Dimensional Analysis - Illustrative Examples Using This Tool

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Submitted on 26-08-2018

Abstract

Dimensional Analysis forms a very significant part of study of physics. The students are supposed to read this prior to themselves making inroads into cardinal areas of physics. Basically the students are taught that $[M]$, $[L]$, $[T]$ are the fundamental dimensions and the dimension of every other quantity can be derived from them. The other key factors which are highlighted are that every equation is dimensionally homogeneous. Dimension analysis helps us to understand link between the units/dimension of physical quantity.

It gives very interesting results and helps to solve various unknown problems which would otherwise require a lot of experimental work. Through this article we shall cite quite a few examples to sensitize the students about some of the additional features of dimensional analysis which generally go-unnoticed.

Key Words: Dimensional analysis; Dimensional homogeneity; Physical quantities.

Introduction

Dimensional analysis is a tool used in physics and engineering for deriving theoretical equations, checking empirical formulae, describing experiments, interpreting results from scale models and performing conversions between different systems of units \cite{1}. Bridgman (1931) stated that, “The principal use of dimensional analysis is to deduce from a study of the dimensions of the variables in any physical system certain limitations on the form of any possible relationship between those variables \cite{2}. It is mainly used to find the relations among physical quantities in complicated physical systems by their dimensions.

Dimensional Analysis studies the properties of observable quantities with dimensions and the properties of mathematical relationships that incorporate them \cite{3}. This analysis is applied in the natural sciences; its principles (dimension, homogeneity, measurement and unity) are key to the formation of scientific thought since they are part of the basic principles of science. Compliance with the principles of Dimensional
Analysis, and in particular the principle of dimensional homogeneity, is a basic prerequisite for proper mathematical modelling.

Many researchers have applied dimensional analysis as an analytical tool in various fields such as Geography [1], Biology [4, 5, 6], Economics [7,8,9] and other fields. A book by Don S. Lemons [10] covers the methods, history and formalisation of the field, and provides physics and engineering applications through the mathematical methods of dimensional analysis.

An historical outline of dimension analysis is given by Huntley [11] who credits Newton with the discovery of the “principle of similitude” and Fourier with its development into present method. Several general treatments are available in the literature [2, 12, 13, 14, 15]. Use of the special symbols M, L, and T to denote the dimensions of mass, length, and time has become standard [16]. Physical dimensions refer to the measurement systems to characterise certain objects. Each physical dimension has several empirical scales of measurements and they are called “units”. There are seven fundamental physical dimensions, namely mass M, length L, time T, temperature Θ, electric current I(or charge Q), amount of substance mol and luminous intensity I. The corresponding units defined by SI (International System of Units) are kilogram, metre, second, kelvin, ampere, mole and candela respectively. All other physical quantities are combinations of these fundamental quantities.[17]. The general procedure of applying dimensional analysis is given by W. Shen [17] and others [18,19].

The fundamental purpose of the present research article is to introduce the basic principles of Dimensional Analysis in the context of the real physical problems by citing few examples to sensitize undergraduate students about the additional features of Dimensional Analysis.

**Dimensional Analysis from Student’s perspective: Misconceptions and Applications**

The students are taught to derive equations such as

\[ T = 2\pi \sqrt{\frac{l}{g}} \quad \text{................. (1)} \]

for the time period of oscillation of a simple pendulum, by assuming that ‘\( T \)’ is a function of ‘\( l \)’ and ‘\( g \)’ by expressing

\[ T = k l^a g^b \quad \text{..................(2)} \]

Where, \( k \) = a dimensionless constant and \( a, b \) are pure numbers.

We arrive at the result (Equation (1)) which is restricted to the extent that the value of the constant \( k \) cannot be fixed.

Students are exposed to similar exercises on introduction of \( \varepsilon_0 \) and \( \mu_0 \) in connection with the electric and magnetic units and their links with [M], [L], [T]. While the above referred exercises are useful for the students, it has been felt that dimensional analysis has lot more to offer. As a matter of fact lot of physics, mathematical techniques can be made to evolve from Dimensional Analysis. Like in Eq (1) we have used the product format, but have we pondered over the fact that quantities with different dimensions can be multiplied (e.g. mass x velocity = momentum), or one can be divided by the other (e.g. density = mass/volume), but they cannot be added or subtracted!

Through this article we shall cite quite a few examples as mentioned earlier to sensitize the students about some of the additional features of Dimensional Analysis.
Let us start by narrating an incident given in the following section.

