initial time. If the initial parameters are known, it is easy to find the parameters for the system at a later time. This is achieved by finding unknowns from the solutions describing the system using initial conditions [4]. Initial conditions are also known as Cauchy conditions. The value of particular unknown function and appropriate number of its derivatives are used to find the solution [5]. Since the wave equation is a second order partial differential equation, displacement and its derivative - velocity are considered as initial conditions. We had already defined the initial displacement of the membrane as \( f(r) \) and initial velocity as \( g(r) \) and we choose three cases of boundary conditions

3.1 Different choices of boundary conditions

1. Case 1
\[ f(r) = 0 \]
\[ g(r) \neq 0 \]
When our initial displacement is zero, the constants \( a_m \) are all zero. The complete solution for vibration of axis symmetric circular membrane from equation (4) is
\[ u(r, t) = \sum_{m=1}^\infty J_0(k_m r) b_m \sin(c k_m t) \] (9)

2. Case 2
\[ f(r) \neq 0 \]
\[ g(r) = 0 \]
When the initial velocity is zero all \( b_m \) values are zero then from equation (4)
\[ u(r, t) = \sum_{m=1}^\infty J_0(k_m r) a_m \cos(c k_m t) \] (10)

3. Case 3
\[ f(r) \neq 0 \]
\[ g(r) \neq 0 \]
In this case $a_m$ and $b_m$ values are non-zero and hence the complete solution is equation (4) itself.

### 3.2 Different types of boundary conditions

In our study we choose two different forms of initial velocity functions

\[ g_1(r) = A J_0 \left( \frac{a_m r}{R} \right) \]  
\[ g_2(r) = B (R - r) \]  

(11)  
(12)

The 3D plot of two initial velocity functions are shown in figure 1. Initial displacement functions used in our study are

\[ f_1(r) = C(R^2 - r^2)^2 \]  
\[ f_2(r) = D(R - r)^2 \]  

(13)  
(14)

The constants A, B, C and D will be properly chosen to match the dimensionality. We have chosen $f_1(r)$ and $f_2(r)$ such that they satisfy boundary conditions, at $r=R$, $f_1(r) = f_2(r) = 0$. The initial displacement functions are plotted in figure 2.

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**Figure 1:** Initial velocity functions.

**Figure 2:** Initial displacement functions.
4 Mode determination using the different initial velocities

Example 1
Let us take the first initial velocity
\[ \varphi_1(r) = AJ_0\left(\frac{\alpha_m r}{R}\right) \]

On substituting the function in Eq (8) we get
\[ b_m = \frac{2}{c\alpha_m R} \int_0^R rAJ_0^2\left(\frac{\alpha_m r}{R}\right) dr \]

Let
\[ \frac{\alpha_m r}{R} = x \]

So
\[ \frac{\alpha_m dr}{R} = dx \]

when \( r = 0, x = 0 \) and when \( r = R, x = \alpha_m \).

We have the standard integral
\[ \int_0^a zJ_0^2(z)dz = \frac{1}{2}a^2 \left( J_0^2(a) + J_1^2(a) \right) \]

On integration
\[ b_m = \frac{AR}{c\alpha_m} \left[ J_0^2(\alpha_m) + J_1^2(\alpha_m) \right] \]

We have from boundary condition
\[ J_0(\alpha_m) = 0 \]
\[ b_m = \frac{AR}{c\alpha_m} \] (15)

\[ u(r,t) = \sum_{m=1}^{\infty} \frac{AR}{c\alpha_m} J_0(k_m r) \sin(ck_m t) \]

Maximum displacement is produced when the drum is struck at centre. At \( r = 0 \) we have \( J_0(0) = 1 \) and hence our expression for deflection of the membrane becomes
\[ u(r,t) = \sum_{m=1}^{\infty} \frac{AR}{c\alpha_m} \sin(ck_m t) \]

Using the equation (6) we have
\[ u(r,t) = \sum_{m=1}^{\infty} \frac{AR}{c\alpha_m} \sin\left(\frac{c\alpha_m r}{R} \right) \] (16)

We have from Eq (4)

Figure 3: First three modes plotted for first initial velocity function.

\[ ck_m = \omega_m \]

Multiplying numerator and denominator with \( R \) we get
\[ ck_m \frac{R}{R} = \omega_m \]
and from Eq (6) we get
\[ \frac{c \alpha_m}{R} = \omega_m \]
So frequency is
\[ f_m = \frac{c \alpha_m}{2\pi R} \] (17)
Here \( c, \alpha_m \) and \( R \) are constants. From equation (17) it is clear that the frequency of different modes of vibration is independent of initial velocity and initial displacement.

5 Calculation of amplitudes for different modes of Timila

To verify the results obtained above, we will apply them to a percussion drum largely used in Kerala, called Timila[7]. Timila has a long resonator body which is made of jack wood. The instrument consists of circular membranes on both heads and are not loaded. The drum does not give a sense of pitch but is used as a rhythmic drum. The main strokes produced by the timila are ‘tha’ and ‘thom’. The instrument is played with palms of both hands. The body of timila has a length of 20.5 in and the diameter of the drum is 6.5 in. When the drum is played, the head vibrates in different modes and these vibrations are transferred to the air molecules in the resonator and the sound is produced. The peculiar construction and playing style adds beauty to the instrument.

