Relativistic Rocket, Its Equation of Motion and Solution for two Special Cases

Somnath Datta

Professor of Physics (Retired),
National Council of Educational Research and Training,
New Delhi-110016
Res: 656, “Snehalata”, 13th Main, 4th Stage, T K Layout,
Mysore 570009, India
datta.som@gmail.com;
http://sites.google.com/site/physicsforpleasure

Submitted on 21-10-2018

Abstract

This article adopts 4-dimensional Minkowski formalism to obtain the equation of motion of a relativistic rocket, i.e., a vehicle in which the exhaust particles are ejected with a fixed relativistic velocity \( u \) opposite the direction of the motion of the rocket, which is taken to be the \( x \) direction. We obtained the equation of motion in the instantaneous rest frame of the rocket, labeled as \( S_0 \), and then converted this equation of motion into the ground frame \( S \). As a prerequisite to this derivation we also reviewed the mass equation of the rocket, i.e., the relationship between the instantaneous rest mass \( M \) of the rocket and its velocity \( v \). We subjected both equations, i.e., the mass equation and the equation of motion to the N.R. test, i.e., the requirement that the forms they assume when \( v \ll c \) (where \( c \) is the velocity of light) converge to their Non Relativistic counterparts. We obtained the solution of the equation of motion in two special cases, namely (i) \( u = c/3 \), and (ii) \( u = c \), and made a plot of the \( v - t \) relationship for both cases. It is seen that the \( v - t \) plot for the case (i) nearly follows the corresponding N.R. counterpart, i.e., \( u \ll c \), up to \( v \approx 0.5c \).

1 Relativistic Rocket

A relativistic rocket, in principle and for all theoretical calculations, is the same familiar rocket the students have studied in their mechanics books [1, 2], with the difference that the exhaust gas is ejected with a “relativistic speed” \( u \) and, as a consequence, the
rocket accelerates to a relativistic speed in due time. What we call relativistic speed is roughly the range: \( c/3 \lesssim u \leq c \), where \( c \) is the speed of light. Because of the relativistic velocities involved in this case Newtonian mechanics breaks down, and we have to use Minkowskian mechanics, in particular Minkowskian equation of motion.

A relativistic rocket, i.e., a space-ship propelled by ejected gas to a relativistic speed is hardly a reality. However, one can still think of matter-antimatter annihilation rockets, pion rockets, for intellectual entertainment\[3\]. The impracticability of operating such contrivances has been illustrated through actual calculations by some authors\[4, 5\], using the example of a photon rocket (i.e., a spaceship propelled by a beam of photons.) Even then the purpose behind our spending time on such an object is somewhat pedagogical. The exercises we are going to undertake are intended to sharpen ones understanding of Minkowskian Equations of Motion, employing 4-vectors.

The mass equation we have derived for a relativistic rocket (see Eq. 28), and the momentum-energy conservation principles used to arrive at it have been covered by several authors\[6, 7, 8, 9\]. Pomeranz wrote several papers on this subject\[10, 11, 12\]. We have written the Equation of Motion (EoM) for the rocket in two equivalent forms as Eqs.(46) and (47). The second form agrees with the 1969 paper (but not with the 1964 paper) of Pomeranz.

Some features of this article that may kindle a special interest in a student or a teacher of special relativity are the following.

1. We have subjected the two important equations derived in this article, namely, (a) the mass velocity equation (28), often referred to as the Ackeret equation\[6\], and (b) the EoM (49) to the N.R. Test, by which we mean that all relativistic equations that have a Non-Relativistic (N.R.) analog must converge to their N.R. counterparts when \( v \ll c \).

2. Taking \( u \) as the ejection velocity of the emitted gas/radiation, we have obtained two special solutions of the EoM, corresponding to (i) \( u = c/3 \), and (ii) \( u = c \). We have plotted the velocity-time relation for both the cases, and shown that the plot for the case (i) closely follows the the plot for the corresponding formula for \( v = v(t) \) obtained using N.R. (Newtonian) mechanics, up to \( v \approx 0.5c \).

3. We have adopted a 4-dimensional Minkowskian approach to obtain the EoM of the rocket, using 4-vectors, e.g., 4-velocity, 4-momentum, 4-acceleration, 4-force. For this purpose we have adopted a mathematical formalism as outlined by Moller\[13\].

In a recent article Bruce\[14\] has demonstrated convergence of the Ackeret equation (28) to its N.R. counterpart Tsiolkovsky
equation (1b), by writing the mass ratio as a product of an infinite series, and obtaining the result using a finite number of terms. In contrast we have demonstrated exact convergence of both the Ackeret equation and the EoM to their N.R. counterparts when $\beta \ll 1$.

2 Symbols and Conventions

All 4-vectors will be presented by a bold letter with a full arrow overhead. While writing its components, the time component, i.e., the $t$-component, will come first, to be followed by its space components in the $x, y, z$ directions. For example, if $\vec{A}$ is a 4-vector, we shall express it as $\vec{A} = (A^t, A^x, A^y, A^z) = (A^0, \mathbf{A})$, where $\mathbf{A} = (A^x, A^y, A^z)$ constitutes a 3-vector. Since the motion of the rocket will be only one-dimensional, confined to the $x$-direction, the $y, z$ components will be absent. The same 4-vector $\vec{A}$ will have only $t$ and $x$ components, i.e., $\vec{A} = (A^t, A^x)$.

The “ground frame” of reference, from which the motion of the rocket is observed will be denoted as $S$. The instantaneous rest frame (IRF) will be denoted by $S_o$.

Time and space components of a typical 4-vector $\vec{A}$, corresponding to motion in the $x$-direction, will be written as $(A^t, A^x)$ with respect to $S$, and as $(A^t, A^x)$ with respect to $S_o$. Note the prime tag “′” attached to the IRF $S_o$.