**Dimension of the argument of the exponential function**

It is about a student from physics honours, while attempting a question on Maxwellian distribution of velocities, he could recollect the exponential factor as $e^{-bu^2}$, but he was confused about the value of ‘$b$’, that is whether it is $\frac{m}{2kT}$ or $\frac{2kT}{m}$. As he was feeling restless about it, a small help was offered and while doing so, it was explained to him that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \quad -\infty < x < \infty . \quad (3)$$

So if ‘$x$’ has a dimension, you have to add quantities having different dimensions which is absurd, so ‘$x$’ must be dimensionless. This was something which the student had not realised earlier. So he could immediately make out that ‘$bu^2$’ must be dimensionless, and if $b = \frac{m}{2kT}$, then both the numerator and denominator of ‘$b$’ would have the dimensions of energy which would make ‘$b$’ dimensionless. A student needs to imbibe this fact as a concept and should take every opportunity to verify this. Some illustrative cases are the discharge through an R-C circuit or an L-R circuit. The equations are-

$$q = q_o e^{-t/cR} \quad \text{.................(4)}$$
$$i = i_o e^{-Rt/L} \quad \text{.................(5)}$$

It can be verified that $CR$ or $\frac{L}{R}$ has the dimension of time. We shall now cite few more examples

**Dimension of ‘$h$’**

‘$h$’ has the same dimension as that of action, i.e (position ) x (momentum). The said quantities are canonically conjugate. It is given by $ML^2T^{-1}$, which also happens to be the dimension of angular momentum. This must have been taken into consideration by Prof. Niels Bohr while making his path breaking postulate about the stationary orbits. In order to standardize the orbits, he had to take into account a physical quantity which is conserved. The electronic orbit in an atom is caused by a central force hence, the angular momentum is a conserved quantity. So orbits can be standardised with the help of angular momentum.

Now, in order to express the condition for stationary orbit, angular momentum has to be equated with a quantity through which discreteness of the orbit, deemed to be the conceivable minimum possible value, gets reflected. Over and above it should have the dimension $ML^2T^{-1}$.

All the above conditions are fulfilled by $h$ or $\hbar$ and we have the mathematical statement of the postulate (with symbols having usual meanings) as,

$$mvr = \frac{nh}{2\pi} \quad \text{......................(6)}$$
Or $$mvr = n\hbar \quad \text{......................(7)}$$
with $n = 1,2,3,\ldots$.

Thus we see that the dimensional analysis of ‘$h$’ plays a crucial role in the framing of Niels Bohr’s postulation of stationary orbits. A similar exercise can also be done by relating dimensional analysis with the uncertainty principle and the simple fact that the Commutator of two hermitian operators is skew-hermitian to arrive at the schrodinger’s operator formalism. Now we will deal with three problems.

**Problem 1**

To obtain a reasonable estimate of the time it takes for the sand to run out fully through a timer (see Figure 1)
Solution

The rate of flow of sand through the constriction (aperture) can be assumed to be uniform and the total time of flow, $T_A$, is proportional to the volume of the sand, $V$

$$V = \frac{\pi h r^2}{3}$$

$\alpha$ is the semi vertical angle of the inverted cone

$$\frac{\pi h \tan^2 \alpha}{3}$$

Therefore $V \alpha h^3$  (8)

The required time, say, $T_A$, may also depend on the acceleration due to gravity $g$, the diameter of the aperture, $d$, and the density of sand, $\rho$.

So

$$T_A \propto h^3 \times f(g,d, \rho)$$ (9)

Now, among $g, d, \rho$ only ‘$g$’ involves dimension of time, it is indeed $L_T^{-2}$. So the function ‘$f$’ has to be proportional to the reciprocal of square root of ‘$g$’. Now $\frac{1}{\sqrt{g}}$ has the dimension $\frac{T}{L^{1/2}}$. We already have $L^3$ as a multiplier. So effectively the involved dimension of $L$ is given by $L^{3/2}$. And $T_A$ cannot depend on $\rho$ which is $m_L^{-3}$.

So

$$T_A \propto L^{-5/2}$$

Therefore,

$$\text{Altogether } T_A \propto \frac{h^3 g^{-1/2}}{d^{5/2}} = \frac{h^3}{\sqrt{gd^5}}$$ (10)

The constant of proportionality is a dimensionless number and can be assumed for the sake of simplicity to be of the order of one.

So an estimate for $T_A$ is

$$\frac{h^3}{\sqrt{gd^5}}$$ (11)

If $h= 10 \text{cm}$, $d= 1 \text{mm}$ and $g= 10 \text{ms}^{-2}$

Then

$$T_A = \frac{(0.1)^3 \times m^3}{\sqrt{[10 \text{ ms}^{-2}] \times (0.001m)^5}}$$

$$= \frac{0.001 \text{ sec}}{\sqrt{10 \times 10^{-15}}}$$

$$= 10^4 \text{ sec} = 2.78 \text{ hr}$$

We have ignored the diameter of the grains of the sand in the analysis, but the trickiest part was to select the parameters relevant for dimensional analysis.