We will find the frequency of modes of circular membrane with following parameters - \( R=0.082 \text{m} \) (the radius of Indian rhythmic drum timila [7]) and \( c=130.31 \text{m/s} \). The values of \( \alpha_m \), the positive roots of Bessel function of order zero are taken from the book by Enrique A. Gonzalez-Velasco [6]. The conventional animal membrane of timila drum head is nowadays replaced by mylar membrane. The value of mass density of such a membrane is \( 0.26 \text{kg/m}^2 \) [8] and assuming the value typical tension applied as \( 4415 \text{N/m} \), the \( c \) is obtained as \( 130.31 \text{m/s} \). From Eq (15) the amplitude depend on \( \alpha_m \) and we had taken \( n=0 \) hence for various \( \alpha_m \) we get various \( b_m \). The values obtained for \( b_m \) are tabulated below. The numerical calculation of first mode is shown below with \( A=0.1 \text{m/s} \) and \( \alpha_1 = 2.4048 \). We get

\[ b_1 = \frac{0.1 \times 0.082}{130.31 \times 2.4048} = 2.6167 \times 10^{-5} \text{m} \]
The displacements of first three modes are plotted and given in figure 3. For vibration of circular membrane with applied initial velocity, frequency of the modes remain same but the amplitude changes.

**Example 2**

For second velocity function

$$g_2(r) = B(R - r)$$

(18)

$$b_m = B\frac{2}{c\alpha_m R[J_1(\alpha_m)]^2} \int_0^R r(R-r)J_0\left(\frac{\alpha_m r}{R}\right) dr$$

(19)

We get \(b_m\) after integration and simplification as

$$b_m = B\frac{\pi R^2 H_0(\alpha_m)}{c\alpha_m^3 J_1(\alpha_m)}$$

where \(H_0(\alpha_m)\) is the Struve function of order zero. Struve function is the solution of non homogeneous Bessel equation and usually found in integrals involving Bessel function.

Then we get

$$u(r,t) = B \sum_{m=1}^{\infty} \frac{\pi R^2 H_0(\alpha_m)}{c\alpha_m^3 J_1(\alpha_m)} \sin \left(\frac{c\alpha_m R}{R} t\right)$$

For the same parameters as in example 1 the values of \(b_m\) are tabulated. The \(b_1\) value is calculated with \(B = 1\, s^{-1}\), \(J_1(\alpha_m) = 0.5192\) and \(H_0(\alpha_1) = 0.7497\). The values of Bessel function and Struve function are obtained using online calculators. The other parameter values are same as example 1.

$$b_1 = 1 \times \frac{3.14 \times (0.082)^2 \times 0.7497}{130.31 \times (2.4048)^3 \times 0.5192}$$

$$= 1.6823 \times 10^{-5} m$$

<table>
<thead>
<tr>
<th>Mode</th>
<th>(b_m(m))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,0)</td>
<td>(2.6167 \times 10^{-5})</td>
</tr>
<tr>
<td>(2,0)</td>
<td>(1.13995 \times 10^{-5})</td>
</tr>
<tr>
<td>(3,0)</td>
<td>(7.2717 \times 10^{-6})</td>
</tr>
<tr>
<td>(4,0)</td>
<td>(5.3366 \times 10^{-6})</td>
</tr>
<tr>
<td>(5,0)</td>
<td>(4.2145 \times 10^{-6})</td>
</tr>
<tr>
<td>(6,0)</td>
<td>(3.4822 \times 10^{-6})</td>
</tr>
</tbody>
</table>
The first three modes plotted for second initial velocity function are given in figure 4.

6 Mode determination using the different initial displacements

Example 1

We have

$$a_m = \frac{2}{R^2[J_1(\alpha_m)]^2} \int_0^R r f(r) J_0 \left( \frac{\alpha_m r}{R} \right) dr$$

(20)

We have our first displacement function

$$f_1 = C(R^2 - r^2)^2.$$ On substitution and integration we get

$$a_m = C \frac{128 R^4 - 16 R^4 (\alpha_m)^2}{(\alpha_m)^5 J_1(\alpha_m)}$$

The deflection of the membrane struck at centre is

$$u(r, t) = \sum_{m=1}^{\infty} C \frac{128 R^4 - 16 R^4 (\alpha_m)^2}{(\alpha_m)^5 J_1(\alpha_m)} \cos \left( \frac{c \alpha_m R}{R} t \right)$$