3 The Rocket, its Specifications

Let us now take a look at the rocket of our discussion. It is moving along the $x$-axis with velocity $v(t)$ m/s with respect to an inertial frame $S$, which, for fixing the idea, we shall call the Ground Frame (GF). It is ejecting gas at a constant velocity $-u$ m/s and its rest mass at a constant rate $r = \frac{d\mu_0}{d\tau}$ kg/s, relative to its Instantaneous Rest Frame (IRF) $S_o(\Theta)$, thereby generating a Reaction force (in this case a Thrust force) $\vec{R}$. Our purpose is to find a formula for $\vec{R}$, and then the Equation of Motion (EoM), and then a solution of this EoM for two representative cases.

Note that we have labeled the IRF with the extra tag $\Theta$ to stress that it coincides with the rocket frame $R$ at the event $\Theta^0$, which, for fixing the idea can be taken as $\Theta^0: \text{“rocket passes a space station } A^0\text{”}$. Every IRF has to be associated with one, and only one, event $\Theta^0$. This association will be useful in the discussions to follow.

It may be worthwhile to stress at this point, even though the reader is aware of it, that $S$ and $S_o$ are both inertial frames and therefore, the components of any 4-vector w.r.t these two frames can be connected by a Lorentz Transformation. Such connection is not possible between the rest frame $R$ of the rocket in which the rocket is permanently at rest and either $S$ or $S_o$, because $R$ is an accelerating frame.

Three quantities are specified for the assessing the performance of the rocket: $u$, $r$
and \( M_i \equiv \) initial rest mass of the rocket at the instant \( t = 0 \), when it starts with zero velocity. In this article \( M = M(\Theta) \) will stand for the instantaneous rest mass of the rocket at the event \( \Theta \). When written as a function of the “ground time” \( t \), it will be appear as \( M(t) \).

Let us consider two infinitely close events \( \Theta_A \) and \( \Theta_B \) (corresponding to the rocket passing two infinitely close space stations \( A \) and \( B \) on its path), the time-space coordinate differentials between them being \((c \, \delta t, \delta x)\) w.r.t \( S \), and \((c \, \delta \tau, 0)\) w.r.t \( S_0 \). Between these events the rocket ejects a quantity of gas of rest mass \( \delta \mu_0 \). Consequently its own velocity changes (i) from \( v(t) \) to \( v(t) + \delta v \) w.r.t the GF \( S \), (ii) from 0 to \( d\nu' \) w.r.t \( S_0 \), and (iii) its rest mass changes from \( M(t) \) to \( M(t) + \delta M \). Note that the time differential between the events being infinitesimally small, \( \delta \tau \) is the proper time between the events. Also, note that, the rate of emission of the rest gas mass w.r.t the rocket frame is

\[
\delta v = -u \frac{\delta M}{M},
\]

\[
v = u \ln \left( \frac{M_i}{M(v)} \right),
\]

\[
M(v) = e^{-\frac{v}{u}},
\]

\[
T = \text{thrust force} = ru.
\]

\[
r = \frac{d\mu}{dt} = -\frac{dM}{dt},
\]

\[
M(v) \frac{dv}{dt} = T = ru,
\]

\[
M_i e^{-\frac{v}{u}} \frac{dv}{dt} = ru.
\]

Hence, \( v = -u \ln \left( 1 - \frac{rt}{M_i} \right) \).

In line (b) \( M(v) \) is the same as \( M(t) \), since \( v = v(t) \). Line (d) in which \( d\mu \) is the mass of

\[4\] Review of the Non-Relativistic (N.R.) results

We shall briefly review the N.R rocket formulas so that we can compare the relativistic results with their N.R. counterparts. We shall drop the subscript “o” from \( d\mu_0 \), because in the N.R. zone there is no such thing as proper mass. The N.R. formulas can be found in standard books on Mechanics. We shall quote the following formulas

\[\begin{align*}
\delta v &= -u \frac{\delta M}{M}, \\
v &= u \ln \left( \frac{M_i}{M(v)} \right), \\
\frac{M(v)}{M_i} &= e^{-\frac{v}{u}}, \\
T &= \text{thrust force} = ru, \\
r &= \frac{d\mu}{dt} = -\frac{dM}{dt}, \\
M(v) \frac{dv}{dt} &= T = ru, \\
M_i e^{-\frac{v}{u}} \frac{dv}{dt} &= ru.
\end{align*}\]
the gas ejected in time $dt$, is a re-statement of conservation of mass. The relationship between the velocity differential and mass differential shown in line (a) is a consequence of (i) conservation of mass, and (ii) conservation of linear momentum. The mass ratio equation (b), known as Tsiolkovsky rocket equation[15], is obtained by integrating the differentials in line (a). Line (e) represents the EoM of the rocket. Line (f) gives the solution of the EoM, subject to the initial condition: $v = 0$ when $t = 0$.

5 Review of the formulas to be used

Let us be specific about the frames of reference, the symbols for velocity, and the Lorentz factors associated with the velocities. All motions are in the X direction. As already mentioned, the ground frame is $S$, the IRF of the rocket (at $\Theta$) is $S_o$. The velocity of the rocket, and hence that of $S_o$, is $v(t)$ w.r.t $S$. The velocity of the ejected gas is constant w.r.t. $S_o$ and equal to $-u$. Let $v(t)$ represent the velocity 3-vector of an arbitrary particle having components $(v_x, v_y, v_z)$ w.r.t $S$ and $(v'_x, v'_y, v'_z)$ w.r.t $S_o$.

In order to avoid future confusion let us remind the reader that $v$ and $\nu$ are two different velocities. The former, i.e., “roman $v$” stands for rocket velocity. The latter i.e. “italicized $\nu$” stands for the velocity of any arbitrary particle, and will be needed for defining the space and time components of 4-velocity, 4-acceleration, 4-momentum and 4-force in general.

This distinction between $v$ and $\nu$ will be removed from Sec. 6 downwards, when the particle in question becomes the same as the rocket itself, momentarily at rest in the IRF $S_o$, and moving with the velocity $v$ w.r.t. $S$.