Problem 2

Two light unstretched, identical springs are joined using a small bob of mass $m$. The springs are anchored at the ends and arranged along a straight line, as shown in the Figure 2 below

![Figure 2](image)

The bob is displaced in a direction perpendicular to the line of the springs by 1cm and then released. The period of the resulting vibration of the bob is 2sec. We have to find the period of vibration if the bob were displaced by 2cm before release. The unstretched length of each spring is $L_o (L_o >> 1 \text{cm})$ and the effect of gravity is to be ignored.

Solution: Refer to Figure 3 below

![Figure 3](image)
The stretched length of each spring is

\[ l = \sqrt{l_o^2 + x^2} \]

\[ = l_o \sqrt{1 + \frac{x^2}{l_o^2}} \]

\[ l_o \gg x \]

Therefore \[ l = l_o + \frac{x^2}{2l_o} \quad \text{……………(12)} \]

Tension in the spring is given by

\[ F = k \frac{x^2}{2l_o} \quad \text{………………(13)} \]

The resultant force acting on the spring can be given by

\[ F_R = 2F \cos \theta \]

where \[ \cos \theta = \frac{x}{\sqrt{l_o^2 + x^2}} \]

therefore \[ F_R = \frac{x}{l_o} \quad \text{(because} \ l_o \gg x) \]

\[ = 2k \frac{x^2}{2l_o} \cdot \frac{x}{l_o} \]

\[ = \frac{3kx^3}{l_o^2} \quad \text{……………(14)} \]

The net force on the spring = - \[ \frac{3kx^3}{l_o^2} \quad \text{……………(15)} \]

The negative sign appears because the resultant force acts opposite to the displacement and the resulting equation of motion is

\[ m \ddot{x} = -\frac{3kx^3}{l_o^2} \quad \text{or} \]

\[ \ddot{x} = -C x^3 \quad \text{……………(16)} \]

where \[ C = \frac{k}{ml_o^2} \]

Multiplying both sides of (16) by 2x, we get

\[ 2 \ddot{x} x = -C (2\dot{x} x^3) \]

\[ \frac{d(\dot{x})^2}{dx} = -C \frac{d}{dx} \left( \frac{x^4}{2} \right) \]

Therefore, \[ \dot{x}^2 = -C_0 x^4 + A, \text{where} \ C_0 = \frac{C}{2} \]

and \[ A = \text{a constant} \]

Now \[ \ddot{x} = 0 \], when \[ x = a \] = the amplitude of the bob

\[ \therefore 0 = -C_0 a^4 + A \]

\[ \therefore A = C_0 a^4 \]

\[ \therefore \dot{x}^2 = C_0(a^4 - x^4) \quad \text{……………(17)} \]

\[ \therefore \dot{x} = \sqrt{C_0(a^4 - x^4)} \]

\[ \therefore \frac{dx}{\sqrt{C_0(a^4 - x^4)}} = \frac{1}{\sqrt{C_0}} dt \quad \text{……………(18)} \]

\[ 'x' \text{ attains the maximum value, i.e.} \ x = a, \text{ when} \ t = T/4, \text{ where} \ T \text{ is the time period.} \]

Therefore from (18), we get

\[ \int_0^{T/4} dt = \frac{1}{\sqrt{C_0}} \int_0^a \frac{dx}{\sqrt{a^4 - x^4}} \quad \text{……………(19)} \]

Putting the values of \[ C_0 = \frac{k}{2ml_o^2} \], we get

\[ \frac{T}{4} = l_o \frac{2m}{k} \int_0^a \frac{dx}{\sqrt{a^4 - x^4}} \]

\[ T = l_o \frac{32m}{k} \int_0^a \frac{dx}{\sqrt{a^4 - x^4}} \quad \text{……………(20)} \]

we put \[ u = x/a \]
then \[ dx = a \ du, \text{ when} \ x=0, \ u=0 \]

\[ x=a, \ u = 1 \]

and \[ \sqrt{a^4 - x^4} = a^2 \sqrt{1 - u^4} \]

\[ \therefore T = l_o \frac{32m}{k} \cdot \frac{1}{a} \int_0^1 \frac{du}{\sqrt{1-u^4}} \quad \text{……………(21)} \]

\[ \therefore T = \frac{k_o l}{a} \quad \text{……………(22)} \]
where \( K_o = l_o \sqrt{\frac{32m}{k}} \) .............(23)

and \( I = \int_0^1 \frac{du}{\sqrt{1-u^4}} \) .............(24)

Both \( K_o \) and \( I \) are constants

\[ \therefore T \propto \frac{1}{a} \] .............(25)

Thus if the amplitude is doubled from 1cm to 2cm the time period gets halved. So it will be \( \frac{1}{2} \) of 2s = 1s

Thus without actually working out the integral ‘\( T \)’, we can solve the problem. Incidentally this integral cannot be worked out in a closed form, however the definite integral can be worked out using numerical methods. Its approximate value is 1.3

So, the trick was getting the inverse proportionality between the time period and the amplitude.

