Energy Tensor for Charged Incoherent Dust in its Own Electromagnetic Field

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Abstract

The Energy Tensor $T_{\text{dust}}^{\mu\nu}$ of a distribution of incoherent charged dust is a sum of two constituent Energy Tensors, viz., the Energy Tensor $D^{\mu\nu}$ of the dust taken as a flow of fluid, and the Energy Tensor $M^{\mu\nu}$ of the Electromagnetic field created by the charges in the dust. The first one is obtained by first writing down the Equation of Motion of a perfect fluid, known as Euler’s equation, and upgrading the same to the 4-vector level. The second one is obtained by writing down the Conservation of Energy and Momentum in an Electromagnetic field and combining them into a single equation for the conservation of 4-momentum. It has been shown that the 4-divergence of the first one is equal and opposite to the 4-divergence of the second one. This leads to $\nabla_{\alpha} T_{\text{dust}}^{\alpha\mu} = 0$, which is the relation to be satisfied by all Energy Tensors that can qualify to be the source term in Einstein’s Field Equation for gravity in his General Theory of Relativity.

1 Introduction

Several years ago we wrote an article in this journal[1] titled “Maxwell’s Stress Tensor and Conservation Momentum in Electromagnetic Field”. That article was addressed to students and teachers in Classical Electrodynamics. We didn’t use the relativistic language at that time, and the vectors and tensors employed at that time were ordinary vectors and tensors, one level lower than 4-vectors and 4-tensors used in the context of Relativity. Instead of calling these entities vectors and tensors, we now call them 3-vectors and 3-tensors - thereby denying
them the full status of respectability in the parlance of Relativity.

The question now arises, can we not raise the Maxwell’s 3-tensor \( \hat{T}^{(em)} \) used at that time to the next higher 4-level? In this article we shall examine this possibility.

2 Stress Tensor and the Volume Force Density

Let us briefly review how a stress tensor \( \hat{T} \) is introduced in general. Consider a material medium, made up of solid, liquid or gas. Imagine a volume \( V \) bounded by a closed surface \( S \) carved out in the medium (Fig 1a). Consider a surface element \( da = n \, da \) at a point \( P \) on \( S \). Here \( n \) is a unit outward normal on \( S \) at \( P \). The infinitesimal stress force vector on \( da \) is

\[
dF_n = \mathcal{T}_n \, da = \hat{T} \cdot n \, da = \hat{T} \cdot da.
\]

Here \( \mathcal{T}_n \) is the stress vector on the surface at the point \( P \). The stress force \( F_s \) transmitted on the matter inside the volume \( V \) is the surface integral of the above infinitesimal stress force. The surface integral is then converted into a volume integral using Gauss’s theorem:\[2]\]

\[
F_s = \int_S \hat{T}(r) \cdot n \, da = \int_V \int \nabla \cdot \hat{T}(r) \, d^3r.
\]

From this it follows that the stress tensor \( \hat{T} \) distributed over a surface \( S \) is equivalent to a volume force density \( f_s \), distributed over the volume \( V \) enclosed by the same surface and the two are related by the equation:

\[
f_s = \nabla \cdot \hat{T}.
\]

Now imagine the same closed surface \( S \) enclosing the same volume \( V \). But now instead of a material medium, the space is occupied by vacuum, a non-material medium to which the ancients had referred to as “aether”. There are incoherent charged particles occupying a patch of this space (Fig 1b). 19th century scientists believed that aether could experience the same kind of stress that material media did, and this stress could be transmitted to electric charges and current inside a closed volume, exactly the same way as in the case of regular matter. The stress tensor for the transmission of electromagnetic forces is known as Maxwell’s stress tensor.

Equation (3) forms the starting point for the construction of Maxwell’s stress tensor.

3 Equations of Electrodynamics

The force density on a distribution of charged particle and their currents is given by the Lorentz force equation:\[3, 4]\:

\[
f_{em} = \frac{\partial P}{\partial t} = \rho E + J \times B,
\]

in which \( P \) is the momentum of the charged particles per unit volume, \( (E, B) \) are, respectively, the electric and magnetic fields, \( (\rho, J) \) are, respectively, the charge and current densities. The \( (E, B) \) fields satisfy Maxwell’s
Figure 1: Stress Force on a Volume $V$: (a) in a material medium, (b) in vacuum containing incoherent charged particles.

equations\[5\]:
$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}; \quad \nabla \times E = -\frac{\partial B}{\partial t}; \quad (a)
$$
$$\nabla \cdot B = 0; \quad \nabla \times B = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t}). \quad (b)
$$
The $E$ and $B$ fields are linked together by their time derivatives. If however we set $\frac{\partial}{\partial t} = 0$ in (5), line (a) will be the field equations for Electrostatics, and line (b) for Magnetostatics. They get separated out into separate compartments without any communication between them. It is relatively easy to write the stress tensors for these two special cases.

4 An important Identity

Construction of the stress tensor for electrostatic field, magnetostatic field and time varying electromagnetic field will be facilitated by the following identity\[6\].

$$\nabla \cdot \left[ \mathbf{A} \mathbf{A} - \frac{1}{2} \mathbf{A}^2 \mathbf{1} \right] = (\nabla \cdot \mathbf{A}) \mathbf{A} - \mathbf{A} \times (\nabla \times \mathbf{A}). \quad (6)
$$

Before establishing the above identity we shall need a standard formula\[7\]

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}. \quad (7)
$$

By setting $\mathbf{B} = \mathbf{A}$ in the above formula and get

$$\nabla \left( \frac{1}{2} \mathbf{A}^2 \right) = \mathbf{A} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{A}. \quad (8)
$$

We shall now prove the identity (6).

Proof:

$$\nabla \cdot (\mathbf{A} \mathbf{A}) = \left( e_i \frac{\partial}{\partial x_i} \right) \cdot (e_i e_j A_i A_j)
$$
$$= \frac{\partial}{\partial x_i} (A_i A_j) e_j
$$
$$= \left\{ \left( \frac{\partial A_i}{\partial x_i} \right) A_j + \left( A_i \frac{\partial}{\partial x_i} \right) A_j \right\} e_j
$$
$$= (\nabla \cdot \mathbf{A}) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{A}. \quad (a) \quad (9)
$$

$$\nabla \cdot \left( \frac{1}{2} \mathbf{A}^2 \mathbf{1} \right) = \left( e_i \frac{\partial}{\partial x_i} \right) \cdot \left( \frac{1}{2} e_i e_j A_i^2 \right)
$$
$$= \frac{1}{2} e_i \frac{\partial A_i^2}{\partial x_i} = \nabla \left( \frac{1}{2} A_i^2 \right)
$$
$$= \mathbf{A} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{A}. \quad (b) \quad (10)
The identity (6) follows when we subtract Eq.(b) from Eq.(a).

\[ Q.E.D. \]

Note that we have used Einstein’s summation convention. That is,

\[ e_l \frac{\partial}{\partial x_l} \equiv \sum_{l=1}^{3} e_l \frac{\partial}{\partial x_l}; \]

\[ e_i e_j A_i A_j \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} e_i e_j A_i A_j, \text{ etc.} \]

5 Maxwell’s Stress Tensors for the (E, B) Fields

(a) Maxwell’s Stress tensor for an Electrostatic Field.

