Darboux transformations and New exactly solvable potentials

C. Sivakumar¹, P. T. Varghese² and K. Babu Joseph³

¹ Department of Physics, Maharaja’s college, Ernakulam, Pin-682011, Kerala, India
thrisivc@yahoo.com

² Department of Physics, Rajagiri school of Engineering and Technology, Kerala, India.
varghese@rajagiritech.edu.in

³ Cochin University of Science and Technology, Kochi, Kerala, India.
bb.jsph@gmail.com

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Abstract

A large number of exactly solvable quantum mechanical potential problems are formed using a technique called Darboux transformations (DT). The similarity between the eigen value spectra of the new and original Hamiltonian is shown. We generated exactly solvable potentials from harmonic oscillator potential and Coulomb potential and for the latter we developed a modified DT to get new potential problems.

1 Introduction

Solving the Schrodinger equation exactly for different potentials is one of the fundamental problems of quantum mechanics, some of which are studied at the undergraduate level. The intent of this paper is to present new potentials other than harmonic oscillator potential, Coulomb potential etc., for which the Schrodinger equation has exact solutions. The Darboux transformations (DT) is a classical method for generating new solutions from given solutions. Here by means of DT, we enlarge the class of potentials for which the Schrodinger equation can be exactly solved.

We know that one can construct new exactly solvable potentials of one dimensional Schrodinger equation using a method that is derived from the inverse scattering problems (based on Gelfand-Levitan equation) and represents an integral transformation.
of the solutions of the initial Schrodinger equation\[1\ 2\ 3\ 4\]. Here we propose another method, which is to use a differential transformation suggested by a French mathematician, Gaston Darboux, a century ago.

The fundamental results of Darboux remained unrecognized for a long time. In 1979, Matveev applied Darboux’s theorem to a large class of linear partial differential equations as well as differential-difference and difference-difference linear evolution equations \[5\]. DT also became popular in the study of nonlinear partial differential equations \[6\]. In fact there is some connection between the two ways of constructing new exactly solvable models. This has been studied by Schnizer and Leeb and Samsanov in their papers \[7, 8, 9\]. Mielnik pointed out the use of a factorization method to construct new Hamiltonians with eigen value spectrum coinciding with original Hamiltonian \[10\]. This method is however a special case of DT \[6, 11\]. Some authors proposed a connection between DT and super-symmetric quantum mechanics \[12\ 13\ 14\ 15\]. We are proposing how DT can be effectively used to generate potentials which have exact solutions other than known ones.

The plan of the paper is as follows. In section 2, we present a brief discussion of DT and show that it leads to new exactly solvable potentials for the Schrodinger equation. Sections 3 and 4 deal with two potential problems which are used for generating new solvable models. Section 4 also presents the modification needed for DT in order to apply it to the radial problem. Section 5 contains the conclusions.

## 2 Darboux Transformations

To introduce the technique of DT, consider one dimensional Schrodinger equation in the form

\[-\frac{\partial^2 \Psi(x)}{\partial x^2} + v(x)\Psi(x) = \lambda \Psi(x) \quad (1)\]

where \(v = \frac{2mV}{\hbar^2}\) and \(\lambda = \frac{2mE}{\hbar^2}\). Suppose Eq. (1) is exactly solved for a given potential and we know all of its eigen functions and eigen values for the potential \(v(x)\). Let \(\Psi_1(x, \lambda_1)\) be a solution of Eq. (1). Its logarithmic derivative is defined as

\[\sigma = \frac{1}{\Psi_1} \frac{\partial \Psi_1}{\partial x}\]  \[(2)\]

Now a DT permits one to obtain the general solution of another Schrodinger equation

\[-\frac{\partial^2 \Phi(x)}{\partial x^2} + u(x)\Phi(x) = \lambda \Phi(x) \quad (3)\]

under the DT,

\[\Phi(x) = \left[\frac{d}{dx} - \sigma\right]\Psi(x) \quad (4)\]

\[u(x) = v(x) - 2\frac{d\sigma}{dx} \quad (5)\]

where \(\Psi\) is any arbitrary solution of Eq. (1).

We can state Darboux’s theorem in the following way: the Schrodinger equation, Eq. (1) is covariant with respect to the action of the DT, \(\Psi(x) \rightarrow \Phi(x)\) and \(v(x) \rightarrow u(x)\). Thus Eqs. (1) and (3) are exactly solvable Schrodinger equations for two different potentials \(v(x)\) and \(u(x)\) respectively, having
exactly the same eigen values. But it is obvious that for Eq. (3), the eigen value spectrum exactly coincides with that of Eq. (1) except for the state \( \Psi_1 \). The examples treated in the following sections show that the coincidence between the eigen values of \( u(x) \) and \( v(x) \) is only partial.