We now have the following Lorentz-factors (to be abbreviated as L-factors), corresponding to the velocities to be used.

$$g(u) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}; \quad \gamma(\nu) = \frac{1}{\sqrt{1 - \frac{\nu^2}{c^2}}},$$

$$\Gamma(\nu) = \frac{1}{\sqrt{1 - \frac{\nu^2}{c^2}}}; \quad \Gamma'(\nu') = \frac{1}{\sqrt{1 - \frac{\nu'^2}{c^2}}}.$$

It has been our usual practice[16] to write the L-factor associated with the Lorentz transformation with the lower case symbol $\gamma$ and the L-factor associated with the velocity of a particle with the upper case symbol $\Gamma$. We call them by two different names, viz., Boost Lorentz factor and Kinematic Lorentz factor, respectively. In this particular case $\gamma$ is a boost L-factor (corresponding to the boost $S \rightarrow S_o$), and $\Gamma$, $\Gamma'$, as well as $g$, are Kinematic L-factors.

Note that the other specifications (namely, $u$ and $r$) remaining the same, introduction of the L-factors makes the difference between the N.R. case and the relativistic one.

We shall now start a relativistic approach to the motion of the rocket. We shall begin by a review of the 4-vectors to be used in our discussions, and the basic equations of motion.
The kinematic and dynamical 4-vectors

We shall write velocity, momentum, acceleration, force, as 4-vectors, so that (1) we can write the equations of motion covariantly, and make them valid in all inertial frames; and (2) we are able to transform the time and space components using Lorentz Transformation.

Let us introduce two dimensionless velocities

\[ \beta \equiv \frac{v(t)}{c}; \quad \nu \equiv \frac{v}{c}. \quad (3) \]

The transition \( S \rightarrow S_o \), will be called boost \( S(c\beta, 0, 0)S_o \), and the inverse transition \( S_o \rightarrow S \) boost \( S_o(-c\beta, 0, 0)S \). For our purpose the inverse transition is more important.

Suppose \( \vec{A} \) is a contravariant 4-vector, having components \((A^t, A^x)\) in the IRF \( S_o \), and \((A^t, A^x)\) in the GF \( S \). Then by Lorentz transformation the above components transform as follows:

\[ A^t = \gamma(A'^t + \beta A'^x). \quad (a) \]
\[ A^x = \gamma(A'^x + \beta A'^t). \quad (b) \]

We shall now write the \((t, x)\)-components of 4-velocity \( \vec{V} \), 4-momentum \( \vec{P} \), 4-acceleration \( \vec{A} \), 4-force \( \vec{F} \) and obtain their transformation equations corresponding to \( S_o \rightarrow S \). The arrows “\( \rightarrow \)” in some of the formulas below will imply narrowing down of the \((t, x, y, z)\) components to \((t, x)\) components.

4-velocity

Consider a particle moving with arbitrary velocity \( v(t) \). The Lorentz factor for this velocity is \( \Gamma \), as defined in (2). Between two infinitely close events \( \Theta_A \) and \( \Theta_B \) on the world line of the particle, it undergoes a 4-displacement \( \delta \vec{r} \), as its own clock records a time lapse of \( \delta \tau \), and the clock in any inertial frame \( S \) a time lapse of \( \delta t \). Since \( \delta \tau \) is proper time, it is related to the time \( \delta t \) by the equation\[ \delta \tau = \frac{\delta t}{\Gamma}. \]

The displacement 4-vector \( \delta \vec{r} \equiv (c \delta t, \delta r) \) is the primordial contravariant 4-vector from which other contravariant 4-vectors are derived by multiplication or division with 4-scalars, namely \( m_o \), the rest mass (or proper mass of the particle) and \( \delta \tau \), the proper time between the events \( \Theta_A \) and \( \Theta_B \). By this property all the 4-vectors to follow are contravariant 4-vectors.

The 4-velocity of the particle is defined as

\[ \vec{V} \equiv \lim_{\delta \tau \rightarrow 0} \frac{\delta \vec{r}}{\delta \tau} = \frac{d \vec{r}}{d \tau} = \Gamma \frac{d \vec{r}}{d t} = \Gamma \left( \frac{c d t}{d t}, \frac{d r}{d t} \right) = \Gamma(c, \nu) \rightarrow \Gamma(c, v). \]

Note that \( \nu = \frac{d r}{d t} \) is the velocity 3-vector. Let us apply the Lorentz transformation (4) to the components of \( \vec{V} \).

\[ \Gamma c = \gamma (\Gamma' c + \beta \Gamma' \nu'). \quad (a) \]
\[ \Gamma v = \gamma (\Gamma' \nu' + \beta \Gamma' c). \quad (b) \]

Setting \( v = c \nu \), and simplifying, we get the following two important relations.
\[ \Gamma = \gamma \Gamma' (1 + \beta v'). \quad (a) \]
\[ \nu = \frac{v' + \beta}{1 + v' \beta}. \quad (b) \]

Equation (b) is known by the popular name velocity addition formula.

**4-acceleration**

The 4-acceleration \( \vec{A} \) is defined, from which its components are obtained\[17, 18\], as follows.

\[ \vec{A} \equiv \frac{d\vec{V}}{dt} = \left( \frac{\Gamma^4}{c} (a \cdot v), \frac{\Gamma^4}{c^2} (a \cdot v) + \Gamma^2 a \right) \rightarrow \Gamma^4 a \left( \frac{v}{c}, 1 \right) = \Gamma^4 a (v, 1). \quad (9) \]

Here
\[ a \equiv \frac{dv}{dt} \rightarrow a = \frac{dv}{dt} \quad (10) \]
is the acceleration 3-vector.