Now, another trick can be applied using dimensional analysis at the stage of Eq(16), which is

\[ \ddot{x} = -Cx^3 \]

From this equation, we can suppose that ‘\( T \)’ depends only on ‘\( C \)’ and ‘\( a \)’. Let us write the dependence as

\[ T \propto C^\alpha \times a^\beta \]

\[ \therefore [T] = [C]^\alpha \times [a]^\beta \]

\[ C = \frac{K}{ml_o^2} \]

\[ [C] = \frac{ML^{-2}L^{-1}}{ML^2} = \frac{1}{L^2T^2} \]

\[ [T] = [T]^{-2\alpha} \times [L]^{-2\alpha+\beta} \]

Above is satisfied with \(-2\alpha = 1\) and \(-2\alpha + \beta = 0\)

\( \alpha = -\frac{1}{2} \) and \( \beta = -1 \)

which again implies \( T \propto \frac{1}{a} \); it is identical to (25)

So, dimensional analysis also does the trick and much more elegantly.

Problem 3
To find an estimate of total mass of water present in the different water sources on Earth.

Solution
We can solve this problem by Dimensional Analysis. To solve the problem we have to start by approximating that the amount of water on earth is equal to the water in oceans plus the water in rivers. Initially we try to estimate two quantities, the density of water and the volume of water contained in the oceans, and also the density of water and volume of water contained in the rivers. The relationship we use is

\( (\text{mass})_{\text{total}} = (\text{density})_{\text{ocean}} \times (\text{volume})_{\text{ocean}} + (\text{density})_{\text{river}} \times (\text{volume})_{\text{river}} \) ..............(26)

The problem in solving estimation related problems is to decide the relationship that exists between the physical quantities. For this we can apply dimensional analysis.

Density has the dimension of mass/volume, so our relationship is

\[ (\text{mass})_{\text{total}} = \frac{\text{mass}}{\text{volume}} \times (\text{volume})_{\text{ocean}} + \frac{\text{mass}}{\text{volume}} \times (\text{volume})_{\text{river}} \] ............(27)

The density of fresh water is \( \rho_{\text{water}} = 1.0 \text{ g-cm}^{-3} \); the density of sea water is slightly higher, but the difference will not matter for this estimate.

We can model the volume occupied by the oceans and rivers as if they completely cover the earth, forming the spherical shell (figure 4).

The volume of a spherical shell of radius \( R_{\text{earth}} \) and thickness \( t \) is given by

\[ (\text{volume})_{\text{shell}} = 4\pi R_{\text{earth}}^2 t \] .............(28)

Where \( R_{\text{earth}} \) is the radius of the earth and \( t \) is the average depth of the ocean.
Assuming that the rivers flow into oceans we can ignore the mass of water in the river and also take account of the fact that the oceans cover about 75% of the surface of the earth. So the volume of the ocean is

\[
(volume)_{ocean} = (0.75)4\pi R_{\text{earth}}^2 t \ldots \ldots (29)
\]

Taking the radius of earth as \( R_{\text{earth}} = 6 \times 10^3 \) km and the average depth of the ocean as \( t = 2 \) km,

\[
(mass)_{ocean} = (density)_{\text{water}}(volume)_{ocean} = \rho_{\text{water}} (0.75)(4\pi R_{\text{earth}}^2 t) = 678 \times 10^9 \text{ kg approx}
\]

**Conclusion**

Through this article we have been able to sensitize students about some features of dimensional analysis in addition to those available in standard textbooks by citing examples on period of simple pendulum, dimension of \( h \), canonically conjugate pairs. We have also solved some unknown problems like finding the total mass of water in water resources on earth up to a good approximation and a reasonable estimate of the time it takes for the sand to run out fully through a timer. The students will find it exciting to model real life physics problems on the basis of estimation of independent and dependent variables and find a working formula for it through dimensional analysis. Students would also be able to appreciate the optimum use of dimensional analysis as a powerful mathematical tool in solving various problems.

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