3 New exactly solvable models from the H.O. potential

We know that Schrodinger equation gives exact solutions for the H.O. potential,

\[
V(x) = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2
\]

where \( k \) is the force constant and \( \omega \) is the angular frequency of the particle of mass \( m \). The equation for H.O. is

\[
-\frac{\partial^2 \Psi}{\partial x^2} + \left[ \frac{m^2 \omega^2 x^2}{\hbar^2} \right] \Psi(x) = \frac{2mE}{\hbar^2} \Psi(x) \tag{6}
\]

where \( E \) satisfies \( (n + \frac{1}{2})\hbar \omega, \ (n = 0, 1, 2, ....) \) for the harmonic oscillator potential. Let us now show that DT leads new exactly solvable Schrodinger equation from Eq. (6). The new equation is,

\[
-\frac{\partial^2 \Phi}{\partial x^2} + u(x) \Phi(x) = \lambda \Phi(x) \tag{7}
\]

where \( u(x) \) and \( \Phi(x) \) are defined by DT (4) and (5).

To define logarithmic derivative we choose, first excited state \((n = 1)\) wave function of Eq. (6)

\[
\Psi_1(x, \lambda = \frac{3m\omega}{\hbar}) = \left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}} x \exp \left( -\frac{m\omega x^2}{2\hbar} \right) \tag{8}
\]

Therefore,

\[
\sigma = \frac{1}{\Psi_1} \frac{\partial \Psi_1}{\partial x} = \frac{1}{x} - \frac{m\omega x}{\hbar} \tag{9}
\]

It immediately follows from DT that,

\[
u(x) = \frac{m^2 \omega^2 x^2}{\hbar^2} + \frac{2m \omega}{\hbar} + \frac{2}{x^2} \tag{10}
\]

This is an anharmonic potential and DT gives exact solutions to this potential, provided various \( \Psi(x) \) of Eq. (6) are given. Now we will define \( \Phi(x) \), using Eq. (4) and \( \Psi(x) \) is arbitrarily chosen from Eq. (6).

Let us choose \( \Psi(x) \) as the ground state \((n = 0)\) wave function of the harmonic oscillator.

\[
\Psi(x) = \left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{4}} \frac{1}{x} \exp \left( -\frac{m\omega x^2}{2\hbar} \right) \tag{11}
\]

Hence,

\[
\Phi(x) = -\left[ \frac{m\omega}{\pi \hbar} \right]^{\frac{1}{4}} \frac{1}{x} \exp \left( -\frac{m\omega x^2}{2\hbar} \right) \tag{12}
\]

This represents an exact solution of Eq. (7) and corresponding eigen value is \( \lambda = \frac{m\omega}{\hbar} \).

To check the admissibility condition on the new wave function, we use the normalization condition of the wave function,

\[
\int_{-\infty}^{\infty} \Phi^* \Phi \, dx = N. \]

It is seen that The new wave function \( \Phi(x) \) is non-normalizable.
because of the essential singularity of this wave function at \( x = 0 \). This suggests that even though Eq. (7) is exactly solved, the new wave function \( \Phi(x) \) is quantum mechanically unacceptable. This rules out the existence of the corresponding eigen state \( (\lambda = \frac{m\omega}{\hbar}) \) from the new exactly solvable potential Eq. (10). A similar fate befalls other even order eigen states \( (n = 2, 4, 6, ...) \) as well.

Now let us choose \( \Psi(x) \) corresponding to third excited state \( (n = 3) \),

\[
\Psi(x) = \left[ \frac{m\omega}{\pi\hbar} \right]^\frac{1}{4} \sqrt{\frac{3m\omega}{\hbar}} [x - \frac{2m\omega x^2}{3\hbar}] \exp \left( -\frac{m\omega x^2}{2\hbar} \right) (13)
\]

which yields

\[
\Phi(x) = \left[ \frac{m\omega}{\pi\hbar} \right]^\frac{1}{4} \sqrt{\frac{3m\omega}{\hbar}} [-\frac{4m\omega x^2}{3\hbar}] \exp \left( -\frac{m\omega x^2}{2\hbar} \right) (14)
\]

This \( \Phi(x) \) is normalizable. The corresponding eigen value \( (\lambda) \) can be obtained from Eq. (7) on substitution of \( \Phi(x) \) and \( u(x) \) into that equation, which gives \( \lambda = \frac{7m\omega}{\hbar} \). Similarly it can be shown that if \( \Psi(x) \) chosen to be any solution of Eq. (6) corresponding to \( n = 3, 5, 7, ... \), the new solutions \( \Phi(x) \) are normalizable.