Let us now apply LT (4) to the \( x \)-component of \( \vec{A} \).

\[ A^x = \gamma (A'^x + \beta A'^i). \]

or, \( \Gamma^4 a = \gamma \Gamma^4 a' (1 + \beta v'). \) \( (11) \)

Let us now suppose that the moving particle, being observed from \( S \) and \( S_0 \), is the rocket itself. In other words the particle is at rest in \( S_0 \), so that \( v' = 0, \Gamma' = 1, \gamma = \Gamma \). In this case the boost L-factor is identical with the kinematic L-factor w.r.t. \( S \). Also, we shall set \( a' = a_o = \) acceleration of the rocket in its rest frame, which can be constant or variable. Then the above equation takes an important form\[17\]

\[ a = \frac{a_o}{\Gamma^3}. \quad (12) \]

For a particle subjected to a finite acceleration \( a_o \) in its rest frame (due to some external force, e.g., an external electric field), its acceleration w.r.t. any inertial frame vanishes as \( v \rightarrow c \), and consequently \( \Gamma \rightarrow \infty \). The above equation protects the rocket from reaching or exceeding the speed of light, even under an ever-continuing acceleration in its rest frame.

**4-momentum**

Let us consider a particle of rest mass \( m_o \) moving with 4-velocity \( \vec{V} \). The 4-momentum \( \vec{P} \) of the particle is then defined as\[18\]:

\[ \vec{P} \equiv m_o \vec{V} = m_o \Gamma (c, v) \rightarrow m_o \Gamma (c, v). \quad (13) \]

If we write
\[ m = \text{relativistic mass} \]
\[ = \Gamma m_o, \quad (a) \]
\[ p = \text{relativistic momentum} \]
\[ = \Gamma m_o v = m v, \quad (b) \]
\[ E = \text{total energy} \]
\[ = \Gamma m_o c^2 = mc^2, \quad (c) \]
then
\[ \vec{P} = \left( E, \frac{E}{c} \cdot p \right). \quad (15) \]

For a mass-less particle, e.g., photon,
\[ E = |p| c. \quad (16) \]

Hence,
\[ \vec{P} = \left( \frac{E}{c}, \frac{E}{c} \cdot n \right) \text{ for a photon}, \quad (17) \]

where \( n \) is the direction of propagation of the photon.

**4-force**
The 4-force \( \vec{F} \), termed Minkowski 4-force is defined in such a way that it will satisfy Minkowski’s Equation of Motion\[18]:
\[ \vec{F} = \frac{d\vec{P}}{d\tau} = \Gamma \frac{d\vec{P}}{dt}. \] (18)

It follows from (15) that
\[ \vec{F} = \Gamma \left( \frac{1}{c} \frac{dE}{dt}, \frac{d\vec{p}}{dt} \right) = \Gamma \left( \frac{\Pi}{c}, \vec{F} \right) \rightarrow \Gamma \left( \frac{\Pi}{c}, \vec{F} \right) \] (19)

where \( \vec{F} = \frac{d\vec{p}}{dt} \) is the 3-force as in Newtonian Mechanics, and \( \Pi \) (Capital-pi) stands for the \textit{power} received by the particle (same as energy received by the particle per unit time), due to (i) work done on it by external forces, and/or (ii) by absorption of radiation or heat (thereby changing its rest mass). In the case of a particle whose rest mass does not change, \( \Pi \) is the same as the power delivered by the force \( \vec{F} \), as shown in Eq. (22) below.

Consider a point particle with a constant rest mass \( m_o \). Due to (13), the equation of motion (18) becomes
\[ \vec{F} = m_o \vec{\ddot{A}} = m_o \frac{d\vec{V}}{d\tau}. \] (20)

In this case the 4-force \( \vec{F} \) is orthogonal to the 4-velocity \( \vec{V} \),
\[ \vec{F} \cdot \vec{V} = 0, \] (21)
and the 4-force takes the form\[18]:
\[ \vec{F} = \Gamma \left( \frac{F \cdot v}{c}, F \right) \rightarrow \vec{F} = \Gamma \left( \frac{F \nu}{c}, F \right). \] (22)

Let us note here that the time-space components of the 4-force written in the form (22) is a consequence of the orthogonality between \( \vec{F} \) and \( \vec{V} \), which in turn is due to the 4-force written in the form (20).

In the sequel we shall write the EoM in the form (20), in which \( \vec{F} \) will be replaced by the reaction 4-force \( \vec{R} \), and \( m_o \) by the instantaneous rest mass \( M \) of the rocket, which is variable. See Eq. (43). Therefore \( \vec{R} \) will have time-space components as in (22).

6 Relativistic mass equation

Even though the relativistic mass formula is well known (known as Ackeret\[6\] equation), we shall make a brief review with two purposes: (1) some of the formulas developed in the process will be required in the sequel; (2) we shall demonstrate that the mass equation and the EoM (to be derived soon) converge to the corresponding formulas (1b) and (1e) in the N.R. limit.

At this point we shall change the velocity symbol for the rocket from \( v \) to \( v \) w.r.t. \( S \), and from \( v' \) to \( v' \) w.r.t. \( S_0 \). Also we shall set \( \beta = \frac{v}{c} \).

As noted in paragraph 5 of Sec. 3 the rocket velocity changes from 0 to \( \delta v' \), and it ejects a quantity of gas of \textit{rest mass} \( \delta \mu_0 \) from the event \( \Theta_A \) to the event \( \Theta_B \) in the IRF \( S_0 \). We shall write the components of the 4-momentum of the rocket at \( \Theta_A \) and \( \Theta_B \), and of the gas ejected between these events - all of them in \( S_0 \).

At this point we draw the attention of
the reader to what we wrote in paragraph 4 of Sec. 3. We emphasize once again that \( M = M(\Theta) = M(t) = M(\tau) \) is the instantaneous rest mass of the rocket at the event \( \Theta \) and hence, is a 4-scalar.