The stress tensor follows when we set E for A in (6), and use the corresponding field equations, by setting \( \frac{\partial}{\partial t} = 0 \) in line (a) of (5):

\[ \nabla \cdot E = \rho / \varepsilon_0; \quad \nabla \times E = 0. \]

\[ f_e = \rho E = \nabla \cdot \mathbf{\hat{T}}_e, \quad (a) \]

where \( \mathbf{\hat{T}}_e = \varepsilon_0 \left[ EE - \frac{1}{2} E^2 \mathbf{\hat{1}} \right]. \quad (b) \] (10)

(b) Maxwell’s Stress tensor for a Magnetostatic Field.

The stress tensor follows when we set B for A in (6), and use the corresponding field equations, by setting \( \frac{\partial}{\partial t} = 0 \) in line (b) of (5):

\[ \nabla \cdot B = 0; \quad \nabla \times B = \mu_0 J. \]

\[ f_m = J \times B = \nabla \cdot \mathbf{\hat{T}}_m, \quad (a) \]

where \( \mathbf{\hat{T}}_m = \frac{1}{\mu_0} \left[ BB - \frac{1}{2} B^2 \mathbf{\hat{1}} \right]. \quad (b) \] (11)

(c) Maxwell’s Stress tensor for an Electromagnetic Field.

Now let us see what happens when we define

\[ \mathbf{\hat{T}}_{(em)} \stackrel{\text{def}}{=} \mathbf{\hat{T}}_e + \mathbf{\hat{T}}_m \]

\[ = \varepsilon_0 \left[ EE - \frac{1}{2} E^2 \mathbf{\hat{1}} \right] + \frac{1}{\mu_0} \left[ BB - \frac{1}{2} B^2 \mathbf{\hat{1}} \right], \quad (12) \]

and use Maxwell’s equations (5). Note that \( \frac{1}{\mu_0} = \varepsilon_0 c^2. \)

We shall do the work in two stages:

(i) set E for A in (6), and use line (a) of (5);

(ii) set B for A in (6), and use line (b) of (5).

\[ \nabla \cdot \mathbf{\hat{T}}_{(e)} = \nabla \cdot \left[ \varepsilon_0 \left( EE - \frac{1}{2} E^2 \mathbf{\hat{1}} \right) \right] = \varepsilon_0 \left[ (\nabla \cdot E) E - E \times (\nabla \times E) \right] \]

\[ = \rho E + \left\{ \varepsilon_0 E \times \frac{\partial B}{\partial t} \right\}, \quad (a) \]

\[ \nabla \cdot \mathbf{\hat{T}}_{(m)} = \nabla \cdot \left[ \frac{1}{\mu_0} \left( BB - \frac{1}{2} B^2 \mathbf{\hat{1}} \right) \right] = \frac{1}{\mu_0} \left[ (\nabla \cdot B) B - B \times (\nabla \times B) \right] \]

\[ = -B \times (J + \varepsilon_0 \frac{\partial E}{\partial t}) \]

\[ = J \times B + \left\{ \varepsilon_0 \frac{\partial E}{\partial t} \times B \right\}. \quad (b) \] (13)

\[ \nabla \cdot \mathbf{\hat{T}}_{(em)} = (\rho E + J \times B) + \left\{ \frac{\partial}{\partial t} (\varepsilon_0 E \times B) \right\}. \quad (c) \]
There are extra terms, for which we have used “braces” {...} for emphasis, that have come as a surprise, because they did not appear in Eqs.\(^{(10)}\) and \(^{(11)}\). The extra term in the line (c) of the above equations represents Field momentum, just as the first term represents Mechanical momentum (i.e., the momentum of the charged particles.) See Eq. \(^{(4)}\).

In order to understand this term we have to take a look at Conservation of Energy and Momentum in an Electromagnetic field, which we have taken up in Sec. \(^{8}\).

\section{4-vectors}

We are now entering the domain of Minkowski’s Space Time\(^{[8]}\) (MST). We shall familiarize the reader with our conventions and symbols, as outlined in a previous article\(^{[9]}\).

Corresponding to a 3-vector \(\mathbf{A}\) there will be a 4-vector \(\mathbf{A} = e^\mu_{\mu} A^\mu\), where \(A^\mu = (A^0, A^1, A^2, A^3)\), and \(\{e^\mu_{\mu}; (\mu = 0, 1, 2, 3)\}\) are the base vectors of the MST. We are taking the time component as the 0-th component and the space components as \((x, y, z) = (1, 2, 3)\) components of the 4-vector, and adopting the signature (+ - - -), as implied by Eq. \(^{(15)}\).

Consider the motion of a point particle in Fig.\(^{2}\). At the lower part of the figure we have shown its Physical Trajectory \(C\) in \(E^3\), the Euclidean 3-space. In the upper part we have shown its World Line \(\Gamma\) in the 4-dimensional Minkowski space \(M^4\), suppressing the Z axis. P \((x, y, z)\) and Q \((x + dx, y + dy, z + dz)\) are two infinitesimally close points on the trajectory \(C\), reached by the particle at times \(t\) and \(t + dt\) respectively. The infinitesimal 4-displacement from \(\Theta_P\) to \(\Theta_Q\) is

\[
d\mathbf{r} = e^\mu_{\mu} dx^\mu = (c dt, dx) = (dx^0, dx^1, dx^2, dx^3).
\]

It is the “primordial” contravariant 4-vector from which all other “truly” contravariant 4-vectors are generated, by multiplication with scalars and differentiation. Contravariant vectors are identified by superscripts for each of their four components. The vectors of mechanics we shall introduce soon are all contravariant vectors.

The norm of the 4-displacement \(dx^\mu\) is

\[
ds^2 = c^2 dt^2 - dr^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{15}
\]

and is therefore a 4-scalar. In the instantaneous rest frame (IRF) of the particle \(d\mathbf{r} = 0\), and \(dt \overset{\text{def}}{=} d\tau\). Therefore,

\[
ds^2 = c^2 d\tau^2, \text{ so that, } d\tau = ds/c, \quad \text{(in the IRF).} \tag{16}
\]

Since \(ds\) is a 4-scalar, i.e., invariant under all Lorentz transformations, and \(c\) is a universal constant, it follows that \(d\tau\) is a 4-scalar.

The time interval \(dt\) is called the Lab time between the events \(\Theta_P\) and \(\Theta_Q\), because it is measured by an observer at rest in the Laboratory, while the particle under observation is moving with the velocity \(\mathbf{u}\). The time interval \(d\tau\), measured in the rest frame of the particle, is called the proper time between the
same pair of events. The relation between the two is given by the equation

$$dt = \Gamma\,d\tau, \quad \text{where} \quad \Gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \quad (17)$$

There are two Lorentz factors one encounters while reading a chapter on Relativistic Mechanics\[10\]. One of them is the boost Lorentz factor $\gamma$ used in all Lorentz transformations (of the coordinates of an event, or in the Lorentz transformation of the components of a 4-vector). The other one is the dynamical Lorentz factor $\Gamma$, as defined in Eq. (17). It is used in writing the expression for 4-velocity, 4-momentum, 4-force. See Eq. (18).

Let $u$ be the velocity of a particle of rest mass $m_0$, $p$ its momentum, $F$ the force acting on it, and $\mathcal{E}$ its total energy (mass energy + kinetic energy). The first three quantities will have their 4-dimensional counterparts: 4-velocity, 4-momentum, 4-force (or Minkowski force).