We calculated the numerical value of $a_1$ with same parameter values as in example 1 and 2 in section 4 and the value of $C$ is taken as $10m^{-3}$. So

$$a_1 = 10 \left( 0.082 \right)^4 \left[ 128 - 16 \times (2.4048)^2 \right] \left( 2.4048 \right)^5 \times 0.5192$$

$$= 3.8406 \times 10^{-4}m$$

<table>
<thead>
<tr>
<th>Mode</th>
<th>$b_m(m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,0)</td>
<td>$1.6823 \times 10^{-5}$</td>
</tr>
<tr>
<td>(2,0)</td>
<td>$6.41695 \times 10^{-7}$</td>
</tr>
<tr>
<td>(3,0)</td>
<td>$3.1662 \times 10^{-7}$</td>
</tr>
<tr>
<td>(4,0)</td>
<td>$7.5916 \times 10^{-8}$</td>
</tr>
<tr>
<td>(5,0)</td>
<td>$5.8671 \times 10^{-8}$</td>
</tr>
<tr>
<td>(6,0)</td>
<td>$2.2306 \times 10^{-8}$</td>
</tr>
</tbody>
</table>
the first three modes are plotted in figure 5.

**Example 2**

![Figure 5: First three modes plotted in figure 5.](image)

Now consider the second displacement function \( f_2 = D(R - r)^2 \). On substitution in equation for \( a_m \), we get

\[
a_m = \frac{2D}{R^2 [J_1(\alpha_m)]^2} \int_0^R r(R - r)^2 J_0 \left( \frac{\alpha_m r}{R} \right) dr
\]

On integration we get

\[
a_m = D \left[ \frac{2\pi R^2 H_0(\alpha_m)}{(\alpha_m)^2 J_1(\alpha_m)} - \frac{8R^2}{(\alpha_m)^3 J_1(\alpha_m)} \right]
\]

Here we get the complete solution

\[
u(r, t) = D \sum_{m=1}^{\infty} \cos \left( \frac{\alpha_m r}{R} t \right)
\]

We obtained the value of amplitude \( a_1 \) with same parameters as above and with \( D = 0.01 m^{-1} \)

\[
a_1 = 0.01 \left[ \frac{2 \times 3.14 \times (0.082)^2 \times 0.7497}{(2.4048)^2 \times 0.5192} \right] - 0.01 \left[ \frac{8 \times (0.082)^2}{(2.4048)^3 \times 0.5192} \right] = 3.094 \times 10^{-5} m
\]

<table>
<thead>
<tr>
<th>Mode</th>
<th>( a_m(m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,0)</td>
<td>( 3.094 \times 10^{-5} )</td>
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<tr>
<td>(2,0)</td>
<td>( 1.863 \times 10^{-5} )</td>
</tr>
<tr>
<td>(3,0)</td>
<td>( 4.082 \times 10^{-6} )</td>
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<td>(4,0)</td>
<td>( 3.744 \times 10^{-6} )</td>
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<tr>
<td>(5,0)</td>
<td>( 1.500 \times 10^{-6} )</td>
</tr>
<tr>
<td>(6,0)</td>
<td>( 1.536 \times 10^{-6} )</td>
</tr>
</tbody>
</table>

The first three modes plotted are shown in figure 6.

**7 Modes with both Initial Displacement and Initial Velocity**

In order to find the maximum displacement for case 3, we choose same initial velocity function and initial displacement function
as given in (11) and (14). For a circular membrane struck at centre the complete solution is arrived from equations (4), (16) and (21) as

$$u(r,t) = \sum_{m=1}^{\infty} D \cos \left( \frac{c_\alpha m}{R} t \right)$$

$$\begin{align*}
\frac{2\pi R^2 H_0(\alpha_m)}{(\alpha_m)^2 J_1(\alpha_m)} & - \frac{8R^2}{(\alpha_m)^3 J_1(\alpha_m)} \\
+ \sum_{m=1}^{\infty} \frac{AR}{\alpha_m} \sin \left( \frac{\alpha_m}{R} t \right)
\end{align*}$$

In this case amplitude is the sum of $a_m$ and $b_m$. The amplitudes of first six modes are given in the table below.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$a_m + b_m(m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,0)</td>
<td>$5.7107 \times 10^{-5}$</td>
</tr>
<tr>
<td>(2,0)</td>
<td>$3.00295 \times 10^{-5}$</td>
</tr>
<tr>
<td>(3,0)</td>
<td>$1.1354 \times 10^{-5}$</td>
</tr>
<tr>
<td>(4,0)</td>
<td>$9.0806 \times 10^{-6}$</td>
</tr>
<tr>
<td>(5,0)</td>
<td>$5.7145 \times 10^{-6}$</td>
</tr>
<tr>
<td>(6,0)</td>
<td>$5.0182 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

The first three modes are plotted in figure 7.

8 Conclusion

The circular membrane vibration is studied for different initial velocities and initial displacements. The amplitudes of first six modes are found and the displacements of first three modes are plotted. It is seen that the mode shape remains invariant for any applied initial displacement and velocity. The parameter that changes with the application of initial displacement and velocity is the amplitude of vibration of modes. This study once again show the importance of modes in any vibration. We had assumed that the density of the membrane as uniform. In future we aim to solve vibration of membranes with non uniform densities.
References