Thus the initial Schrodinger equation (6) for the H.O. potential leads to a new exactly solvable equation (7) for an anharmonic potential \( u(x) \) given by Eq. (10). When we assume solutions, \( \Psi(x) \) of initial equation, we can solve Eq. (7) to give new solutions \( \Phi(x) \). Thus we have two exactly solvable potentials instead of one. Even though the eigen value spectrum of \( u(x) \) coincides with that of H.O. potential, only those eigen states corresponding to \( n = 3, 5, 7, ... \) are quantum mechanically acceptable. Thus the coincidence of eigenvalues is only partial. In the above example we have taken first excited state wave function of Eq. (6) to form a new solvable potential \( u(x) \). Similarly we can take any solution of Eq. (6) to define the new exactly solvable potential. This leads to a large class of exactly solvable potentials for the Schrodinger equation. We should mention one more point regarding the above idea. We got a new exactly solvable equation (7) from (6) provided its wave functions are known. Similarly we can form another exactly solvable model from eq. (7), when all of its solutions are known. This procedure is called the iteration of DT. This can be further extended. Thus starting from a single Schrodinger equation exactly solvable potential, we can generate a wide class of exactly solvable potentials by DT.
4 Coulomb potential (Hydrogen atom)

For a two particle central potential problem, the general solution of Schrodinger equation [17] has a radial and an angular part such as \( \Psi(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi) \). The radial part \( R(r) \) satisfies radial wave equation,

\[
- \frac{d^2 R}{dr^2} - \frac{2}{r} \frac{dR}{dr} + v(r) R(r) = \lambda R(r) \tag{15}
\]

where \( v = \frac{2\mu V(r)}{\hbar^2} + \frac{l(l+1)}{r^2} \) and \( \lambda = \frac{2\mu E}{\hbar^2} \); \( \mu \) is the reduced mass of the two particle system and \( \frac{l(l+1)}{r^2} \) is the centrifugal potential. For Hydrogen atom, Coulomb potential (in cgs system of units) is \( V(r) = -\frac{Ze^2}{r} \) and \( E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}, \ n = 1, 2, 3, ... \) (Z is the atomic number). If we use the expression for the classical Bohr radius of the atom \( a = \frac{\hbar^2}{\mu e^2} \), then

\[
v(r) = -\frac{2Z}{ar} + \frac{l(l+1)}{r^2} \tag{16}
\]

and

\[
\lambda = -\frac{Z^2}{a^2 n^2} \tag{17}
\]

4.1 Modification for Darboux transformation

In order to apply DT to Eq. (15), the usual transformation in the case of Schrodinger equation needs a slight modification. Darboux transformed exactly solvable radial equation can be written as,

\[
- \frac{d^2 \chi(r)}{dr^2} - \frac{2}{r} \frac{d\chi(r)}{dr} + u(r) \chi(r) = \lambda \chi(r) \tag{18}
\]

where

\[
\chi(r) = \frac{d}{dr} - \sigma \tag{19}
\]

by the Darboux transformation. Where the logarithmic derivative \( \sigma \) is defined as

\[
\sigma = \frac{1}{R_1} \frac{dR_1}{dr} \tag{20}
\]

\( R_1(r) \) is a particular solution of Eq. (15) for \( \lambda = \lambda_1 \). \( R(r) \) is arbitrarily chosen from the solution set of Eq. (15). \( R(r) \) can be any radial wave function of Eq. (15) except that used to define logarithmic derivative).