We shall review/summarize the relevant formulas to be needed in the sequel\[10\].

\begin{align*}
\vec{U} &= \gamma \vec{U}^\mu = \gamma \frac{dx^\mu}{d\tau} \\
&= \gamma \, \Gamma \frac{dx^\mu}{d\tau} = \Gamma(c, u). \quad (a) \\
\vec{P} &= \gamma \vec{P}^\mu = m_0 \vec{U} = m_0 \Gamma(c, u). \quad (b) \\
m &= \Gamma m_0 = \text{relativistic mass}; \quad (c) \\
\mathcal{E} &= mc^2 = \text{energy}; \quad (d) \\
\vec{p} &= m\,u = 3\text{-momentum.} \quad (e) \\
\vec{F} &= \gamma \vec{F}^\mu = \frac{1}{c} \frac{d\mathcal{E}}{d\tau} \frac{d\vec{p}}{d\tau} \\
&= \Gamma \left( \frac{1}{c} \frac{d\mathcal{E}}{d\tau} \frac{d\vec{p}}{d\tau} \right) = \Gamma \left( \frac{\vec{F}}{c}, F \right). \quad (f)
\end{align*}

(18)
In line (f) we have set $\frac{dc}{dt} = \Pi$ (Capital pi). It stands for the power received by the particle (same as energy received by the particle per unit time), due to (i) work done on it by external forces: $\Pi = \mathbf{F} \cdot \mathbf{u}$, and/or (ii) by absorption of radiation or heat (thereby changing its rest mass). A force $\mathbf{F}$ which does not change the rest mass of the particle comes under case (i).

7 Minkowski Volume Force Density

Now we consider a stream of incoherent particles constituting a fluid in motion (Fig 3). An infinitesimal volume $\delta V$ (shown colored in the figure), identified at the event point $(x) = (r, t)$ contains a collection of fluid particles, which together possess a rest mass $\delta m_o$, a quantity of charge $\delta q$, and is moving with the velocity $\mathbf{u}(r, t)$ with respect to the Lab frame $S$. The volume occupied by this collection of particles is $\delta V$ in the Lab frame $S$ and $\delta V_o$ in the IRF $S_o$ (so that $\delta V_o$ is the proper volume). Lorentz contraction of the dimension of this box along the direction of $\mathbf{u}$ changes its proper volume $\delta V_o$ to the laboratory volume

$$\delta V = \frac{1}{\Gamma(x)} \delta V_o; \quad \text{Or,} \quad \delta V_o = \Gamma(x) \delta V. \quad (19)$$

Let the 3-force on this collection be $\delta \mathbf{F}(x)$, and the power received $\delta \Pi(x)$. Then according to (18 f), the Minkowski force acting on these particles (inside the proper volume $\delta V_o$) is

$$\delta \mathbf{F}(x) = \Gamma(x) \left( \frac{\delta \Pi(x)}{c}, \frac{\delta \mathbf{F}(x)}{c} \right)$$

$$= \frac{\delta \Pi(x)}{c} \delta V \left( \frac{1}{c} \frac{\delta \mathbf{F}(x)}{\delta V}, \frac{\delta \mathbf{F}(x)}{\delta V} \right)$$

$$= \delta V_o \left( \frac{1}{c} \frac{\delta \mathbf{F}(x)}{\delta V}, \frac{\delta \mathbf{F}(x)}{\delta V} \right). \quad (20)$$

We define Minkowski volume 4-force density $\mathbf{J}(x)$ to be the Minkowski force per unit proper volume - the 3-scalar density $\varpi$ (to be pronounced as var-pi) as the power received per unit lab volume, and 3-vector density $\mathbf{f}(x)$ as the 3-force per unit lab volume, as explained below.

$$\mathbf{J}(x) \equiv \lim_{\delta V_o \to 0} \frac{\delta \mathbf{F}(x)}{\delta V_o} \quad (a)$$

$$\varpi(x) \equiv \lim_{\delta V \to 0} \frac{\delta \Pi(x)}{\delta V}. \quad (b) \quad (21)$$

$$\mathbf{f}(x) \equiv \lim_{\delta V \to 0} \frac{\delta \mathbf{F}(x)}{\delta V}. \quad (c)$$

It follows from (20) and (21) that

$$\delta \mathbf{F}(x) = \delta V_o \mathbf{J}(x), \quad (a)$$

where, $\mathbf{J}(x) = \mathbf{e}_\mu f^\mu = \left( \frac{\varpi}{c}, \mathbf{f}(x) \right) \quad (b) \quad (22)$

is the Minkowski volume 4-force density (already defined.)

An example of Minkowski volume force density is the electromagnetic 4-force density $\mathbf{J}_{em}(x)$ shown in Eq: (36).

The 4-stress tensor is now defined to be a symmetric tensor: $\mathbf{T}(x) = \mathbf{e}_\mu T^{\mu\nu}(x) \mathbf{e}_\nu$ of rank 2, satisfying the requirement:

$$\mathbf{J}(x) \equiv \nabla \cdot \mathbf{T} \Rightarrow f^\mu(x) \equiv \nabla_a T^{a\mu}(x). \quad (23)$$
In the above $\nabla$ is the 4-dimensional differential operator, having components:

$$\nabla = \left( \frac{1}{c} \frac{\partial}{\partial t'}, \nabla \right) = \left( \frac{\partial}{\partial x'^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right).$$  

(24)

We shall call $\hat{T}$ the Minkowski 4-stress tensor.

8 Energy and Momentum

Conservation in One Voice

There are two important theorems that are used to state the conservation laws involving the Electromagnetic forces. We shall state them as two theorems, because they follow directly from Maxwell’s equations.

Consider the same stream of charged particles subjected to electromagnetic forces of their own creation. We shall apply energy and momentum conservation theorems to this system of particles.

(A) The energy theorem, also called Poynting’s theorem[11] is written as

$$\mathbf{E} \cdot \mathbf{J} + \frac{\partial \mathcal{W}}{\partial t} = -\nabla \cdot \mathbf{S}. \quad (25)$$

We have proved the above theorem in Appendix A.1. We interpret the terms appearing in the above equation as follows.
\[ E \cdot J = \text{work done by the field on the fluid particles per unit volume} \]
\[ w = \frac{\varepsilon_0}{2} (E^2 + c^2B^2) = \text{field energy density} \] \hfill (b) \hfill (26)
\[ S = \varepsilon_0 c^2 (E \times B) = \text{field energy flux density} \] \hfill (c)
\[ = \text{Poynting’s vector} \]

All densities alluded to in the context of energy-momentum theorems (25) and (29) are \emph{lab} densities. See comments after Eq. (20).

To justify the above interpretation we integrate over a volume \( V \) bounded by a surface \( S \), and apply Gauss’s theorem, we get:

\[ \int \int \int_V \left( E \cdot J + \frac{\partial w}{\partial t} \right) dv = - \int \int \int_V \nabla \cdot S \ dv = - \int_S S \cdot da. \] \hfill (27)

LHS = rate of change of [mch energy + fld energy] inside \( V \).
RHS = - \text{outflux of fld energy across } S = \text{influx of fld energy across } S.

Therefore, \textit{rate of change of [mch energy + fld energy] per unit volume = influx density of fld energy per unit volume.}

Our interpretation is justified.

\textbf{(B) The momentum theorem[11]}

We proved the following theorem, which follows from Maxwell’s equations, as Eq. (13c).

\[ \begin{align*}
\rho E + J \times B + \left\{ \frac{\partial}{\partial t} (\varepsilon_0 E \times B) \right\} &= \nabla \cdot \hat{T}_{(em)} \\
\text{(28)}
\end{align*} \]

We shall interpret the two terms on the LHS as follows. The \((E, B)\) field exerts a force on the existing charge-current distribution according the Lorentz force equation. The first term represents this force \( f_{em} \), as in Eq. (4), equal to the rate of change of the momentum of the particles per unit volume represented by \( P \), which we shall refer to as \textit{mechanical momentum density}.