The \( (t, x) \) components of the momentum 4-vectors we shall write below will follow from (13), in which we shall set \( m_o = M(t) \). Also, note that the kinetic L-factors are: \( g \) for the ejected gas, and \( \Gamma' = 1 \) for the rocket, since \( v' = 0 \), i.e., the rocket is momentarily at rest in \( S_o \). The 4-vectors written below have only \( (t, x) \) components, and are valid in the IRF \( S_o \).

\[
\begin{align*}
\text{At } \Theta_A : & \quad \vec{P} = M(c, 0) \quad \text{(a)} \\
\text{At } \Theta_B : & \quad \vec{P} + \delta \vec{P} = (M + \delta M)(c, \delta v') \quad \text{(b)} \\
\text{Change in 4-momentum :} & \quad \delta \vec{P} = (\delta Mc, M \delta v') \quad \text{(c)} \\
4\text{-momentum of the ejected gas :} & \quad \delta \vec{P} = \delta \mu_0 g(c, -u). \quad \text{(d)}
\end{align*}
\]

Hence, (in the limit \( dv' \to 0 \)),

\[
\delta v = \left(1 - \frac{v^2}{c^2}\right) dv'.
\]

This transforms Eq. (24d) to

\[
\frac{dv}{1 - \frac{v^2}{c^2}} = -u \frac{dM}{M}.
\]

Integrating (27) from \( t = 0 \) to \( t = t \), setting \( M = M_i \) (\( i \) for “initial”), and \( v = 0 \) at \( t = 0 \), we get [17]

\[
\frac{c}{2} \ln \frac{c + v}{c - v} = -u \ln \frac{M}{M_i}. \quad \text{(28)}
\]

Or, \( \frac{M}{M_i} = \left(\frac{c - v}{c + v}\right)^{\frac{c}{2}} = \left(\frac{1 - \beta}{1 + \beta}\right)^{\frac{c}{2}}. \)

The above equation is referred to as the Ackeret equation [6].

One of the requirements of all Relativistic formulas is that they must converge to the corresponding N.R. counterparts (if such counterparts exist) in the N.R. limit \( v/c \to 0 \)
Proof: We set $\beta = v/c$. Then in the limit $\beta \to 0$

\[
\frac{1}{1-\beta} \to 0 \quad (1+\beta).
\]

Hence, $\frac{1}{(1-\beta) \to 0 \frac{1}{1+\beta}}$

\[
\left(\frac{c-v}{c+v}\right) \frac{1}{(1+\beta)^2} \to \left[(1+\beta)^{-2}\right] = \left[(1+\beta)^{-2}\right] \left(-\frac{v}{u}\right).
\]

By definition $e \equiv$ Euler's number [19]

\[
\lim_{x \to 0} (1+x)^{1/x}.
\]

Hence,

\[
\left(\frac{c-v}{c+v}\right) \frac{1}{(1+\beta)^2} \to e^{-\frac{v}{u}}.
\]

Or,

\[
\frac{M}{M_i} \to e^{-\frac{v}{u}},
\]

(29)

\[
\begin{align*}
\text{At } \Theta_A : & \quad \vec{p} = M(c,0) \quad \text{(a)} \\
\text{At } \Theta_B : & \quad \vec{p} + \delta \vec{p} = (M+\delta M)(c,\delta v') \quad \text{(b)} \\
\text{Change in 4-momentum} & \\
4\text{-mom. of the emitted photons} & \quad \delta \vec{p} = (\delta E_0/c, -\delta E_0/c) \quad \text{(c)} \\
\text{of a for a photon driven rocket[17].} & \quad M/M_i = \sqrt{\frac{c-v}{c+v}} = \sqrt{\frac{1-\beta}{1+\beta}} \quad \text{(32)}
\end{align*}
\]

7 The Thrust 4-Force

The 4-momentum of the rocket at the event $\Theta_A$ can be written as $\vec{P}(\Theta_A) = M(\Theta_A)\vec{V}(\Theta_A)$. We have used Eq. (13), replaced $m_o$ with $M(\Theta_A)$. The time difference between the events $\Theta_A$ and $\Theta_B$ is $\delta t$ w.r.t. $S_i$ and $\delta \tau$ w.r.t. the IRF $S_o$. Differentiating $\vec{P}$ w.r.t. $\tau$ we get

\[
\text{It follows that Eq. (24d) will be valid for } u = c. \text{ As a consequence (28) is also valid for } u = c. \text{ We shall rewrite this for this special case}
\]
\[
\frac{d\vec{P}}{d\tau} = \frac{d}{d\tau}(M\vec{V}) = M\frac{d\vec{V}}{d\tau} + \frac{dM}{d\tau}\vec{V}. \tag{33}
\]

To evaluate \(\frac{dM}{d\tau}\), we use (24 b)

\[
\frac{dM}{d\tau} = -\gamma \frac{d\mu_0}{d\tau} = -gr,
\tag{34}
\]

as defined in paragraph 1 of Sec.3

To get a parallel formula for the photon-driven rocket we refer to (31 b), and get

\[
\frac{dM}{d\tau} = -\frac{1}{c^2} \frac{dE_o}{d\tau}. \tag{35}
\]

We can combine the two equations into one, assuming that the rocket is ejecting relativistic mass, either in the form of matter or in the form of radiation (we shall use the term radiation to mean photons), at the constant rate of \(\varrho\) kg/s in its rest frame.

For matter emission :

\[
\varrho = \lim_{\delta \tau \to 0} \frac{g}{\delta \tau} \frac{\delta \mu_0}{\delta \tau} = g \frac{d\mu_0}{d\tau} = gr. \tag{a}
\]

For photon emission :

\[
\varrho = \lim_{\delta \tau \to 0} \frac{\delta E_o/c^2}{\delta \tau} = \frac{1}{c^2} \frac{dE_o}{d\tau}. \tag{b}
\]

Note that \(r\) is constant by assumption, and \(g\) is constant because \(u\) is so. Hence \(\varrho\) is constant in line (a). We now assume that if photons are ejected to generate the reaction force, then \(\frac{dE_o}{d\tau}\) is also constant in line (b). Then by (34) - (36)

\[
\frac{dM}{d\tau} = -\varrho = \text{constant} \tag{37}
\]

for both matter and radiation.