It can be seen that radial wave equation is covariant under the transformations \( R(r) \to \chi(r) \) and \( v(r) \to u(r) \) only if

\[
u(r) = \frac{d\sigma}{dr} - \frac{2}{r} \frac{d\sigma}{dr} + \frac{2}{r^2} \tag{21}
\]

To arrive at this, substitute \( \chi(r) \) from Eq. (19) into Eq. (18) and making use of Eq. (15) we get,

\[
[u(r) - \nu(r) + \frac{2}{r^2} \frac{d\sigma}{dr} - \frac{2}{r} \frac{dR(r)}{dr} + \frac{d\nu(r)}{dr} + \frac{d^2 \sigma}{dr^2} + \sigma \nu(r) + \frac{2}{r} \frac{d\sigma}{dr} - \sigma u(r)]R(r) = 0 \tag{22}
\]
This equation is meaningful only if the coefficients of \( \frac{dR}{dr} \) and \( R \) are identically zero, which yields the Darboux transformed potential as,

\[
u(r) = v(r) - 2 \frac{d\sigma}{dr} + \frac{2}{r^2} \tag{23}\]

Thus Eq. (17) gives a new exactly solvable radial wave equation for a potential \( u(r) \) under the transformations Eqs. (19) and (21). If the \( l \) values in the definition of the logarithmic derivative and in the expression for \( v(r) \) are different, the potential transformation becomes inadmissible, because any intermediate modification to the potential other than that suggested by Eq. (21) does not come under the purview of Darboux theorem. This means that we cannot apply DT to an \( s \)-state wave function \((l = 0)\) using the logarithmic derivative defined by another \( p \)-state wave function \((l = 1)\) and vice versa.

### 4.2 New exactly solvable models using Coulomb potential

suppose initial radial equation (15) is exactly solved for Hydrogen atom and assume that its solutions \([17]\) are known. Using this reference system we can construct a large number of radial equations exactly solvable potentials by using Eq. (21). Thus Eq. (18) above is an exactly solvable one and substituting for \( u(r) \) and \( \chi(r) \) in this equation, we can also find the value of \( \lambda \). \( u(r) \) and \( v(r) \) have got identical eigen value spectra.

To define the logarithmic derivative, we choose 1\( s \)-wave function \((l = 0, n = 1)\) of hydrogen atom, which is a solution of Eq. (15) ie.,

\[
R_{n_l}(r, \lambda_1) = R_{1s}(r, \lambda_1) = R_{1s}(r, - \frac{Z^2}{a^2}) = 2(\frac{Z}{a})^{\frac{3}{2}} \exp(-\frac{Zr}{2a}) \tag{24}\]

where \( a \) is Bohr radius of the atom. Since \( l = 0 \), \( v(r) = - \frac{2Z}{ar} \)

\[
\sigma = \frac{1}{R_{1s}} \frac{dR_{1s}}{dr} = - \frac{Z}{a} \tag{25}\]

\[
u(r) = - \frac{2Z}{ar} + \frac{2}{r^2} \tag{26}\]

Thus DT provides a new exactly solvable radial equation Eq. (18) for a new potential Eq. (26). This can be exactly solved knowing various radial functions of Eq. (15). The solutions \( \chi(r) \) of new radial equation are given by DT Eq. (19). Let us arbitrarily choose \( R(r) \) as 2\( s \) wave function \((n = 2, l = 0)\) of hydrogen atom radial equation,

\[
R(r) = \frac{1}{\sqrt{2}} \left( \frac{Z}{a} \right)^{\frac{3}{2}} \left( 1 - \frac{Zr}{2a} \right) \exp(-\frac{Zr}{2a}) \tag{27}\]

It immediately follows that

\[
\chi(r) = - \frac{1}{4\sqrt{2}} \left( \frac{Z}{a} \right)^{\frac{3}{2}} \frac{Z^2r}{a^2} \exp(-\frac{Zr}{2a}) \tag{28}\]

From the new radial equation Eq. (18), we will get the corresponding eigen value \( \lambda = \)
$-\frac{\hbar^2}{2m^2}. It is clear from Eq. (17) that this eigen value coincides with that of hydrogen atom radial equation. Other solutions of Eq. (18) can be obtained in a similar way, by using \( R(r) \) as 3s, 4s... wave functions of hydrogen atom.

Thus a wide class of radial equations with exactly solvable potentials can be deduced from hydrogen atom radial equation. In some cases the eigen values of the new and old potentials exactly coincide and in some other cases the coincidence is only partial.

5 Conclusions

We attempted to formulate a large class of one dimensional exactly solvable potentials for the Schrodinger equation by applying Darboux transformations. By applying DT to a reference problem, the class of potentials for which Schrodinger equation can be exactly solved can be enlarged. We can summarise the results as follows:

1. DT can be effectively used to construct an infinite number of exactly solvable systems from a known system.
2. Not all systems obtained by the Darboux procedure are quantum mechanically admissible, but a large number of valid systems can be generated.
3. Darboux transformed Hamiltonian share the eigen value spectrum of the original Hamiltonian only partially.
4. Physical systems representing new solvable potentials to be studied further.

References