However, when these fields starts changing with time they create a propagating em field which carries away energy and momentum. The second term should represent the rate of change of this \textit{field momentum density}, to be represented by the symbol \( g \).

\[ g \overset{\text{def}}{=} \varepsilon_0 (E \times B) = \frac{S}{c^2}. \] \hfill (29)

Now we can rewrite Eq. (28) as

\[ \frac{\partial \mathbf{P}}{\partial t} + \frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \hat{T}_{(em)} \] \hfill (30)

To justify the above interpretation, we shall integrate (30) over a volume \( V \) bounded by a surface \( S \) and apply Gauss’s
\[
\frac{d}{dt} \left( \iiint_V P \, d^3r \right) + \frac{d}{dt} \left( \iiint_V g \, d^3r \right) = \iiint_V \left( \nabla \cdot \hat{T}_{(em)} \right) \, dv = \int_S \hat{T}_{(em)} \cdot \mathbf{n}(\mathbf{r}) \, da. \tag{31}
\]

LHS = The Rate of change of [Mch momentum + Fld momentum] inside \(V\).

RHS = Total em force transmitted across \(S\) = Influx of fld momentum across \(S\).

Hence, we interpret Eq. (31) as saying that

Rate of change of [Mch momentum + Fld momentum] per unit volume = Influx density of Fld momentum per unit volume.

Our interpretation is justified.

Eqs. (25) and (30) are two equations expressing conservation of Energy and Momentum separately. The spirit of relativity will demand that they should be integrated into a single equation, unifying conservation of energy and momentum as a conservation of 4-momentum. As a first step towards this we rewrite Eqs. (25) and (30) in such a way that the left side will represent the charged particles and the right side the em field.

\[
\begin{align*}
\mathbf{E} \cdot \mathbf{J} & = - \left[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{S} \right]. \quad (a) \\
\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} & = - \frac{\partial \mathbf{g}}{\partial t} + \nabla \cdot \hat{T}_{(em)} \\
& = - \left[ \frac{\partial \mathbf{g}}{\partial t} + \nabla \cdot \hat{\Phi}_{(em)} \right]. \quad (b)
\end{align*}
\]

In the last equation \(\hat{\Phi}_{(em)}\) is the momentum "outflux density", equal and opposite to momentum "influx density \(\hat{T}_{(em)}\).

Lines (a) and (b) of Eq. (32) represent the time component and the space components of one 4-vector equality.

The right side terms can be combined into a 4-vector, which we shall define to be the negative 4-divergence of a 4-tensor \(\mathbf{M}\), namely the Maxwell's Energy 4-tensor. This tensor is an upgradation of the Maxwell's stress \(\hat{T}_{(em)}\) defined in Eq. (13c), except that \((-\hat{T}_{(em)})\), defined as \(\hat{\Phi}_{(em)}\), forms the 3 \(\times\) 3 core of this upgradation. The 4 \(\times\) 4 components of this tensor will be written as \(M^{\mu \nu}\). The time and space components of the new 4-vector are:

\[
\begin{align*}
\text{Time component:} & \quad \frac{1}{\epsilon} \left( \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{S} \right) \overset{\text{def}}{=} \nabla_\alpha M^{\alpha 0} \quad (a) \\
\text{Space component:} & \quad \left[ \frac{\partial \mathbf{g}}{\partial t} + \nabla \cdot \hat{\Phi}_{(em)} \right]_k \overset{\text{def}}{=} \nabla_\alpha M^{\alpha k}, \quad k = 1, 2, 3. \quad (b)
\end{align*}
\]
The subscript $k$ on the left side implies $x, y, z$ components of the vector corresponding to $k = 1, 2, 3$ respectively.

It is now easy to identify the 16 components of the Minkowski Energy 4-tensor $\mathbf{M}$ by taking a close look at Eqs. $(33)$, and recalling the components of the operator $\nabla_{\alpha}$ shown in $(24)$. Eq.$(33\, a)$ yields the components of the column 0, and Eq.$(33\, b)$ the components of the columns $k = 1, 2, 3$. For help see Appendix A.3.

$$M^{\mu\nu}(x) = 
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & \frac{w}{c} & \Phi^{11}_{em} & \Phi^{12}_{em} & \Phi^{13}_{em} \\
1 & \frac{S_x}{c} & \Phi^{21}_{em} & \Phi^{22}_{em} & \Phi^{23}_{em} \\
2 & \frac{S_y}{c} & \Phi^{31}_{em} & \Phi^{32}_{em} & \Phi^{33}_{em} \\
3 & \frac{S_z}{c} & \Phi^{11}_{em} & \Phi^{12}_{em} & \Phi^{13}_{em}
\end{pmatrix}$$

(34)

We have written $\Phi^{11}_{em}, \Phi^{12}_{em}, ...$ to mean $\Phi^{xx}_{em}, \Phi^{xy}_{em}, ...$ respectively. Note that $M^{\mu\nu}$ is symmetric and traceless.

$$M^{\mu\nu} = M^{\nu\mu},$$
$$M^{\mu}_{\mu} = 0.$$  (35)

Both properties are Lorentz invariant, i.e., same in all inertial frames.

The left side terms be combined into another 4-vector, namely $\mathbf{f}_{em}$, the Lorentz force density (per unit proper volume). It can be expressed in terms of the Electromagnetic Field tensor $F^{\mu\nu}$:

$$\mathbf{f}_{em}(x) \overset{\text{def}}{=} \left( \frac{1}{c} E \cdot J, \rho E + J \times B \right) = e_{\mu} \left( \frac{1}{c} F^{\mu\alpha} J_{\alpha} \right),$$  (36)

where $F^{\mu\alpha}, J_{\alpha}$ are, respectively, the Electromagnetic Field 4-Tensor and the Electric current density 4-vector. The above equality follows from the definition of the field tensor $F^{\mu\nu}$ by its relation to the Lorentz force on a particle of charge $q$, as has been shown in Appendix A.4.

The conservation equations for 4-Momentum, appearing disjointedly as $(25)$ and $(30)$, will now join into the following single 4-equation (12):

$$\frac{1}{c} F^{\mu\alpha} J_{\alpha} = -\nabla_{\beta} M^{\beta\mu}(x).$$  (37)

9 Euler’s (Non-Relativistic) Equation of Motion for a Perfect Fluid

Our objective now is to construct the energy tensor of the simplest “closed system”. The term “closed” in this context means that the system is self-contained in all its dynamical behavior, i.e., all dynamical processes take place due to forces of interaction within the system, there being no scope for exchange of energy and momentum with anything outside. The total energy and the total momentum of a closed system are therefore fully conserved.

A closed system contains both matter and forces. The only kind of classical forces that can receive relativistic treatment are electromagnetic forces. Before linking up matter with electromagnetic forces, we shall consider an oversimplified model which consists of matter in the form of per-
fect fluid - sometimes also called "classical fluid" - moving under the influence of internal and external forces whose origin we need not specify at this moment. We shall first lend a non-relativistic treatment to this fluid, so that transition to a relativistic formalism becomes smooth in the next section. The equation of motion of this perfect fluid is known as Euler’s equation.

By perfect fluid we mean a fluid which does not offer any viscous forces, which as the reader knows, causes shear stresses in the fluid. A perfect fluid, whether at rest or in motion, can sustain only normal compressive stresses inside, familiarly known as “pressure”.