We now go back to (33), and rewrite it as follows.

\[
M\frac{d\vec{V}}{d\tau} = \frac{d\vec{P}}{d\tau} - \frac{dM}{d\tau}\vec{V} = \frac{d\vec{P}}{d\tau} + \varrho \vec{V} \equiv \vec{R} \tag{a}
\]

where \(\vec{R} \equiv \frac{d\vec{P}}{d\tau} + \varrho \vec{V}\) (b)

is the “Reaction 4-Force”, or better still the Thrust 4-Force\(^{18}\). However, we are using the symbol \(R\) instead of \(T\), because the latter symbol can be confused with time.

In the following equations we write the \((t, x)\) components of the 4-vectors in \(\text{S}_o\). Using (24 a), (23 d), (36 a):

\[
\frac{d\vec{P}}{dt} = -\frac{d\vec{P}}{d\tau} = -g \frac{d\mu_0}{d\tau}(c, -u) = -q(c, -u).
\]

from (6) :

\[
\vec{V} = (c, 0),
\]

since \(v' = 0, \Gamma' = 1\).

from (38 b) :

\[
\vec{R} = -q(c, -u) + q(c, 0) = q(0, u) = (0, R).
\]

(39)

In other words, the Reaction 4-force has the following components w.r.t \(\text{S}_o\):

\[
\vec{R} = (R^t, R^x) = (0, R)
\]

where \(R = qu = gru = \text{Reaction 3-force, w.r.t.}\ S_o\).

(40)

Note that the \((t, x)\)-components of the Reaction 4-force \(\vec{R}\) in \(\text{S}_o\) are in agreement with (22).

We shall now find the \((t, x)\) components of the Reaction 4-force: \(\vec{R} = (R^0, R^x)\), in the ground frame \(S\), applying Lorentz
transformation Eq. (4), corresponding to the boost: \( S_0(-c\beta, 0, 0)S \), to the \((t, x)\) components of \( \mathbf{\hat{R}} \) in the IRF \( S_0 \), shown in (40).

\[
\begin{align*}
\mathbf{\hat{R}}^t &= \Gamma(\mathbf{\hat{R}}^t + \beta \mathbf{\hat{R}}^x) = \Gamma \mathbf{\hat{R}}. \quad (a) \\
\mathbf{\hat{R}}^x &= \Gamma(\mathbf{\hat{R}}^x + \beta \mathbf{\hat{R}}^t) = \Gamma \mathbf{\hat{R}}. \quad (b)
\end{align*}
\]

(41)

Note that the \((t, x)\)-components of the Reaction 4-force \( \mathbf{-R} \) in \( S_0 \) are in agreement with (22), which we rewrite in the present context as

\[
\mathbf{-R} = (\mathbf{R}^t, \mathbf{R}^x) = \Gamma \left( \frac{\mathbf{R}v}{c}, \mathbf{R} \right). \quad (42)
\]

Referring back to Eq. (36)

- For radiation emission \( R = \varrho c = \frac{1}{c} \frac{dE_o}{dt} \).

- For matter emission \( R = \varrho u = gru = gT \), where \( T = ru \) is the same thrust force of non-relativistic mechanics. See Eq. (1c). It changes to \( R = gT \) as it enters the relativistic domain.

8 The Equation of Motion

We return to Eq. (38 a) and write the equation of motion \[18\]

\[
M \frac{d}{d\tau} \mathbf{\hat{V}} = \mathbf{\hat{R}}, \quad (43)
\]

where \( M = M(\Theta) \) is the instantaneous rest mass of the rocket at the event \( \Theta \), and is a 4-scalar. All we now have to do is to write the \( x \)-component of the 4-vectors on either side of the equation, and simplify the same to obtain the acceleration \( a = \frac{dv}{dt} \) of the rocket in the GF \( S \). We shall, however, find it convenient to work out the acceleration \( a_o = \frac{dv'}{d\tau} \) in the IRF \( S_0 \), and convert this acceleration to \( a \) using Eq (12).

Consider the \( x \)-component of \( \mathbf{\hat{V}} \) using (6). The kinematic quantities in \( S_0 \) will be identified with “prime”. Then,

\[
\frac{dV^x}{d\tau} = \frac{d(\Gamma v')}{d\tau} = \Gamma \frac{dv'}{d\tau} + \frac{d\Gamma}{d\tau} v' = \frac{dv'}{d\tau} = a_o, \quad (44)
\]

since \( v' = \) instantaneous velocity of the rocket in \( S_0 = 0. \) Consequently \( \Gamma' = 1. \)

From (40), the \( x \)-component of \( \mathbf{\hat{R}} \) is \( R^x = R = \varrho u. \) We thus get a simple looking equation of motion, which is valid in \( S_o. \)

\[
M(\Theta) a_o = \varrho u = \text{constant.} \quad (45)
\]

Mass \( \times \) acceleration is constant. But mass is not constant. Hence the acceleration in IRF \( S_0 \) is not constant.

We shall write the EoM in the ground frame \( S \), by converting \( a_o \rightarrow a \), the acceleration in the ground frame \( S \), using [12], which gives \( a_o = \Gamma^3 a. \)

\[
M(\Theta) \Gamma^3 a = \varrho u = \text{constant.} \quad (a)
\]

Or, \( M(\Theta) \frac{d\varrho}{dt} = \varrho u = \text{constant.} \quad (b) \)

(46)

An alternative version of Eq. (46 b) is

\[
M(\Theta) \frac{d}{dt} \left( \frac{\Gamma v}{c} \right) = \varrho u = -\frac{dM}{d\tau} u = -\Gamma \frac{dM}{d\tau} u = \text{constant,} \quad (47)
\]

in which Pomeranz wrote the EoM in his 1969 paper[12]. To see the equivalence between the two forms one has to show that

\[
\frac{d\Gamma}{dt} = \frac{\Gamma^3 v}{c^2} \frac{dv}{dt}.
\]
Now we rewrite Eq. (46 b), using the mass equation (28).
\[ \left( \frac{c - v}{c + v} \right)^{\frac{n}{2}} \Gamma^3 \frac{dv}{dt} = \frac{\rho u}{M_i} = \text{constant}, \] (48)
where \( M_i \) is the initial (rest) mass of the rocket.