Let us consider a fluid in streamline motion as previously illustrated in Fig. 3. This fluid is characterized by a velocity field \( \mathbf{u}(\mathbf{r}, t) \) and a fluid mass density \( \sigma(\mathbf{r}, t) \), both of which, in general, are unsteady fields, i.e., functions of \( t \) as well. The divergence of \( \mathbf{u} \) is called dilatation, a term we shall explain with the help of Figs. 4(a),(b).

We have shown a stream of fluid in motion, inside of which we have marked out a volume \( V \) at time \( t \). Since the fluid particles on the surface \( S \) of \( V \) have different velocities \( \mathbf{u}(\mathbf{r}, t) \), the boundary \( S \) not only moves with the particles lying on it, but also changes to a different shape \( S' \) (shown with broken line) at the time \( t + dt \). Consequently \( V \) will also change to a different volume, say \( V' \).

Consider a film of fluid particles lying over a tiny area \( \delta a \) centred at the point \( P \). These particles move a tiny distance \( \mathbf{u} \, dt \) from \( P \) to \( P' \) in time \( dt \). In this time a volume of fluid \( \delta v \) flows out from \( V \), crossing the tiny surface area \( da \). The volume that flows out is \( \delta v = |\mathbf{u} \cdot \mathbf{n}| \delta a \, dt \).

There are certain regions of \( S \), say at \( P \), where \( \mathbf{u} \cdot \mathbf{n} \) is positive, and the outflux (i.e., volume outflow) is positive. There are some other regions, say, at \( Q \), where \( \mathbf{u} \cdot \mathbf{n} \) is negative, and the outflux is negative. The net outflux of fluid volume is the surface integral of \( \mathbf{u} \) over the boundary surface \( S \). This can be written as

\[
dV = V' - V = \left[ \int_S (\mathbf{u} \cdot \mathbf{n}) \, da \right] \, dt
\]

where we have used Gauss’s theorem to convert the surface integral to a volume integral. We reduce the finite volume \( V \) to an infinitesimal volume \( \delta V \), thereby avoid integration, and get

\[
d(\delta V) = [(\nabla \cdot \mathbf{u}) \, \delta V] \, dt.
\]

Therefore,

\[
\frac{1}{\delta V} \frac{d(\delta V)}{dt} = \nabla \cdot \mathbf{u}.
\]

In other words, \( \nabla \cdot \mathbf{u} \) is the rate of change of volume per unit volume - or, more compactly dilatation.

Now we take up equation of motion proper. Consider a fluid element consisting
of an infinitesimal collection of fluid particles moving along the stream (Fig. 4c). At time \( t \) its centre of mass is located at \( P \) where it occupies a volume \( \delta V \). The mass of this element is \( \delta m = \sigma(\mathbf{r}, t) \delta V \), its momentum \( \delta \mathbf{p} = \delta m \mathbf{u}((\mathbf{r}, t)) \) and the force impressed on it \( \delta \mathbf{F} = f(\mathbf{r}, t) \delta V \), where \( f(\mathbf{r}, t) \) represents the volume force density. Applying Newton’s Second Law of motion to this fluid element,

\[
\frac{d}{dt}(\delta \mathbf{p}) = \delta \mathbf{F}, \quad (a)
\]

or,

\[
\frac{d}{dt}[\delta m \mathbf{u}(\mathbf{r}, t)] = f(\mathbf{r}, t) \delta V. \quad (b)
\]

Note that in the above equation \( \frac{d}{dt} \) represents convective derivative whose meaning we shall explain with a more general example. Let there exist a certain field \( f(x, y, z, t) \) in the fluid (eg, temperature, fluid velocity, pressure). The value of this field at the location of the particle changes from \( f(x, y, z, t) \) to \( f(x + dx, y + dy, z + dz, t + dt) \) as the particle moves with velocity \( \mathbf{u} \) from the location \( \mathbf{r} = (x, y, z) \) at time \( t \) to take up a new location \( \mathbf{r} + \delta \mathbf{r} = (x + dx, y + dy, z + dz) \) at time \( t + dt \). The net change is

\[
\int df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial t} dt
\]

\[
= \left[ \frac{\partial f}{\partial x} u_x + \frac{\partial f}{\partial y} u_y + \frac{\partial f}{\partial z} u_z + \frac{\partial f}{\partial t} \right] dt
\]

\[
= \left( \mathbf{u} \cdot \nabla + \frac{\partial}{\partial t} \right) f(x, y, z, t) dt
\]

\[
= \left( \frac{df}{dt} \right)_c dt. \quad (42)
\]

We have attached a subscript “c” to stress that the time rates of the changes of physical quantities in motion are given by their Convective Derivatives.

\[
\frac{df}{dt} \equiv \left( \frac{df}{dt} \right)_c \overset{\text{def}}{=} \left( \mathbf{u} \cdot \nabla + \frac{\partial}{\partial t} \right) f(x, y, z, t). \quad (43)
\]

Using Eqs. (40) and (43), we establish a few relations for future reference.
where $\sigma$ represents any density function, of which the mass density is particular example.

In N.R. physics mass is conserved. Consider the two terms in the first equality in line (a). The first term, if positive, means increase in mass in $dV$ due to density fluctuation. The second term, if positive, means increase in mass due to volume fluctuation. However, both of them cannot be positive. Increase in one term is nullified by decrease in the other. Together they represent zero change. We get back the mass conservation equation, known as continuity equation.

$$\frac{d}{dt}[\sigma \delta V] = 0, \quad \Rightarrow \quad \frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{u}) = 0. \quad (45)$$

We shall convert Eq. (44) to a momentum equation. Replace the scalar density $\sigma$ with the density of the $x$-component of momentum $\sigma u_x$ in the above equation, and get
d$$\frac{d}{dt}[(\sigma u_x) \delta V] = \left[\frac{\partial (\sigma u_x)}{\partial t} + \nabla \cdot (\sigma u_x \mathbf{u})\right] \delta V. \quad (46)$$

The above relation holds for all the three components $u_x, u_y, u_z$. Multiplying the components with $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ and adding them together, we get

gether, we get

$$\frac{d}{dt}[\sigma \mathbf{u} \delta V] = \left[\frac{\partial (\sigma \mathbf{u})}{\partial t} + \nabla \cdot (\sigma \mathbf{u} \mathbf{u})\right] \delta V.$$

Going back to Eq. (41), noting that $\delta m\mathbf{u}(r,t) = \sigma \mathbf{u}\delta V$ and using (47) we get the general equation of motion for the fluid:

$$\frac{\partial}{\partial t}(\sigma \mathbf{u}) + \nabla \cdot (\sigma \mathbf{u} \mathbf{u}) = \mathbf{f}. \quad (48)$$

Eq. (48) is the general equation of motion of a fluid, to be referred to as the Euler’s Equation.

### 10 Relativistic Equation of Motion for a Continuous Incoherent Media

We shall upgrade the $E^3$ version of the fluid equation of motion (48) to $M^4$. The starting point of the former was (41a). The starting point of the latter will be the $M^4$ version of this equation, i.e., Eq. (18f) in which we set $\delta \mathbf{P} \rightarrow \delta \mathbf{P}$ and $\delta \mathbf{F} \rightarrow \delta \mathbf{F}$.

$$\frac{d(\delta \mathbf{P})}{d\tau} = \delta \mathbf{F}. \quad (49)$$

Here $\delta \mathbf{P}$ is the 4-momentum of the mass content of the same fluid volume $\delta V$ considered in Sec. 9 and $\delta \mathbf{F}$ is the Minkowski force on this volume.

We shall write the left and the right side the above equation, using (18b) and (22b):
\[ \delta \mathbf{P} = \mathbf{e}_\mu \delta p^\mu; \quad \text{so that} \]
\[ \frac{d(\delta \mathbf{P})}{d\tau} = \mathbf{e}_\mu \left( \frac{d(\delta p^\mu)}{d\tau} \right), \quad (a) \]
\[ \delta \mathbf{F} = f(x) \delta V_o = \mathbf{e}_\mu (f^\mu(x) \delta V_o). \quad (b) \]

In line (b) we make use of the definition of \( f \) given in (22).

Then the EoM (49) can be written in the form:
\[ \frac{d \delta p^\mu}{d\tau} = f^\mu(x) \delta V_o \]
\[ \equiv \left( \frac{\omega}{c}, f \right) \delta V_o. \quad (a) \]
\[ \text{Or,} \quad \frac{d \delta p^\mu}{dt} = \left( \frac{\omega}{c}, f \right) \delta V. \quad (b) \]

To go from the line (a) to the line (b), we divided each side with \( \Gamma \), and recalled Eqs. (17) and (19).

At this point let us be aware that mass is not conserved in relativistic mechanics. Mass conservation is violated, even if infinitely, in all real situations. Mass of a system changes when chemical reactions take place, when atoms absorb or emit light, when a gas expands or is compressed. Even for the perfect fluid, whose dynamics was given a relatively simple non-relativistic treatment in Sec 6.7, its mass is continuously changing because of the work being done by fluid pressure. This effect has to be taken into consideration.

To make our task manageable we shall think of a moving fluid in which the constituent particles - atoms, molecules, nuclei, electrons - whatever they may be, remain in their original ground states through the dynamical processes, and, hence, donot emit or absorb light, so that their rest masses do not change. The particles are charged, and the electromagnetic field created by their charges determine their Equation of Motion.

Let Fig. 3 represent a segment of this flowing fluid. An infinitesimal volume \( \delta V \) of this fluid, at the event point \( (x) \), possesses a rest mass \( \delta m_0 \), which is the sum of the rest masses of all the constituent particles inside \( \delta V \). That is, \( \delta m_0 = \Sigma \delta N m_{oi} \), where \( \delta N \) is the number of particles inside \( \delta V \) and \( m_{oi} \) is the rest mass of the \( i \)-th particle in this infinitesimal collection. Let \( \sigma_0 \) stand for proper density of rest mass, which we define as:
\[ \sigma_0(x) = \lim_{\delta V_o \to 0} \frac{\delta m_0}{\delta V_o} \]

where \( \delta V_o \) is the proper volume of the above collection, i.e., volume measured in the instantaneous rest frame. In contrast to \( \sigma_0 \), we use another symbol \( \sigma \) to denote density of relativistic mass in the observer’s frame \( S \).

Seen from the observer’s frame, the above collection of \( \delta N \) particles are now confined within a smaller volume \( \delta V = \delta V_o / \Gamma \) and the relativistic mass of this collection is \( \delta m = \Gamma \delta m_0 \). Therefore,
\[ \sigma(x) = \lim_{\delta V \to 0} \frac{\delta m}{\delta V} = \lim_{\delta V_o \to 0} \frac{\Gamma \delta m_0}{(\delta V_o / \Gamma)} = \Gamma^2 \sigma_0(x). \]

We shall work out the equations of motion of the energy and momentum content of the volume \( \delta V \). The relativistic mass of this volume is:
\[ \delta m = \sigma(x)\delta V. \]
According the formula (18 e) the 4-momentum of the mass content within this volume is $\delta p^\mu = (\delta p^0, \delta \mathbf{p})$ where

$$
\delta p^0 = \delta m c = (\sigma \delta V) c.
$$

$$
\delta \mathbf{p} = \delta m \mathbf{u} = (\sigma \delta V) \mathbf{u}.
$$

(55)

Let us now go back to the equation of motion (51 b). We shall expand the left hand side corresponding to $\mu = 0$, using the time component of $\delta p^\mu$ as given in (55). With some help from (44):

$$
d \frac{\delta p^0}{dt} = c \frac{d}{dt} (\sigma \delta V) = c \left[ \frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{u}) \right] \delta V.
$$

(56)

and corresponding to $\mu = i = 1, 2, 3$ in a similar way with help from (47):

$$
d \frac{\delta \mathbf{p}}{dt} = \frac{d}{dt} (\sigma \mathbf{u} \delta V) = \left[ \frac{\partial (\sigma \mathbf{u})}{\partial t} + \nabla \cdot (\sigma \mathbf{u} \mathbf{u}) \right] \delta V.
$$

(57)

The equations of motion (51 b) will then become:

$$
\begin{bmatrix}
\frac{\partial (c^2 \sigma)}{\partial t} + \nabla \cdot (c \sigma \mathbf{u}) \\
\frac{\partial (\sigma \mathbf{u})}{\partial t} + \nabla \cdot (\sigma \mathbf{u} \mathbf{u})
\end{bmatrix}
= \frac{\partial \mathbf{u}}{\partial t}. \quad \text{(a)}
$$

$$
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0 & c^2 & cu_x & cu_y & cu_z \\
1 & ux_c & u_x^2 & u_xu_y & u_xu_z \\
2 & uy_c & u_yu_x & u_y^2 & u_yu_z \\
3 & uz_c & u_zu_x & u_zu_y & u_z^2
\end{bmatrix}

(58)

Eq. (a) is the Energy equation and (b) is the Momentum equation. As in the case of the Electromagnetic field in Sec. 8 we shall integrate these two disjointed equations into a single equation. Let us go back to (18 f), which gives the Minkowski’s Equation of Motion, and Eq. (18 f) a) which gives the 4-velocity $U^\mu$ in which $\mathbf{u}$ is the 3-velocity space component: $U^\mu = \Gamma(c, \mathbf{u})$. Because of the relation given in (53), the left sides of lines (a) and (b) of Eq. (58) combine to form a single expression: $\nabla_\beta [\sigma_0 (U^\beta U^\mu)]$.

In the same way, thanks to (22 b), the right sides of lines (a) and (b) of Eq. (58) combine to form a single expression: $f^\mu$. Therefore, the two lines of Eq. (58) now become one line, a single relation between two 4-vectors:

$$
\nabla_\beta [\sigma_0 (U^\beta U^\mu)] = f^\mu.
$$

(59)

We now define the Energy Tensor of the Incoherent Fluid (also called Incoherent Dust) as

$$
D^{\mu\nu} \overset{\text{def}}{=} \sigma_0 U^\mu U^\nu,
$$

(60)

It is now very easy to identify the $4 \times 4$ components of $D^{\mu\nu}$:

$$
D^{\mu\nu} = \sigma_0 c^2 \times
$$

$$
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0 & c^2 & cu_x & cu_y & cu_z \\
1 & ux_c & u_x^2 & u_xu_y & u_xu_z \\
2 & uy_c & u_yu_x & u_y^2 & u_yu_z \\
3 & uz_c & u_zu_x & u_zu_y & u_z^2
\end{bmatrix}

(61)

The EoM, written as (59) takes the beautiful comprehensive form:

$$
\nabla_\mu D^{\mu\nu} = f^\nu.
$$

(62)

11 Energy Tensor for a System of Charged Incoherent Fluid

We prepared the ground-work for this section in Setion 8 in particular through
Eq. (37). Before proceeding further we shall recognize the following two volume 4-force densities.

\[ \mathbf{f}_{\text{fld} \rightarrow \text{mat}} = (f_{\text{fld} \rightarrow \text{mat}, 0}, f_{\text{fld} \rightarrow \text{mat}}) \]

= 4-force per unit proper volume (u.p.v)

from the em fld on the particles in the dust.

\[ \mathbf{f}_{\text{mat} \rightarrow \text{fld}} = (f_{\text{mat} \rightarrow \text{fld}, 0}, f_{\text{mat} \rightarrow \text{fld}}) \]

= 4-force per unit proper volume (u.p.v)

from the particles in the dust on the em fld.