We shall set \( \beta = v/c \), and \( c/u = n \) in the index of the leftmost factor in Eq. (48). Here \( n \geq 1 \) is a positive real number greater than or equal to 1. \( n = 1 \) corresponds to \( u = c \). On the other extreme \( n \to \infty \) would converge to the N.R. EoM shown in (1e).

Let us rewrite the mass equation (28), setting \( v = c\beta \), \( c/u = n \).
\[ M = \left( \frac{1 - \beta}{1 + \beta} \right)^{\frac{n}{2}}. \] (50)

We assume that the rocket has no payload, all its mass will ultimately be ejected out to provide the thrust. In other words, the rocket operates until \( M \to 0 \), which happens when \( \beta \to 1 \).

We shall now show that the above EoM (49) will converge to the Non Relativistic EoM as given in (1e). We shall set \( \rho = gr \) as per (36 a), \( \beta \to 0 \) and use the definition of Euler’s number \( e \), as in (29).

**Proof:**

\[ \left[ \frac{(1 - \beta)^{\frac{n-1}{2}}}{(1 + \beta)^{\frac{n+1}{2}}} \right] \frac{d\beta}{dt} \to 0 \left[ \frac{1}{(1 + \beta)^{\frac{n+1}{2}}} \right] \left[ \frac{1}{(1 + \beta)^{\frac{n+1}{2}}} \right] \]
\[ = \frac{1}{(1+\beta)^n} - \frac{1}{(1+\beta)^{3/2}} \]
\[ = \frac{1}{(1+\beta)^{3/2}} \frac{v}{u} = \frac{1}{e^{v/u}}. \]

Substituting this in (49) we get:
\[ M_i e^{-v/u} \frac{dv}{dt} = \rho u = gru = ru, \text{ since } g \to 1. \] (51)

Thus we get back (1e).

**Q.E.D.**

### 9 Solution of the EoM in special cases

**Example 1.** Set \( n = 3 \), implying \( u = c/3 \).

The reason for choosing \( n = 3 \) is two fold: (1) The EoM shown in (49) will assume the simplest form, the numerator within the square brackets becoming 1; (2) We are now at the threshold of transition from the non-relativistic (N.R.) to the relativistic domain, the L-factor is very close to 1, in fact \( g = \frac{3}{2\sqrt{2}} = 1.06 \). Our results obtained here should be close to the N.R. results, so that we may feel comfortable that we are on right track.

We specialize the EoM (49) for this special case:
\[
\left[ \frac{1}{(1 + \beta)^3} \right] \frac{d\beta}{dt} = \frac{\varrho}{3M_i} \equiv k_3 = \text{constant.} \tag{52}
\]

Integration, subject to the initial condition: \( \beta = 0 \) when \( t = 0 \) leads to the following solution:

\[
\beta = \left[ \frac{1}{\sqrt{1 - 2k_3t}} - 1 \right]; \quad 0 \leq t \leq t_c. \tag{53}
\]

The reader can verify the answer by differentiating \( \beta \) w.r.t \( t \).

We have written \( t_c \) to mean \textit{“critical time”} when \( M \to 0 \) as explained below Eq. (50). In other words, \( t_c \) is the time at “burn out”, assuming that the rocket has no payload, all its mass has ultimately been ejected out to provide the thrust. This happens when \( \beta \to 1 \). From Eq. (53):

\[
\left[ \frac{1}{\sqrt{1 - 2k_3t_c}} - 1 \right] = 1
\]

Or,

\[
1 - 2k_3t_c = \frac{1}{4}; \quad \Rightarrow \quad k_3t_c = \frac{3}{8}
\]

Or,

\[
\frac{et_c}{M_i} = \frac{9}{8},
\]

for the N.R. case:

\[
\frac{et_c}{M_i} = 1.
\]

We have plotted the velocity-time relation (rather the \( \beta-t \) relation) in Fig.1(a), using Gnuplot. On the same graph we have also plotted the N.R. equation (51f).

We have set \( M_i = 1 \) kg, \( r = 1 \) kg/s (see specifications in Sec. 3). Setting \( u/c = 1/3 \) in the first of the equations in (2) we get:

\[
g = \frac{3}{\sqrt{8}} = 1.06. \quad \text{Hence } \varrho \text{ (defined in Eq. 56a)} = gr = 1.06 \text{ kg/m.} \]

Therefore \( k_3 = \frac{6}{3M_i} = 0.3535 \). From Eq. (54) the critical time is:

\[
t_c = \frac{3}{8 \times 0.3535} = 1.06 \text{ s which has been set as the upper limit on the } t \text{ axis.}
\]

The two plots, \textit{Relativistic} and \textit{Non-relativistic} almost coincide up to \( t \approx 0.8 \) s, \( \beta \approx 0.5 \).

In Fig.1(b) we have plotted \( \frac{d\beta}{dt} = k_3(1 + \beta)^3 = 0.3535(1 + \beta)^3 \). However, in this case we set the vertical axis to represent the independent variable \( \beta \), aligning it (tic-wise) with the vertical \( \beta \) axis of Fig.(a). The horizontal axis, pointing to the left, represents the dependent variable \( \frac{d\beta}{dt} \). We achieved this configuration by first plotting \( \frac{d\beta}{dt} \) versus \( \beta \) the usual way, then turning the plot anticlockwise by 90°. Our objective here has been to check whether the slope of the \( \beta-t \) curve in Fig (a) is corroborated by the measure of \( \frac{d\beta}{dt} \) in Fig (b). In order to judge the correspondence, we marked four selected points on the curve (a) and their corresponding points on (b), wrote the values of \( \frac{d\beta}{dt} \) for these points on the upper horizontal axis of the plot box. Fair correspondence between these values in Fig(b) and the corresponding slopes in Fig (a) is discernible, suggesting that Eq. (53) is the solution of the EoM (52).