(63)

Let us now understand the effect of above two 4-force densities.

Rate of change of matter 4-momentum per u.p.v

\[ = \left[ \frac{1}{c} \mathbf{E} \cdot \mathbf{J} + \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \right] \]

\[ = \frac{1}{c} F^{\alpha \beta} J_\alpha = \mathbf{f}_{\text{fld} \rightarrow \text{mat}}. \]

(64)

Rate of change of field 4-momentum per u.p.v

\[ = \left[ \frac{1}{c} \left( \frac{\partial \mathbf{E}}{\partial t} + \nabla \cdot \mathbf{S} \right) \right] \]

\[ + \left( \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \mathbf{\Phi}_{\text{(em)}} \right) \]

\[ = \nabla_\beta M^{\alpha \beta} (x) = \mathbf{f}_{\text{mat} \rightarrow \text{fld}} \]

(65)

We can now go back to (57) and rewrite the 4-Momentum conservation equation as[12]:

\[ \mathbf{f}_{\text{fld} \rightarrow \text{mat}} = -\mathbf{f}_{\text{mat} \rightarrow \text{fld}}. \]

(65)

The above equation represents a generalization of Newton’s 3rd Law of motion for the 3-forces of action and reaction to the 4-forces of action and reaction between a charged fluid media and its own electromagnetic field.

The EoM of the charged dust is given by Eq. (52), in which the “force” \( f^\mu \) is now the electromagnetic force on matter, i.e.,

\[ f^\mu_{\text{fld} \rightarrow \text{mat}} = f^\mu_{\text{em}}, \]

as given in (36), coming from the charge-current density \( J^\mu \) present in the matter itself. The EoM is now written as

\[ \nabla_\alpha D^{\alpha \mu} (x) = f^\mu_{\text{fld} \rightarrow \text{mat}} \]

\[ = -f^\mu_{\text{mat} \rightarrow \text{fld}} = -\nabla_\alpha M^{\alpha \mu} (x). \]

(66)

The system consisting of the matter (represented by \( D^{\alpha \mu} \)) and its own em fld (represented by \( M^{\alpha \mu} \)) is now a closed system. Its Energy Tensor is

\[ T^{\alpha \mu}_{\text{dust}} (x) = D^{\alpha \mu} (x) + M^{\alpha \mu} (x), \]

satisfying

\[ \nabla_\alpha T^{\alpha \mu}_{\text{dust}} = 0. \]

(68)

In Newton’s theory of gravitation a massive star, or a massive planet is the source of Gravitation. In Einstein’s General Theory of Relativity mass is replaced by energy. However, energy itself has no respectable status, because energy is the time component of 4-momentum. Hence Energy is replaced by 4-momentum, and energy density (analogous to mass density) by the Energy Tensor, which is loosely the density of 4-momentum. Since a star is an isolated object, its energy tensor must have zero 4-divergence. Eq. (67) gives the simplest example of such an energy tensor, and Eq. (68) tells us the desirable property of such a source of gravitation.

References

[1] Somnath Datta, Maxwell’s Stress Tensor and Conservation of Momentum in Electromagnetic Field, Physics Education,
www.physed.in, Indian Association of Physics Teachers, Vol 30, No.3, (July-Sep 2014), Article Number 1, 42 pages.

[2] Somnath Datta, as cited above, Eq.(65).


[5] David J. Griffiths, as cited above, p. 344


[7] See for example Vector Formulas compiled in: David Griffiths, as cited above.


A Appendix

A.1 Energy Conservation in Electromagnetic Field

We shall prove Poynting’s theorem as given in Eq. (25) using Maxwell’s equations (5). The electric current density appears in Eq. (5b).

\[
E \cdot J = E \cdot \varepsilon_0 c \left\{ \nabla \times c B (r, t) - \frac{\partial E (r,t)}{\partial t} \right\}
\]

However, \( E \cdot \nabla \times B = B \cdot \nabla \times E - \nabla \cdot (E \times B) \) (an identity)

Hence, \( E \cdot J = \varepsilon_0 c^2 \left\{ B \cdot \nabla \times E - \nabla \cdot (E \times B) \right\} - \varepsilon_0 E \cdot \frac{\partial E (r,t)}{\partial t} \)

\[= \varepsilon_0 \varepsilon_0 \frac{\partial}{\partial t} (E^2 + c^2 B^2) - \nabla \cdot (E \times B) \]

\[= -\frac{\partial w}{\partial t} - \nabla \cdot S.\]

Q.E.D.

A.2 Examples of Lowering and Raising an index

Ex.1

\[
V_\mu = V^\nu g_{\nu \mu} = \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} V^0, -V^1, -V^2, -V^3 \end{pmatrix}
\]

(A.1)

\[
A^\mu = g^{\mu \nu} A_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A^0 \\ -A^1 \\ -A^2 \\ -A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ -A^1 \\ -A^2 \\ -A^3 \end{pmatrix}
\]

(A.2)

Lowering or Raising \( \Rightarrow \) No change in the time component, sign change in the space component.

Ex.2 Let

\[
F^{\mu \nu} = \begin{pmatrix} f^{00} & f^{01} & f^{02} & f^{03} \\ f^{10} & f^{11} & f^{12} & f^{13} \\ f^{20} & f^{21} & f^{22} & f^{23} \\ f^{30} & f^{31} & f^{32} & f^{33} \end{pmatrix}
\]

(A.3)
be a contravariant 4-tensor. We shall lower only the first index $\mu$, then only the second index $\nu$, then both indices $\mu, \nu$.

$$F_{\mu}^{\nu} = g_{\nu \alpha} F^{\alpha \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} f^{00} & f^{01} & f^{02} & f^{03} \\ f^{10} & f^{11} & f^{12} & f^{13} \\ f^{20} & f^{21} & f^{22} & f^{23} \\ f^{30} & f^{31} & f^{32} & f^{33} \end{pmatrix}$$

(A. 4)

First index lowered $\Rightarrow$ No change in row 0. Sign change in rows 1,2,3.

$$F_{\mu}^{\nu} = F^{\mu \alpha} g_{\nu \alpha} = \begin{pmatrix} f^{00} & f^{01} & f^{02} & f^{03} \\ f^{10} & f^{11} & f^{12} & f^{13} \\ f^{20} & f^{21} & f^{22} & f^{23} \\ f^{30} & f^{31} & f^{32} & f^{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(A. 5)

Second index lowered $\Rightarrow$ No change in col 0. Sign change in cols 1,2,3.

$$F_{\mu \nu} = g_{\mu \alpha} F^{\alpha \nu} = \begin{pmatrix} f^{00} & f^{01} & f^{02} & f^{03} \\ f^{10} & f^{11} & f^{12} & f^{13} \\ f^{20} & f^{21} & f^{22} & f^{23} \\ f^{30} & f^{31} & f^{32} & f^{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(A. 6)

Both indices lowered $\Rightarrow$ No change in $\{00, kj, jk\}$ components. Sign change in $\{0k, k0\}$ components.