\textbf{Example 2.} \hspace{1cm} Set \( n = 1 \), implying \( u = c \).

This is the photon rocket mentioned in the Introduction. In this case a jet of photons flowing out from the tail end of the rocket is serving as the propellant. We specialize the
EoM (49) for this special case:

\[
\left[ \frac{1}{(1-\beta)(1+\beta)^2} \right] \frac{d\beta}{dt} = \frac{q}{M_i} \equiv k_o = \text{constant.} \\
(55)
\]

Integrating from \( t = 0 \) when \( \beta = 0 \) to \( t = \beta \):

\[
\int_{0}^{\beta} \frac{d\beta}{(1-\beta)(1+\beta)^2} = k_o t, \\
(56)
\]

we get

\[
\frac{\beta}{2(1+\beta)} + \ln \sqrt{\frac{1+\beta}{1-\beta}} = k_o t = \left( \frac{q}{M_i} \right) t.
\]

Or,

\[
t = \left( \frac{M_i}{q} \right) \left[ \frac{\beta}{2(1+\beta)} + \ln \sqrt{\frac{1+\beta}{1-\beta}} \right], \\
(57)
\]

The reader can verify the solution by differentiating \( t \) w.r.t \( \beta \).

We have plotted the velocity-time relation (rather the \( \beta-t \) relation) in Fig.2 (a). However in Eq. (57) \( t \) is a function of \( \beta \). Hence, using Gnuplot we first got \( \beta \) as the horizontal axis and \( t \) as the vertical axis. In order to reverse their roles we transformed the plot by (i) a rotation through 90° in the anticlockwise direction, followed by (ii) a reflection about the vertical axis (i.e. about the new \( \beta \) axis).

In Fig.2 (b) we have plotted \( \frac{d\beta}{dt} \) (its axis pointing left) versus \( \beta \) (its axis pointing upward). The procedure, objective, and explanations are the same as in Example 1.

It should be noted that in this case \( q \) is given by (36 b), which we rewrite and interpret as follows.
\[ \dot{\beta} = \frac{1}{c^2} \frac{dE_o}{d\tau} = \frac{1}{c^2} \times \text{radiative power emitted} \]
from the tail end of the rocket \hspace{1cm} (58)

For the plottings we have taken \( M_i = 1 \) kg, \( \varrho = 1 \) kg/s.

How long does the rocket operate? Until \( \beta \to 1 \), as mentioned below Eq. (50), and therefore, by (57), until \( t \to \infty \).

It is seen from the plot in Fig.2(a) that \( \beta \) approaches unity or, \( v \) approaches \( c \) asymptotically.

10 Summary

1. It has been assumed that the rocket is emitting relativistic mass (in the form of gas/photons) with a constant speed \( u \), opposite to the direction of the motion of the rocket.

2. The rate of ejection of relativistic mass (for both matter and photons) is a constant, represented by \( \varrho \), and defined in Eq. (36).

3. The mass-velocity relation (Ackeret formula) (28) was rederived, and its validity for both matter and radiation was established. See Eq. (31).
4. The mass-velocity formula \((28)\) has been shown to converge to the corresponding N.R. formula \((1b)\). See steps in \((29)\).

5. The 4-dimensional (Minkowskian) equation of motion (EoM), shown in the frame-independent form \((43)\), involves instantaneous mass of the rocket \(M\) (which is a 4-scalar), instantaneous 4-velocity \(\vec{V}\) and the Reaction (Thrust) 4-vector \(\vec{R}\).

6. The same EoM takes the form \((45)\) in the instantaneous rest frame \(S_0\), and, upon Lorentz transformation, the form \((46)\) in the ground frame \(S\).

7. With simple manipulations, the EoM in \(S\) changes to a simpler looking equation \((49)\), where \(n = c/u\).

8. The same EoM \((49)\) has been shown to converge to the corresponding N.R. formula \((1e)\).

9. The EoM \((49)\) was specialized for two cases, namely, (i) \(n = 3\), corresponding to \(u = c/3\), and written as \((53)\), and (ii) \(n = 1\) corresponding to \(u = c\) written as \((57)\).

10. The corresponding EoMs were solved to obtain the relationship between \(v\) and \(t\), written as Eq \((53)\) for the case (i); and Eq. \((57)\) for the case (ii). The velocity-time relations have been plotted for both cases, as shown in Figs \(1\) and \(2\) respectively. For the case (i) we have set \(M_i = 1\) kg; \(r = 1\) kg/s. For the case (ii) we have set \(M_i = 1\) kg; \(q = 1\) kg/s. The following features can be easily noticed. For the case (i) the plot of the relativistic formula \((53)\) follows the corresponding N.R. formula \((1f)\) closely, upto \(t \approx 0.5\) s, or \(\beta \approx 0.8\). For the case (ii), \(\beta \to 1\) asymptotically.

Acknowledgments

The author is grateful to American Association of Physics Teachers, in particular, Prof Rogers Fuller, Associate Director of Membership, American Association of Physics Teachers, and to Harold Q and Charolette Mae Fuller for granting him Fuller Fund Membership of the American Association of Physics Teachers. As a privilege of this membership, which has been continuing since 2001, the author could get online access to all the past issues of the American Journal of Physics, which made it possible to write this article.

References


[17] Wolfgang Rindler, as cited above, p. 64, Eq. (3.2); p. 70, Eq. (3.16); p. 101; p. 127, Ex. 6.13.

[18] C. Moller, as cited above, p. 101, Eq. (4.42); p. 102, Eq. (4.50); p. 103, Eqs. (4.55) (4.54); p. 105, Eqs. (4.61), (4.64).