**Ex. 3** Trace of the contravariant tensor $F^{\mu \nu}$ is defined as $F^{\mu}_{\mu}$; sum over $\mu$. Going back to (A. 5),

$$\text{Tr}\{F\} = F^{\mu}_{\mu} = f^{00} - (f^{11} + f^{22} + f^{33}) = \text{sum of the diagonal elements of } F^{\mu}_{\mu}.$$  

(A. 7)
A.3 Components of Maxwell’s Stress 3-Tensor and Maxwell’s 4 Tensor, and their Traces

Maxwell’s 3-Tensor was written in a short form in Eq. (13). We shall now write down the $3 \times 3$ components of this tensor. The reader should verify them.

$$\mathbf{T}_{em}^{xx} = \frac{\varepsilon_0}{2} [(E_x^2 - E_y^2 - E_z^2) + c^2(B_x^2 - B_y^2 - B_z^2)];$$
$$\mathbf{T}_{em}^{yy} = \frac{\varepsilon_0}{2} [(E_y^2 - E_z^2 - E_x^2) + c^2(B_y^2 - B_z^2 - B_x^2)];$$
$$\mathbf{T}_{em}^{zz} = \frac{\varepsilon_0}{2} [(E_z^2 - E_x^2 - E_y^2) + c^2(B_z^2 - B_x^2 - B_y^2)];$$
$$\mathbf{T}_{em}^{xy} = \mathbf{\Phi}_{em}^{xy} = \varepsilon_0 [E_x E_y + c^2 B_x B_y];$$
$$\mathbf{T}_{em}^{yz} = \mathbf{\Phi}_{em}^{yz} = \varepsilon_0 [E_y E_z + c^2 B_y B_z];$$
$$\mathbf{T}_{em}^{zx} = \mathbf{\Phi}_{em}^{zx} = \varepsilon_0 [E_z E_x + c^2 B_z B_x].$$

We can now write the trace of the Maxwell 3-tensor.

$$\text{Tr}\{\mathbf{T}_{em}\} = \mathbf{T}_{em}^{xx} + \mathbf{T}_{em}^{yy} + \mathbf{T}_{em}^{zz} = -\frac{\varepsilon_0}{2} (E^2 + c^2 B^2)$$

Maxwell’s 4-Tensor was defined by Eq. (33). We shall use the same equation to identify all the components of $M^{\mu \nu}(x)$.

From (33): $\frac{1}{c} \left( \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} \right) = \nabla \cdot \mathbf{M}.$

Or, $\frac{\partial w}{\partial t} + \frac{\partial}{\partial x^j} (S_j/c) = \frac{\partial M_0^0}{\partial t} + \frac{\partial}{\partial x^j} M_j^0$ (sum over $j$). (A.10)

Hence, $M_0^0 = w; \ M_j^0 = S_j/c.$

In the following we shall write $\mathbf{\Phi}_{em}^{11}, \mathbf{\Phi}_{em}^{12}, \ldots$ to mean $\mathbf{\Phi}_{em}^{xx}, \mathbf{\Phi}_{em}^{xy}, \ldots$ respectively.

From (33): $\left[ \frac{\partial x^k}{\partial t} + \nabla \cdot \mathbf{\Phi}_{(em)} \right]_k = \nabla \cdot \mathbf{M}^k; \ k = 1, 2, 3.$

Or, $\frac{\partial c g_k}{\partial t} + \frac{\partial}{\partial x^j} (\mathbf{\Phi}_{em}^{jk}) = \frac{\partial M_0^k}{\partial t} + \frac{\partial}{\partial x^j} (M_j^k); \ \text{sum over} \ j \ k = 1, 2, 3.$ (A.11)

Hence, $M_0^k = c g_k; \ M_j^k = \mathbf{\Phi}_{em}^{jk}.$

We can now write all the $4 \times 4$ components of $M^{\mu \nu}(x)$.

$$M^{\mu \nu}(x) = \begin{pmatrix}
0 & 1 & 2 & 3 \\
\frac{w}{c} & \mathbf{\Phi}_{em}^{11} & \mathbf{\Phi}_{em}^{12} & \mathbf{\Phi}_{em}^{13} \\
\frac{S_x}{c} & \mathbf{\Phi}_{em}^{21} & \mathbf{\Phi}_{em}^{22} & \mathbf{\Phi}_{em}^{23} \\
\frac{S_z}{c} & \mathbf{\Phi}_{em}^{31} & \mathbf{\Phi}_{em}^{32} & \mathbf{\Phi}_{em}^{33}
\end{pmatrix}$$

(A.12)

Because of Eq. (29), $c g_k = \frac{S_k}{c},$ and the tensor is symmetric.
The trace of the Maxwell’s 4 tensor follows from (A.7) and (A.9).

\[
\text{Tr}\{M\} = M^\mu_\mu = M^{00} - (M^{11} + M^{22} + M^{33}) = w - (\Phi^{11}_\text{em} + \Phi^{11}_\text{em} + \Phi^{11}_\text{em}) = w + (T^{xx}_\text{em} + T^{yy}_\text{em} + T^{zz}_\text{em}) = w - \frac{\varepsilon_0}{2}(E^2 + c^2B^2) = 0.
\]

(A.13)

A.4 EM Field Tensor

The force experienced by a particle carrying an electric charge \(q\) is a velocity-dependent force, called \text{Lorentz Force}, written as:

\[
\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})
\]

(A.14)

where \(\mathbf{u}\) is the velocity of the charged particle at the event point \((x)\). The above equation also serves as the definition of the the \text{Electric field} \(\mathbf{E}\) and the \text{Magnetic field} \(\mathbf{B}\) at the location of the particle. Let us write the Lorentz factor for the particle’s velocity:

\[
\Gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \quad \text{(A.15)}
\]

Substituting the Lorentz force (A.14) in (18 f), the time and space components of the corresponding Minkowski force \(F^\mu\) are now obtained compactly as:

\[
\vec{\mathbf{F}} = e^\mu_{\mathbf{r}} F^\mu = q\Gamma \left(\frac{1}{c} \mathbf{E} \cdot \mathbf{u}, \mathbf{E} + \mathbf{u} \times \mathbf{B}\right), \quad \text{(A.16)}
\]

The above equation tells us that the Minkowski 4-force acting on a charged particle \(q\) is a linear function of its 4-velocity, and therefore can be written as:

\[
F^\mu = \frac{q}{c} F^\mu_\nu U_\nu. \quad \text{(A.17)}
\]

and in an expanded form as:

\[
\begin{align*}
F^0 &= \frac{\Gamma}{c} \mathbf{F} \cdot \mathbf{u} = \frac{q\Gamma}{c} (E_x u_x + E_y u_y + E_z u_z) \\
F^1 &= \Gamma F_x = \frac{q\Gamma}{c} (E_x c + cB_z u_y - B_y u_z) \\
F^2 &= \Gamma F_y = \frac{q\Gamma}{c} (E_y c + cB_x u_z - B_z u_x) \\
F^3 &= \Gamma F_z = \frac{q\Gamma}{c} (E_z c + cB_y u_x - B_x u_y)
\end{align*}
\]

(A.18)

Note that the second equality in Eq. (36) is obtained in the same way as Eq. (A.17) is derived from (A.16).